Chapter 6

Higher Dimensional Cosmological Model Universe with Quadratic Equation of State in Lyra Geometry

[4]

6.1 Introduction:

In the General theory of relativity, Einstein described gravitation in terms of geometry. The field equations of Einstein's admit/represent only non static cosmological models for nonzero energy density. Based on the cosmological principle, Einstein introduced the cosmological constant into his field equations in order to obtain a static model of the universe. Later, Weyl (1918a) proposed a more general theory in which electromagnetism together with gravitation is described geometrically. He showed how one can introduce a vector field in the Riemannian space-time with an intrinsic geometrical significance. But this theory was based on non-integrability of length of vector under parallel displacement so that it had some unsatisfactory features and hence this theory did not gain general acceptance, which is known as Weyl's geometry still today. After having these concepts, Lyra (1951) suggested a modification of Riemannian geometry, which may also be considered as a modification of Weyl's geometry, by introducing a gauge function into the structureless manifold which removes the non-integrability condition of the length of a vector under parallel transport and a cosmological constant is naturally introduced from the geometry. This modified Riemannian

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geometry is known as Lyra's geometry. Halford (1970, 1972) pointed out that in the normal general relativistic treatment the constant displacement vector field ϕ_k in Lyra's geometry plays the role of cosmological constant and the scalar-tensor treatment based on Lyra's geometry predicts the same effect, within observational limits, as far as the classical solar system test are concerned (as in the Einstein's theory of relativity).

In 1984 Marciano (1984) has suggested that the experimental observation of fundamental constant with varying time could produce the evidence of extra dimensions. Also in the same year 1984 Witten (1984) shown that the cosmos at its early stage of evolution might have had a higher dimensional era. In fact with the passage of time the standard dimensions are expanded while the extra dimensions are shrinking to the Plankian dimension, which is beyond our ability to detect with the presently available experimental facilities. This fact has attracted many researchers to investigate problems in the field of higher dimensions.

Since our universe is expanding with acceleration as suggested by [Silk (1989); Perlmutter (1999); Riess et al. (1998); Jimenez et al. (2003)] but the satisfactory explanation about physical mechanism and driving force of accelerated expansion of the universe is yet to achieve as human mind has not achieved perfection. It is supposed that the dark energy is responsible to produce sufficient acceleration in late time of evolution of the universe. Therefore it is essential to study the fundamental nature of the dark energy. Several approaches have been made to understand the dark energy. The cosmological constant is assumed to be the simplest candidate of dark energy. Therefore various cosmologists suggested many alternative models with different equations of state, scalar fields, with Lyra Geometry, f(R) Lagrangian, additional space dimensions and many others [Bhamra (1974); Karade (1978); Sen and Vanstone (1972); Reddy and Innaiah (1986); Asgar and Ansary (2014b); Sahoo and Sivakumar (2015); Amendola et al. (2007); Felice and Tsujikawa (2005); Sotiriou and Faraoni (2010)].

In order to study dark energy and general relativistic dynamics for different cosmological models, Quadratic equation of state plays a vital role. The general form of quadratic equation of state is given by

$$p = p_0 + \alpha\rho + \beta\rho^2 \ ,$$

where p_0, α, β are the parameters. This equation represents the first term of the Taylor's expansion of any equation of state of the form $p = p(\rho)$ about $\rho = 0$.

Dark energy universe with different equations of state has been discussed by Nojiri and Odintsov (2004, 2005a), Nojiri et al. (2005b, 2011), Capozziello et al. (2006a) and Bamba et al. (2012) describing dark energy or unified dark matter.

Ananda and Bruni (2005) discussed the cosmological models by considering different form of non-linear quadratic equation of state. They have shown that the behaviour of the anisotropy at the singularity found in the brane scenario can be recreated in the general relativistic context by considering general form of quadratic equation of state. Also by considering quadratic equation of state of the form

$$p = \alpha\rho + \frac{\rho^2}{\rho_c} ,$$

also Ananda and Bruni (2006) have discussed the anisotropic homogeneous and inhomogeneous cosmological models in general relativity and tried to isotropize the universe at early times when the initial singularity is approached. In our present study, we have consider the quadratic equation of state of the form

$$p = \alpha\rho^2 - \rho ,$$

where $\alpha \neq 0$ is a constant quantity but we can take $p_0 = 0$ to avoid complexities in our calculations. This will not affect the quadratic nature of the equation of state.

