Chapter 4

Higher Dimensional LRS Bianchi Type-I Cosmological Model Universe Interacting with Perfect Fluid in Lyra Geometry

[3]

4.1 Introduction:

Einstein developed his general theory of relativity, where gravitation is described in terms of geometry. Based on the cosmological principle, Einstein introduced the cosmological constant into his field equations in order to obtain a static model of the universe because without the cosmological term his field equations admit only non static cosmological models for nonzero energy density. Later, Weyl (1918) proposed a more general theory in which electromagnetism is also described geometrically. He showed how one can introduce a vector field in the Riemannian space-time with an intrinsic geometrical significance. But this theory was based on non-integrability of length transfer so that it had some unsatisfactory features and hence this theory did not gain general acceptance, which is known as Weyl's geometry still today. After having these concepts, Lyra (1951) suggested a modification of Riemannian geometry, which may also be considered as a modification of Weyl's geometry, by introducing a gauge function into the structureless manifold which removes the non-integrability

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condition of the length of a vector under parallel transport and a cosmological constant is naturally introduced from the geometry. Halford (1970, 1972) pointed out that in the normal general relativistic treatment the constant displacement vector field ϕ_i in Lyra's geometry plays the role of cosmological constant and the scalar-tensor treatment based on Lyra's geometry predicts the same effect, within observational limits, as far as the classical solar system test are concerned (as in the Einstein's theory of relativity).

In the past and recent years many prominent researchers like Berman and Gomide (1988); Johri and Desikan (1994); Beesham (1993); Reddy et al. (2006); Reddy et al. (2007a); Reddy et al. (2007b); Adhav et al. (2008); Rao and Kumari (2012a); Bermann (1983); Singh and Agarwal (1993); Rahaman et al. (2005); Rao et al. (2008a); Adhav (2011); Zia and Singh (2012); Mollah et al. (2015); Darabi (2013); Panigrahi and Nayak (2014); Singh (2015); Asgar and Ansary (2014a,b); Sahoo and Mishra (2015); Adhav et al. (2015); Rao et al. (2015) have investigated and proposed different cosmological models and ideas of the universe within the framework of Lyra's geometry and other theories of relativity in different context. But the main problem in Astrophysics is the discovery, about two decades ago, that our Universe expansion is accelerating, instead of showing down as predicted by the Big Bang theory [Silk (1989)]. Observational evidence for accelerated expansion in the universe has been growing during this period [Ostriker (1995); Bagla et al. (1996); Efstathiou1 et al. (1990)]. Independent confirmation using observations of high red shift supernovae [Garnavich (1998); Perlmutter et al. (1999); Tonry et al. (2003); Barris et al. (2004); Riess et al. (2004); Astier et al. (2005)] along with observations of cosmic microwave background radiation (CMB) [Melchiorri et al. (2000); Spergel et al. (2003); Komatsu et al. (2009)] and large scale structure [Percival et al. (2007)] have made this result more acceptable to the community. In fact, the recent observations of Type SNeIa supernova, CMB anisotropies the large scale galaxies structures of universe and Sachs Wolf effects have led to the idea that our universe undergoes accelerated expansion at the present epoch tending to a de-Sitter space-time as predicted by inflation theory [Riess et al. (1998); Schmidt et al. (1998); Steinhardt et al. (1999); Persic et al. (1996); Cunha (2009)]

Moreover, solutions of Einstein field equations in higher dimensional space times are believed to be physical relevance possibly at extremely early times before the universe underwent the compactification transitions. As a result now-a-days higher dimensional theory is receiving great attention in both Cosmology and Particle Physics. Particle Physicists and cosmologists predicted to exist in GUT (Grand Unified Theory). Using a suitable scalar field it was shown that the phase transitions on the early universe can give rise to such objects which are nothing but the topological knots in the vacuum expectation value of the scalar field and most of their energy is concentrated in a small region. As the necessity of study of higher-dimensional space-time in this field aiming to unify gravity with other interactions, the concept of extra dimension is relevant in cosmology, particularly for the early stage of universe and theoretically the present four dimensional stage of the universe might have been preceded by a multi-dimensional stage.

