CHAPTER 4

FUZZY CLOSURE AND FUZZY POINT

4.1 INTRODUCTION

Three years after Zadeh [86] introduced the notion of a fuzzy set in his seminal paper in 1965, the first paper of fuzzy topology introduced by Chang [20], appeared in 1968. Atanassov [6] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Coker [22] came out with the concept called "intuitionistic fuzzy topological spaces". Tong[81, 82] introduced and investigated strong forms of open sets called strong regular open, θ-open and δ-open sets respectively. Mashour et al ([60, 61]), Levine ([44, 45]), Njastad [62], Abd El-Monsef et al [1], Arya and Noiri ([69, 70]), Bhattacharyya and Lahiri [16], Palaniappan and Rao [71], Maki et al [54, 55, 56, 57] and Dontchev [27] introduced respectively pre open sets, semi-open sets, g-open sets, α-open sets, β-open sets, generalized semi-open sets, semi-generalized open sets,

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The results discussed in this chapter have published in four different research papers in International Journals

1. Basumatary B., Fuzzy Interior and Fuzzy Closure with Extended Definition of Fuzzy Set, *Int. J. Computational System Engineering*, Vol-2, No-2, 2015.
2. Basumatary B., A Note on Fuzzy Closure of Fuzzy Set, *JPMNT*, Vol-3, issue-4, pp.35-39,(2015).
3. Basumatary B., A note on relation between Fuzzy interior and Fuzzy closure with extended definition of Fuzzy set ,*Dimorian Review e-journal* Vol. 3, No.2, pp8-13.
4. Singh K. P. & Basumatary B., A Note on Quasi-Coincidence for Fuzzy Points of Fuzzy Topology on the Basis of Reference Function, *I. J. Math. Sc. & Computing*, Vol-2, No-3, 2016.

regular generalized open sets, generalized preopen sets and generalized semipreopen sets which are some weak forms of open sets. Various types of sets have been studied in (Dontchev [27], Maki [52, 53, 54, 55]). Ganster and Reily [33] and Balachandran et al [7] have introduced locally closed sets and generalized locally closed sets which are weaker than open and closed sets.

Now let us discuss on some definition and properties of interior and closure of a crisp set of classical topology.

4.2 TOPOLOGY

Let X be a nonempty set and τ be a family of subset in X satisfying the following axioms

(T1) $0_X, 1_X \in \tau$

(T2) $G_1 \cap G_2 \in \tau$, for any $G_1, G_2 \in \tau$

(T3) $\bigcup G_i \in \tau$, for any arbitrary family $\{G_i : G_i \in \tau, i \in I\}$.

In this case the pair (X,τ) is called a topological space any member in τ is known as open set in X and clearly every element of τ^C is said to be closed set.

4.2.1 CLOSURE OF A SET

Let (X,τ) be a topological space and A be subset of X, then closure of A is defined as

$Cl(A) = \bigcap \{Q : Q \text{ is closed set in X and } A \subseteq Q\}$

Example: Let X={a, b, c, d} and let τ={ϕ, {a}, {b, c}, {a, d}, {a, b, c}, X}, then (X, τ) be topological space. Let A={b}, B={a, b}, D={b, c, d}, we try to find closures for the sets A, B and D.

Here closed subsets of X are X, {b, c, d}, {a, d}, {b, c}, {d}, ϕ.

Hence Cl(A)={b, c}. Cl(B)=X, because X is the only closed set containing B.

Cl(D)={b, c, d}.

4.2.1.1 Theorem: Let (X,τ) be a topological space and A be subset of X. Then A is closed if and only if Cl(A)=A.

4.2.1.2 SOME THEOREMS ON CLOSURES OF A SET

Let (X,τ) be a topological space and A, B be subsets of X. Then

(i) Cl(X)=X, Cl(ϕ)=ϕ

(ii) $A \subseteq Cl(A)$

(iii) $A \subseteq B \Rightarrow Cl(A) \subseteq Cl(B)$

(iv) $Cl(A \cup B) = Cl(A) \cup Cl(B)$

(v) Cl(Cl(A))=Cl(A).

4.3. A NEW APPROACH TO THE DEFINITION OF FUZZY TOPOLOGY

4.3.1 Definition:

A fuzzy topology on a nonempty set X is a family τ of fuzzy set in X satisfying the following axioms

(T1) $0_X, 1_X \in \tau$

(T2) $G_1 \cap G_2 \in \tau$, for any $G_1, G_2 \in \tau$

(T3) $\cup G_i \in \tau$, for any arbitrary family $\{G_i : G_i \in \tau, i \in I\}$.