Many authors like Chavanis (2013a), Maharaj and Takisa (2012), Rahaman et al. (2009a), Feroze and Siddiqui (2011) have studied cosmological models based on quadratic equation of state under different circumstances.

Recently Reddy et al. (2015), Adhav et al. (2015), Rao et al. (2015) studied Bianchi type cosmological models with quadratic equation of state in general and modified theories of relativity.

Motivated from the above mentioned research, here we have investigated five dimensional Bianchi type-I cosmological models in the framework of Lyra geometry with quadratic equation of state. Physical and geometrical properties of the model are discussed.

6.2 Field Equations and their Solutions:

The field equations based on Lyra's Geometry as obtained by Sen (1957) and Sen & Dunn (1971) in normal gauge are written as:

$$R_{ij} - \frac{1}{2}g_{ij}R + \frac{3}{2}\phi_i\phi_j - \frac{3}{4}g_{ij}\phi^k\phi_k = T_{ij} \tag{6.1}$$

where we use the units in which $\frac{8\pi G}{c^4} = 1$ [Wesson 1992; Baysal et. al. 2001; Bali and Dev 2002] and ϕ_i is the displacement vector given by

$$\phi_i = (0,0,0,0,\beta(t)) \tag{6.2}$$

Here we consider the five dimensional LRS-Bianchi type-I axially symmetric metric in the form

$$ds^2 = A^2dx^2 + B^2(dy^2 + dz^2) + C^2d\psi^2 - dt^2 \tag{6.3}$$

with the convention that $x^1 = x, x^2 = y, x^3 = z, x^4 = \psi, x^5 = t$ where A, B and C are functions of time t only. Here the extra coordinate is taken to be space like.

The energy momentum tensor T_{ij} for perfect fluid is given by

$$T_{ij} = (\rho + p)u_iu_j + pg_{ij} \tag{6.4}$$

where, ρ is the energy density, p is the pressure and u^i is the five velocity vector given by

$$u^i = (0,0,0,0,1) \tag{6.5}$$

which satisfies

$$g_{ij}u^iu^j = u^iu_i = -1 \tag{6.6}$$

Also, we have assume an equation of state (EoS) in the general form $p = p(\rho)$ for the matter of distribution.

In this case we have considered it in the quadratic form as

$$p = \alpha\rho^2 - \rho \tag{6.7}$$

where α is a constant and strictly $\alpha \neq 0$ [Reddy et. al. 2015].

In comoving coordinate system, we have from (6.4)

$$T_1^1 = T_2^2 = T_3^3 = T_4^4 = -p \ ; \ T_5^5 = \rho \ and \ T_j^i = 0 \ for \ all \ i \neq j \tag{6.8}$$

Using commoving coordinate system, the Einstein's equations (6.1) and (6.2) for the metric (6.3) with the help of equation (6.4)-(6.8) can be written as

$$\frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}\dot{B}}{AB} + 2\frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} - \frac{3}{4}\beta^2 = -\rho \tag{6.9}$$

$$2\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}^2}{B^2} + 2\frac{\dot{B}\dot{C}}{BC} + \frac{3}{4}\beta^2 = -p \tag{6.10}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC} + \frac{3}{4}\beta^2 = -p \tag{6.11}$$

$$\frac{\ddot{A}}{A} + 2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}\dot{B}}{AB} + \frac{3}{4}\beta^2 = -p \tag{6.12}$$

where, the overhead dot $(.)$ denote the derivative with respect to time t.