So in this chapter we discussed about the higher dimensional cosmological models in Lyra geometry by considering locally rotationally symmetric (LRS) Bianchi Type-I metric with the use of deceleration parameter and certain physical assumption to find out the solutions which are realistic with the observational facts.

4.2 Field Equations and Their Solutions:

Here we consider the five dimensional plane symmetric metric in the form

$$ds^2 = A^2(dx^2 - dt^2) + B^2(dy^2 + dz^2) + C^2 dm^2 \tag{4.1}$$

with the convention $x^1 = x, x^2 = y, x^3 = z, x^4 = m, x^5 = t$ where A, B and C are functions of time t only. Here the extra coordinate is taken to be time like.

Einstein's field equations based on Lyra's geometry as used by Sen (1957) and Sen and Dunn (1971) is-

$$R_{ij} - \frac{1}{2}g_{ij}R + \frac{3}{2}\phi_i\phi_j - \frac{3}{4}g_{ij}\phi^k\phi_k = -8\pi T_{ij} \tag{4.2}$$

where ϕ_i is the displacement vector given by-

$$\phi_i = (0,0,0,0,\beta(t)) \tag{4.3}$$

and T_{ij} is the energy momentum tensor for the perfect fluid given by

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij} \tag{4.4}$$

where, ρ is the energy density, p is the pressure and u^i is the five velocity vector given by

$$u^i = (0,0,0,0,\frac{1}{A}) \tag{4.5}$$

Also let, $x^i = (\frac{1}{A}, 0, 0, 0, 0)$ so that

$$g_{ij}u^i u^j = -1 = -x^i x_i \quad and \quad u^i x_i = 0 \tag{4.6}$$

In comoving coordinate system, we have from (4.4)

$$\begin{aligned} &T_1^1 = T_2^2 = T_3^3 = T_4^4 = -p \ ; \ T_5^5 = \rho \ \ and \\ &T_j^i = 0 \ \ for \ \ i \neq j \end{aligned} \tag{4.7}$$

Using equations (4.3)-(4.7) , the surviving field equations of the equations (4.2) for the metric (4.1) are obtained as

$$2\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}^2}{B^2} + 2\frac{\dot{B}\dot{C}}{BC} - 2\frac{\dot{A}\dot{B}}{AB} - \frac{\dot{A}\dot{C}}{AC} + \frac{3}{4}\beta^2 = A^2 p \tag{4.8}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \frac{\dot{A}^2}{A^2} + \frac{\dot{B}\dot{C}}{BC} + \frac{3}{4}\beta^2 = A^2 p \tag{4.9}$$

$$\frac{\ddot{A}}{A} + 2\frac{\ddot{B}}{B} - \frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{3}{4}\beta^2 = A^2 p \tag{4.10}$$

$$\frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}\dot{B}}{AB} + 2\frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} - \frac{3}{4}\beta^2 = -\rho A^2 \tag{4.11}$$

Now from (4.9) and (4.10) we have

$$\frac{\dot{B}}{B} - \frac{\dot{C}}{C} = \frac{k_1}{B^2 C} \tag{4.12}$$

where $k_1 > 0$ is an integrating constant.

Since the field equations (4.8)-(4.11) are highly non linear, so in order to obtain the exact solution of the equations (4.8)-(4.11), we use the following scale transformations as used by Reddy and Venkateswarlu (1989)

$$A = e^a \ , \ B = e^b \ , \ C = e^c \quad and \quad dt = AB^2 C dT \tag{4.13}$$