In this case the pair (X,τ) is called a fuzzy topological space and any fuzzy set in τ is known as fuzzy open set in X and clearly every element of τ^C is said to be closed set.

Here $1_x=\{x, 1, 0; x\epsilon X\}$ and $0_X=\{x, \mu(x), \mu(x); x\epsilon X\}$.

Example

Let X={a, b}

Let A={(a, 0.5, 0), (b, 0.6, 0)}, B={(a, 0.8, 0), (b, 0.7, 0)}.

Then the family $\tau=\{0_X, 1_X, A, B\}$ is fuzzy topology, here $1_x=\{x, 1, 0; x\epsilon X\}$ and

$0_X=\{x, 0, 0; x\epsilon X\}$.

4.4 FUZZY CLOSURE ON THE BASIS OF REFERENCE FUNCTION

Let (X, τ) be fuzzy topology also $A=\{x, \mu_A(x), \gamma_A(x); x\epsilon X\}$ be fuzzy set on X.

Then closure of a fuzzy set A is defined as the intersection of all the closed subsets containing A, denoted it as cl(A) and is defined as follows

$Cl(A) = \cap \{Q : Q \text{ is closed set in X and } A \subseteq Q\}$

$=\{x, \min(\mu_{iA}), \max(\gamma_{iA}); x\epsilon X\}$

Here $\min(\mu_{iA})$ is the membership function, $\max(\gamma_{iA})$ is the reference function of fuzzy set A and their difference that is [$\min(\mu_{iA}) - \max(\gamma_{iA})$] is the membership value.

Example:

Let X= {a, b} and

A= { {a, 0.2, 0}, {b, 0.1, 0}}

B= {{a, 0.4, 0}, {b, 0.3, 0}}.

Then the family $\tau=\{1_X, 0_X, A, B\}$ of fuzzy set in X is fuzzy topology on X.

Let C={{a, 0.5, 0.4}, {b, 0.5, 0.3}}, where in fuzzy set C the membership value of a is 0.1 and the membership value of b is 0.2.

Then G_1={{a, 1, 0.2}, {b, 1, 0.1}} and G_2= {{a, 1, 0.4}, {b, 1, 0.3}} are closed sets.

Therefore cl(C)=$G_1 \cap G_2$={{a, 1, 0.4}, {b, 1, 0.3}}.

4.4.1 Theorems on fuzzy closure:

Let (X, τ) be fuzzy topology, then

(i) $cl(1_X)=1_X, cl(0_X)=0_X$

It is clear to see that the proposition is true when we expressed fuzzy set on the basis of reference function.

(ii) $A \subseteq cl(A)$

It is clear to see that the proposition is true when we expressed fuzzy set on the basis of reference function.

Example:

Let X= {a, b} and

A= {{a, 0.2, 0}, {b, 0.1, 0}}

B= {{a, 0.4, 0}, {b, 0.3, 0}}.

Then the family $\tau=\{1_X, 0_X, A, B\}$ of fuzzy set in X is fuzzy topology on X.

Let C={{a, 0.5, 0.4}, {b, 0.5, 0.3}}, where in fuzzy set C the membership value of a is 0.1 and the membership value of b is 0.2.

Then G_1={{a, 1, 0.2}, {b, 1, 0.1}} and G_2= {{a, 1, 0.4}, {b, 1, 0.3}} are closed sets.

Therefore cl(C)=$G_1 \cap G_2$={{a, 1, 0.4}, {b, 1, 0.3}}.

From this example it is clear $C \subseteq$ cl(C).

(iii) cl(cl(A))=cl(A)

Example:

Let X= {a, b} and

A= { {a, 0.2, 0}, {b, 0.1, 0}}

B= {{a, 0.4, 0}, {b, 0.3, 0}}.

Then the family $\tau=\{1_X, 0_X, A, B\}$ of fuzzy set in X is fuzzy topology on X.

Let C={{a, 0.5, 0.4}, {b, 0.5, 0.3}}, where in fuzzy set C the membership value of a is 0.1and the membership value of b is 0.2.

Then G_1={{a, 1, 0.2}, {b, 1, 0.1}} and G_2= {{a, 1, 0.4}, {b, 1, 0.3}} are closed sets.

Therefore cl(C)=$G_1 \cap G_2$={{a, 1, 0.4}, {b, 1, 0.3}}.

Now if we consider cl(C)=D, then cl(D)={{a, 1, 0.4}, {b, 1, 0.3} }=cl(C).

Hence cl(cl(C))=cl(C).