Let us define the spatial volume V and average scale factor $R(t)$ for axially symmetric LRS Bianchi type-I metric as

$$V = R^4(t) = AB^2C \tag{6.13}$$

The mean Hubble's Parameter H is given by

$$H = \frac{\dot{R}}{R} = \frac{1}{4}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) \tag{6.14}$$

The scalar expansion θ and shear scalar σ^2 are given by

$$\theta = \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} + \frac{\dot{C}}{C} \tag{6.15}$$

and

$$\sigma^2 = \frac{1}{2}\left(\sum_{i=1}^{4} H_i^2 - 4H^2\right) = \frac{1}{2}\left(\frac{\dot{A}^2}{A^2} + 2\frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2} - \frac{\theta^2}{4}\right) \tag{6.16}$$

The average anisotropy parameter Δ is defined as

$$\Delta = \frac{1}{4}\left[\sum_{i=1}^{4}\left(\frac{H_i - H}{H}\right)^2,\right] \tag{6.17}$$

where $H_i\,; i = 1,2,3,4$ represents the directional Hubble's parameter in the directions of x, y, z and ψ respectively, which are given by

$$H_x = \frac{\dot{A}}{A}\ ,\ H_y = H_z = \frac{\dot{B}}{B}\ ,\ H_\psi = \frac{\dot{C}}{C},$$

and $\Delta = 0$ corresponds to isotropic expansion.

Also, the deceleration parameter q is defined as

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 \tag{6.18}$$

Since equations (6.9)-(6.12) represents a system of four independent (simultaneous) equations involving six unknowns viz. A, B, C, β, ρ and p , so in order to obtain deterministic solution of the above system of equations we need to two more physical equations [conditions] or mathematical equations [relations] involving these unknowns.

Out of these two additional physical conditions we may consider

(i) One physical condition as the quadratic equation of state (6.7) given above and

(ii) the scalar expansion θ is proportional to the shear scalar σ^2 so that we may consider as

$$C = B^n \tag{6.19}$$

From Equations (6.10)-(6.12) we have

$$\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} + \frac{\dot{B}^2}{B^2} + \frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}\dot{C}}{BC} - \frac{\dot{A}\dot{C}}{AC} = 0 \tag{6.20}$$

and

$$\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}-\frac{\dot{B}\dot{C}}{BC}+\frac{\dot{A}\dot{C}}{AC}-\frac{\dot{B}^2}{B^2}=0 \tag{6.21}$$

Equation (6.19) together with the equations (6.20) and (6.21) give us-

$$\frac{\ddot{B}}{B}+(n+1)\left(\frac{\dot{B}}{B}\right)^2+\frac{\dot{A}\dot{B}}{AB}=0 \quad if \quad n\neq 1 \tag{6.22}$$

$$\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}+(n+1)\frac{\dot{A}\dot{B}}{AB}-(n+1)\left(\frac{\dot{B}}{B}\right)^2=0 \tag{6.23}$$

Adding (6.22) and (6.23) we get

$$\frac{\dot{A}}{A}\left\{\frac{\ddot{A}}{A}+(n+2)\frac{\dot{B}}{B}\right\}=0 \tag{6.24}$$

This yields the following three cases:

Case (I) :

$$\frac{\ddot{A}}{A}+(n+2)\frac{\dot{B}}{B}=0$$

Case (II) :

$$\frac{\dot{A}}{A}=0\Rightarrow \dot{A}=0$$

Case (II) :

$$\frac{\dot{A}}{A}=0 \quad and \quad \frac{\ddot{A}}{A}+(n+2)\frac{\dot{B}}{B}=0\ .$$

Out of the above three cases the solutions given by Case (I) and Case (III) are not so interesting and realistic solutions in some parameters so we intend to determine the cosmological models with quadratic equation of state for the Cases (II).

Here in this case

$$\frac{\dot{A}}{A}=0\Rightarrow \dot{A}=0\Rightarrow A=constant=k_4$$

Using this value of A in equations (6.22) and (6.19) we have

$$A = k_4$$

$$B = [(n+2)(k_6 t + k_7)]^{\frac{1}{n+2}}$$

$$C = [(n+2)(k_6 t + k_7)]^{\frac{n}{n+2}}$$

where k_4, k_5, k_6 and k_7 are constants.