Using transformations (4.13) in (4.8)-(4.11) we have

$$2b'' + c'' - 4a'b' - 2b'c' - 2c'a' - b'^2 = pe^{2(2a+2b+c)} - \frac{3}{4}\beta^2 e^{2(a+2b+c)} \tag{4.14}$$

$$a'' + b'' + c'' - 3a'b' - 2b'c' - 2c'a' - a'^2 - b'^2 = pe^{2(2a+2b+c)} - \frac{3}{4}\beta^2 e^{2(a+2b+c)} \tag{4.15}$$

$$a'' + 2b'' - 4a'b' - 2b'c' - c'a' - a'^2 - b'^2 = pe^{2(2a+2b+c)} - \frac{3}{4}\beta^2 e^{2(a+2b+c)} \tag{4.16}$$

$$2a'b' + 2b'c' + c'a' + b'^2 = -\rho e^{2(2a+2b+c)} + \frac{3}{4}\beta^2 e^{2(a+2b+c)} \tag{4.17}$$

where dashes denote derivative with respect to time T.

Solving equations (4.14)-(4.16) we have

$$a = b = c \tag{4.18}$$

Therefore from equation (4.13) we have

$$A = B = C \tag{4.19}$$

By using (4.19) in (4.8)-(4.11) we have

$$3\frac{\ddot{A}}{A} + \frac{3}{4}\beta^2 = A^2 p \tag{4.20}$$

$$6\frac{\dot{A}^2}{A^2} - \frac{3}{4}\beta^2 = -A^2 \rho \tag{4.21}$$

Since there are two independent equations involving four unknowns A, β , p and ρ, so in order get deterministic solutions of the above set of highly nonlinear equations (4.20)-(4.21) here we are using the special law of variation of Hubble's parameter proposed by Bermann (1983) that gives constant deceleration parameter as

$$q = -\frac{R\ddot{R}}{\dot{R}^2} \tag{4.22}$$

where q is a constant and

$$R = (A^2B^2C)^{\frac{1}{4}} \tag{4.23}$$

is the overall scale factor.

Here the constant q is taken as negative so the model is an accelerating model of the universe.

Solving equation (4.22) we have

$$R = (\alpha t + \gamma)^{\frac{1}{q+1}} \tag{4.24}$$

where $\alpha \neq 0$ and γ are constants and $q+1 \neq 0$.

By using (4.19) in equation (4.23) we have

$$A = R^{\frac{4}{5}} \tag{4.25}$$

Therefore from (4.19), (4.24) and (4.25) we have

$$A = B = C = R^{\frac{4}{5}} = (\alpha t + \gamma)^{\frac{4}{5(q+1)}} \tag{4.26}$$

Now we are considering the following two cases:

Case - I : When β is a constant :

From (4.20) and (4.21) we have

$$p = \frac{3}{4}\frac{\beta^2}{(\alpha t + \gamma)^{\frac{8}{5(q+1)}}} - \frac{12\alpha^2(5q+1)}{25(q+1)^2(\alpha t + \gamma)^{\frac{8}{5(q+1)}+2}} \tag{4.27}$$

$$\rho = \frac{3}{4}\frac{\beta^2}{(\alpha t+\gamma)^{\frac{8}{5(q+1)}}} - \frac{96\alpha^2}{25(q+1)^2(\alpha t+\gamma)^{\frac{8}{5(q+1)}+2}} \tag{4.28}$$

Here the integrating constants α and γ are to be chosen in such a way that ρ and p are non-negative.

Therefore the metric (4.1) can be written as

$$ds^2 = (\alpha t+\gamma)^{\frac{8}{5(q+1)}}[dx^2+dy^2+dz^2+dm^2-dt^2] \tag{4.29}$$

The above equation (4.29) together with the equation (4.27) and (4.28) will be the exact 5-D LRS Bianchi type-I perfect fluid cosmological model in Lyra Geometry when β is a constant.