(iv) $A \subseteq B \Rightarrow cl(A) \subseteq cl(B)$

Example:

Let X= {a, b} and

A= { {a, 0.2, 0}, {b, 0.1, 0}}

B= {{a, 0.4, 0}, {b, 0.3, 0}}.

Then the family τ={1_X, 0_X, A, B} of fuzzy set in X is fuzzy topology on X.

Let C={{a, 0.5, 0.4}, {b, 0.5, 0.3}}, where in fuzzy set C the membership value of "a" is 0.1 and the membership value of "b" is 0.2.

D={{a, 0.6, 0.3}, {b, 0.5, 0.2}}, where in fuzzy set C the membership value of "a" is 0.3 and the membership value of "b" is 0.3.

Clearly from our definition of fuzzy set $C \subseteq D$.

Now cl(C)= {{a, 1, 0.4}, {b, 1, 0.3}} and cl(D)= {{a, 1, 0.2}, {b, 1, 0.1}}, which shows that cl (C)$\subseteq$cl (D).

Hence C$\subseteq$D $\Rightarrow$cl (C)$\subseteq$cl (D).

(v) cl(A$\cup$B)= cl(A)$\cup$cl(B)

Example:

Let X= {a, b} and

A= { {a, 0.2, 0}, {b, 0.1, 0}}

B= {{a, 0.4, 0}, {b, 0.3, 0}}.

Then the family τ={1_X, 0_X, A, B} of fuzzy set in X is fuzzy topology on X.

Let C={{a, 0.5, 0.4}, {b, 0.5, 0.3}}, where in fuzzy set C the membership value of "a" is 0.1 and the membership value of "b" is 0.2.

D={{a, 0.6, 0.3}, {b, 0.5, 0.2}}, where in fuzzy set C the membership value of "a" is 0.3 and the membership value of "b" is 0.3.

Now cl(C) = {{a, 1, 0.4}, {b, 1, 0.3}} and cl(D)= {{a, 1, 0.2}, {b, 1, 0.1}}.

Also C$\cup$D = {{a, 0.5, 0.4}, {b, 0.5, 0.3}} $\cup$ {{a, 0.6, 0.3}, {b, 0.5, 0.2}}

= {{a, 0.6, 0.3}, {b, 0.5, 0.2}}

So Cl(C$\cup$D) = {{a, 1, 0.2}, {b, 1, 0.1}}.

$Cl(C) \cup Cl(D)$ = {{a, 1, 0.4}, {b, 1, 0.3}} $\cup$ {{a, 1, 0.2}, {b, 1, 0.1}}.

= {{a, 1, 0.2}, {b, 1, 0.1}}.

Hence $Cl(C \cup D) = Cl(C) \cup Cl(D)$.

4.4.2 Proposition:

Let A={x, $\mu_A(x)$, $\gamma_A(x)$; xϵX} be fuzzy set in fuzzy topological space (X, τ). If A is fuzzy closed set then A=cl(A).

4.5 Theorem:

Let X be a non empty set. Consider an operator c: $I^X \rightarrow I^X$ satisfying the following condition

i) $c(0_X)= 0_X$

ii) $A \subseteq c(A)$, where A is fuzzy subset of X.

iii) $c(A \cup B)= c(A) \cup c(B)$, where A and B are fuzzy subsets of X,

then δ={A: c(A)=A, A is fuzzy subset of X} is fuzzy topology on X.

Also if operator c also fulfils the condition

iv) c(c(A))=c(A), for any fuzzy set in X.

Then fuzzy topology δ defined above, in fuzzy topology (X, δ), cl(A)=c(A).

It is clearly seen that the above theorem is true when we apply our extended definition of fuzzy set.

4.6 FUZZY POINT ON THE BASIS OF REFERENCE FUNCTION:

Let X be a non empty set and p be a fixed element of X. Let $r\epsilon(0, 1)$ and $s\epsilon[0, 1)$ such that $r\text{-}s\langle 1$, then the fuzzy set $p^r{}_s(y)=\{ x, p_r(x), p_s(x); x\epsilon X\}$ is called fuzzy point in X, where $p_r(x)=r$, when $x=y$, otherwise zero, denotes the membership function and $p_s=r$, when $x=y$, otherwise zero, denotes the reference function.

Note: Let $A=\{x, \mu_A(x), \gamma_A(x); x\epsilon X\}$. The fuzzy point $p^r{}_s=\{ x, p_r(x), p_s(x); x\epsilon X\}$ is contained in A if and only if $\mu_A(x)\geq p_r(x)$ and) $\gamma_A(x)\leq p_s(x)$.