Therefore the equations (6.6)-(6.9) reduces to

$$\frac{(2n+1)k_6^2}{[(n+2)(k_6 t + k_7)]^2} - \frac{3}{4}\beta^2 = -\rho \tag{6.25}$$

and

$$-\frac{(2n+1)k_6^2}{[(n+2)(k_6 t + k_7)]^2} + \frac{3}{4}\beta^2 = -p \tag{6.26}$$

So from equation (6.7), we have

$$\rho = 0 \tag{6.27}$$

and

$$p = 0 \tag{6.28}$$

It means that our model represents a vacuum universe.

The equation (6.25) or (6.26) will give us

$$\frac{3}{4}\beta^2 = \frac{(2n+1)k_6^2}{(n+2)^2 (k_6 t + k_7)^2} \tag{6.29}$$

Now the metric (6.3) can be written as

$$ds^2 = k_4^2 dx^2 + [(n+2)(k_6 t + k_7)]^{\frac{2}{n+2}} (dy^2 + dz^2) + [(n+2)(k_6 t + k_7)]^{\frac{2n}{n+2}} d\psi^2 - dt^2 \quad (6.30)$$

The equation (6.30) represents a perfect fluid cosmological model with quadratic equation of state in Lyra geometry.

From equations (6.13)-(6.17) we have the spatial volume V, Hubble's parameter H, expansion factor θ, Shear scalar σ and anisotropy parameter Δ as

$$V = (n+2)k_4(k_6 t + k_7) \quad (6.31)$$

$$H = \frac{k_6}{4(k_6 t + k_7)} \quad (6.32)$$

$$\theta = 4H = \frac{k_6}{k_6 t + k_7} \quad (6.33)$$

$$\sigma^2 = \frac{(3n^2 - 4n + 4)k_6^2}{8(n+2)^2(k_6 t + k_7)^2} \quad (6.34)$$

$$\frac{\sigma^2}{\theta^2} = \frac{(3n^2 - 4n + 4)}{8(n+2)^2} = constant \quad (6.35)$$

$$q = \frac{k_6^2}{(k_6 t + k_7)^2} - 1 \quad (6.36)$$

6.3 Physical and Geometrical Properties of the Solutions:

We have observed from equation (6.29) that the displacement vector $\beta = \frac{2}{\sqrt{3}} \frac{k_6\sqrt{2n+1}}{n+2}$ when $t = 0$ and $n \neq -2$ i.e. $\beta \neq 0$. Also $\beta \to 0$ when $t \to \infty$. Again from equation (6.31) it is observed that initially when $t = 0$ then $V = k_4 k_7 (n+2)$ if $n \neq -2$ and gradually increases and finally when $t \to \infty$ then $V \to \infty$. Again from equations (6.32) and (6.33) we have seen that at the initial stage i.e. when $t \to 0$ then the Hubble's parameter H and expansion factor θ are constants but as the time progresses gradually they decreases and finally when $t \to \infty$ both H and θ become zero, which means that the Universe is expanding with the increase of time but the rate of expansion becomes slow as time increases Also we have observed that $\frac{\sigma^2}{\theta^2} = \frac{3n^2-4n+4}{8(n+2)^2} = constant \neq 0$ if $n \neq -2$ for all values of t, so our universe is always an anisotropic one. But when $n = -2$ then $\frac{\sigma}{\theta} \to \infty$ for all values of t. Here the shear scalar is finite at $t = 0$ and decreases as t increases. The expansion in the model stops at infinite time. Also from equation (6.36) we see that the deceleration parameter $q < 0$ when $t > 1 - \frac{k_7}{k_6}$ or $t < -1 - \frac{k_7}{k_6}$. It means that the universe is accelerating because acceleration at certain stage in the evolution of the universe implies $q < 0$ for some time so our model is an inflationary.

6.4 Conclusion:

In this paper we studied five dimensional LRS Bianchi type I cosmological model in the framework of Lyra's geometry in presence of perfect fluid, by using quadratic equation of state given by the equation (6.30) which is an inflationary model. Our work analyze the general feature of LRS Bianchi Type-I cosmological model with time dependent displacement vector so the concept of Lyra geometry is still exist even after the infinite times with different ideas and concepts. So it will be interesting to study the different properties of different topological defects within the framework of Lyra geometry and beneficial for further study to investigate the different models of our universe.