Now if we take $q=\beta=0$ then from equation (4.27) and (4.28) we have

$$p = -\frac{12\alpha^2}{25(\alpha t+\gamma)^{\frac{18}{5}}}$$

$$\rho = -\frac{96\alpha^2}{25(\alpha t+\gamma)^{\frac{18}{5}}}$$

Since both ρ and p are negative so from the above two equations we have

$$\rho = 8p$$

which satisfies the general equation of state $p=\delta\rho$

Case - II : When β is a function of t:

There are two independent field equations (4.20) and (4.21) involving three unknowns ρ , p and β . So in order to get deterministic solution we must have to assume a physical or mathematical condition amongst the unknowns. Here we consider the equation of state (i.e. physical condition) as

$$p = \delta\rho \tag{4.30}$$

Case - II-(a) : Dust (or, In coherent matter) distribution [$\delta = 0$, i.e. $p = 0$ and $\rho \neq 0$]:

When $\delta = 0$ then from (4.30) we have

$$p = 0 \tag{4.31}$$

Putting $p = 0$ in (4.20) we have

$$\frac{3}{4}\beta^2 = \frac{12\alpha^2(5q+1)}{25(q+1)^2(\alpha t+\gamma)^2} \tag{4.32}$$

Using (4.32) in (4.21) we have

$$\rho = \frac{12\alpha^2(5q-7)}{25(q+1)^2(\alpha t+\gamma)^{\frac{8}{5(q+1)}+2}} \tag{4.33}$$

The above equation (4.29) together with equations (4.31)-(4.33) will constitute an exact 5-D LRS Bianchi type-I coherent matter distribution model universe in Lyra geometry.

Case - II-(b) : Stiff (or, Zel'dovich) fluid distribution [$\delta = 1$]:

When $\delta = 1$ then from (4.30) we have

$$p = \rho \tag{4.34}$$

When $\delta = 1$ i.e. when $p = \rho$ then we can see that it is not possible to find out a physically meaningful solution for the field equations.

Therefore when β is a function of time t then Bianchi type-I cosmological stiff fluid universe does not exist in this theory.

Case - II-(c) : Disordered distribution of Radiation (or, Radiation Universe) [$\delta = \frac{1}{3}$]:

When $\delta = \frac{1}{3}$ then from (4.30) we have

$$\rho = 3p \tag{4.35}$$

Using $\rho = 3p$ in (4.20) and (4.21) we have

$$p = \frac{6\alpha^2(5q-7)}{25(q+1)^2(\alpha t+\gamma)^{\frac{8}{5(q+1)}+2}} \tag{4.36}$$

$$\rho = \frac{18\alpha^2(5q-7)}{25(q+1)^2(\alpha t+\gamma)^{\frac{8}{5(q+1)}+2}} \tag{4.37}$$

Therefore from (4.20) [or, (4.21)] we have

$$\frac{3}{4}\beta^2 = \frac{6\alpha^2(3q-1)}{5(q+1)^2(\alpha t+\gamma)^2} \tag{4.38}$$

The equation (4.29) together with the equations (4.35)-(4.38) will constitute an exact 5-D LRS Bianchi type-I radiating model universe in Lyra geometry.

Case - II-(d) : Matter distribution in internebular space [$\delta = \frac{2}{3}$ i.e. when $\rho = \frac{3}{2}p$]:

When $\delta = \frac{2}{3}$ then from (4.30) we have

$$\rho = \frac{3}{2}p \tag{4.39}$$

Using $\rho = \frac{3}{2}p$ we have from (4.20) and (4.21)

$$p = \frac{24\alpha^2(5q-7)}{25(q+1)^2(\alpha t+\gamma)^{\frac{8}{5(q+1)}+2}} \tag{4.40}$$

$$\rho = \frac{36\alpha^2(5q-7)}{25(q+1)^2(\alpha t+\gamma)^{\frac{8}{5(q+1)}+2}} \tag{4.41}$$

Therefore from (4.20) [or, (4.21)] we have

$$\frac{3}{4}\beta^2 = \frac{12\alpha^2(15q-13)}{25(q+1)^2(\alpha t+\gamma)^2} \tag{4.42}$$

The equation (4.29) together with the equations (4.39)-(4.42) will constitute an exact 5-D LRS Bianchi type-I cosmological model universe in the matter distribution in internebular space in Lyra geometry.