4.6.1 Definitions:

1. Let A and B are two fuzzy sets in X then A and B are said to be intersecting to each other if and only if there exists a point $x\epsilon X$ such that $A\bigcap B \neq \phi$.

2. Also, two fuzzy sets A and B are said to be equal if and only if $p\epsilon A \Leftrightarrow p\epsilon B$, for fuzzy point p in X.

4.6.1.1 Properties

Let us consider the family of fuzzy sets$\{A_i: i\epsilon I\}$ in X and P be fuzzy point on X. Then

1. If $P\epsilon\bigcap \{A_i: i\epsilon I\}$, then for If $i\epsilon I$ we have If $P\epsilon A_i$.
2. $f(P^C)=(f(P))^C$

It is seen that these properties are easily verified if the complementation is defined on the basis of reference function.

4.7 Definition:

A fuzzy point p is said to be quasi-coincident with the fuzzy set A if $p \supseteq A^C$, denoted by $p\mathit{q}A$.

4.7.1 Proposition Let (X, δ) be fuzzy topology. Let A and B be two fuzzy sets then AqB at $x \Leftrightarrow$ BqA at x.

Proof

Case1

When reference function is zero.

Let A={x, μ_1(x), 0; x∈X} and B={x, μ_2(x), 0; x∈X}

Let AqB at $x \Rightarrow A \supseteq B^C$ at x

$\Rightarrow (B^C)^C \supseteq A^C$ at x

$\Rightarrow B \supseteq A^C$ at x

Hence, BqA at x

Conversely let BqA at x

Now BqA at x $\Rightarrow B \supseteq A^C$ at x

$\Rightarrow (A^C)^C \supseteq B^C$ at x

$\Rightarrow A \supseteq B^C$ at x

Thus AqB at x.

Case 2

When reference function is not zero.

Let A={x, $\mu_1(x)$, $\gamma_1(x)$; x∊X} and B={x, $\mu_2(x)$, $\gamma_2(x)$; x∊X}

And B^C={x,1, $\mu_2(x)$; x∊X}∪{x, $\gamma_2(x)$, 0; x∊X}.

Now

Let AqB at x $\Rightarrow$ $A \supseteq B^C$ at x

$\Rightarrow$ Membership value of A ≥ Membership value of B^C, at x

$\Rightarrow (\mu_1(x)- \gamma_1(x))\geq (1- \mu_2(x)) + \gamma_2(x)$

$\Rightarrow 1-(\mu_1(x)- \gamma_1(x))\leq 1- (1- \mu_2(x)) + \gamma_2(x)$

$\Rightarrow 1-(\mu_1(x)- \gamma_1(x))\leq (\mu_2(x)) - \gamma_2(x)$

$\Rightarrow$ Membership value of $A^C \leq$ Membership value of B

$\Rightarrow$ BqA at x

Conversely let BqA then A^C={x,1, $\mu_1(x)$; x∊X}∪{x, $\gamma_1(x)$, 0; x∊X}.

Now BqA at x $\Rightarrow$ $B \supseteq A^C$ at x

$\Rightarrow$ Membership value of B ≥ Membership value of A^C at x

$\Rightarrow (\mu_2(x)- \gamma_2(x))\geq (1- \mu_1(x)) + \gamma_1(x)$

$\Rightarrow 1-(\mu_2(x)-\gamma_2(x)) \leq 1-(1-\mu_1(x))+\gamma_1(x)$

$\Rightarrow 1-(\mu_2(x)-\gamma_2(x)) \leq (\mu_1(x))-\gamma_1(x)$

$\Rightarrow$ Membership value of $B^C \leq$ Membership value of A, at x

$\Rightarrow$ AqB, at x

Hence AqB at x $\Leftrightarrow$ BqA at x.

4.7.2 Proposition Let (X, δ) be fuzzy topology. Let A and B be two fuzzy sets then AqB $\Leftrightarrow$ BqA.

Proof We can prove this proposition by following prove of the proposition1.

4.7.3. Proposition Let (X, δ) be fuzzy topology. Let A, B and C be fuzzy sets, if $A \subseteq B$ then CqA $\Rightarrow$ CqB.

Proof

Case1

When reference function is zero.

Let A={x, $\mu_1(x)$, 0; x∈X} , B={x, $\mu_2(x)$, 0; x∈X} and C={x, $\mu_3(x)$, 0; x∈X}.

We have $A \subseteq B$ so clearly $\mu_1(x) \leq \mu_2(x)$.

Now

CqA $\Rightarrow C \supseteq A^C$

Since, $A \subseteq B \Rightarrow B^C \subseteq A^C$.

So $CqA \Rightarrow C \supseteq A^C \supseteq B^C$

$\Rightarrow C \supseteq B^C$

$\Rightarrow CqB$

Case 2:

When reference function is not zero.

Let $A=\{x, \mu_1(x), \gamma_1(x); x\epsilon X\}$, $B=\{x, \mu_2(x), \gamma_2(x); x\epsilon X\}$ and $C=\{x, \mu_3(x), \gamma_3(x); x\epsilon X\}$.

Also, $A^C=\{x,1, \mu_1(x); x\epsilon X\} \cup \{x, \gamma_1(x), 0; x\epsilon X\}$, $B^C=\{x,1, \mu_2(x); x\epsilon X\} \cup \{x, \gamma_2(x), 0; x\epsilon X\}$.

Now

$CqA \Rightarrow C \supseteq A^C$

$\Rightarrow$ Membership value of C $\geq$ Membership value of A^C

$\Rightarrow (\mu_3(x)- \gamma_3(x)) \geq (1- \mu_1(x)) + \gamma_1(x)$

Again as

$A \subseteq B \Rightarrow B^C \subseteq A^C$

$\Rightarrow$ Membership value of $B^C \leq$ Membership value of A^C

$\Rightarrow (1- \mu_2(x)) + \gamma_2(x) \leq (1- \mu_1(x)) + \gamma_1(x)$.

Hence

$CqA \Rightarrow (\mu_3(x)- \gamma_3(x)) \geq (1- \mu_1(x)) + \gamma_1(x) \geq (1- \mu_2(x)) + \gamma_2(x)$

$\Rightarrow$ ($\mu_3(x)$- $\gamma_3(x)$)$\geq$ (1- $\mu_2(x)$) + $\gamma_2(x)$

$\Rightarrow C \supseteq B^C$

$\Rightarrow$ CqB

Therefore when $A \subseteq B$ then CqA $\Rightarrow$ CqB.

4.7.4 Proposition Let (X, δ) be fuzzy topology. Let A, B fuzzy sets, if $A \subseteq B$ then pqA $\Rightarrow$ pqB.

Proof Prove is straightforward.

4.7.5 Proposition Let (X, δ) and (Y, Γ) be two fuzzy topological spaces and let A and B be fuzzy sets. Let f be a function from X to Y then

i. $Aqf^{-1}(B) \Leftrightarrow f(A)qB$

ii. $AqB \Rightarrow f(A)q\, f(B)$

iii. $f^{-1}(A)\ q\ f^{-1}(B) \Rightarrow AqB$

i. Proof

We have $Aqf^{-1}(B) \Rightarrow A \supseteq (f^{-1}(B))^C$

$\Rightarrow A \supseteq (f^{-1}(B^C))$

$\Rightarrow f(A) \supseteq f(f^{-1}(B^C))$

$\Rightarrow f(A) \supseteq B^C$

$\Rightarrow$ f(A)qB

Conversely let f(A)qB.

Now

$f(A)qB \Rightarrow f(A) \supseteq B^C$

$\Rightarrow f^{-1}(f(A)) \supseteq f^{-1}(B^C)$

$\Rightarrow A \supseteq (f^{-1}(B))^C$

$\Rightarrow Aqf^{-1}(B)$

Hence $Aqf^{-1}(B) \Leftrightarrow f(A)qB$.

(ii) Proof

Let $AqB \Rightarrow A \supseteq B^C$

Now

$f(A)(y) = \bigcup \{A(x);\ x \in X: f(x)=y\}$

$\supseteq \bigcap \{B^C(x);\ x \in X: f(x)=y\}$, as $A \supseteq B^C$

$= (\bigcup \{B(x);\ x \in X: f(x)=y\})^C$

$= (f(B))^C$

$\Rightarrow f(A)q\, f(B)$

Hence $AqB \Rightarrow f(A)q\, f(B)$

(iii). Proof

The proved of the proposition is straightforward following prove of i and ii.

4.8 CONCLUSION

The main purpose of this chapter was to represent fuzzy closure on the basis of reference function. We defined the definition of fuzzy closure on the basis of reference function and also given examples. Further, some propositions on fuzzy closures are also discussed with example. Also, in this chapter we discussed on definition of fuzzy point and some of their propositions on the basis of reference function. It is seen that all the propositions on fuzzy closure and fuzzy points can be explained very interestingly with the help of our new definition of fuzzy set on the basis of reference function.