In all the cases (II)-(a)-(d) the reality condition $\rho > 0$ is obtained as

$$q > \frac{7}{5} \tag{4.43}$$

Now in all the above four cases we see that the value of the deceleration parameter $q > 0$ i.e. our model is an accelerating one.

4.3 Physical and Geometrical Properties of the Solutions:

Here the spatial volume V and the average scale factor $R(t)$ for the Bianchi type-I plane symmetric metric (4.1) defined by $V = R^4(t) = (-g)^{\frac{1}{2}} = A^2B^2C = A^5$ of the model are given by

$$V = (\alpha t + \gamma)^{\frac{4}{q+1}} \tag{4.44}$$

and

$$R(t) = (\alpha t + \gamma)^{\frac{1}{q+1}} \tag{4.45}$$

We observed that the volume V is increasing with the increase of time if $q+1 > 0$ i.e. if $q > -1$ and the volume V is decreasing with the increase of time and tend to zero as $t \to \infty$ if $q+1 < 0$ i.e. if $q < -1$. Also the scale factor R is increasing with the increase of time if $q+1 > 0$ i.e. if $q > -1$ and the scale factor R is decreasing with the increase of time and tend to zero as $t \to \infty$ if $q+1 < 0$ i.e. if $q < -1$

Also the mean Hubble's parameter H is obtained as

$$H = \frac{\alpha}{(q+1)(\alpha t + \gamma)} \tag{4.46}$$

From the above equation (4.46) it has been observed that in the initial stage when $t = 0$, $H = \alpha/[(q+1)\gamma]$. Again the value of H decreases with the increase of time t and finally H

becomes zero whenever $t \to \infty$. Also the Hubble's parameter H becomes infinite whenever $q = -1$ or $t = -\frac{\gamma}{\alpha}$.

The expansion factor θ calculated for the flow vector u^i is given by

$$\theta = u^i_{;i} = \frac{1}{A}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) = 4\frac{\dot{A}}{A^2} = \frac{16\alpha}{5(q+1)(\alpha t + \gamma)^{\frac{4}{5(q+1)}+1}} \tag{4.47}$$

The model has a singularity at $t = -\frac{\gamma}{\alpha}$ and the scalar expansion $\theta \to 0$ as time $t \to \infty$ if $q > -1/5$.

The components of the shear scalar σ for the metric (4.1) are given by

$$\sigma^1_1 = \frac{1}{A}\left(\frac{\dot{A}}{A} - \frac{A\theta}{4}\right)$$

$$\sigma^2_2 = \frac{1}{A}\left(\frac{\dot{B}}{B} - \frac{A\theta}{4}\right)$$

$$\sigma^3_3 = \frac{1}{A}\left(\frac{\dot{B}}{B} - \frac{A\theta}{4}\right)$$

$$\sigma^4_4 = \frac{1}{A}\left(\frac{\dot{C}}{C} - \frac{A\theta}{4}\right)$$

$$\sigma^5_5 = 0$$

Therefore the shear scalar σ for the metric (4.1) is given by

$$\sigma^2 = \frac{1}{2}\sigma^{ij}\sigma_{ij} = \frac{1}{2}[(\sigma^1_1)^2 + (\sigma^2_2)^2 + (\sigma^3_3)^2 + (\sigma^4_4)^2 + (\sigma^5_5)^2] = 0 \tag{4.48}$$

Since $\sigma^2 = 0$ so our model universe is shear free. Also since $\frac{\sigma}{\theta} = 0$ for all values of t so our model universe is always an isotropic one.

4.4 Conclusion

In this chapter, we have considered a LRS Bianchi type I cosmological model universe interacting with perfect fluid in the context of Lyra's geometry by using constant deceleration parameter. Here we have discussed different distributions like dust, stiff fluid, disordered distribution and Matter distribution in internabular space and it is observed that our model universe is always an isotropic one.