Chapter 2

Interaction of Gravitational field and Brans-Dicke field in R/W universe containing Dark Energy like fluid

2.1 Introduction

Since many forms of dark energy are always accompanied and inter-related with a scalar field we are motivated to see whether the Brans-Dicke scalar field can manifest some form of dark energy and what roles it can play in causing the accelerated expansion of the universe. We are also motivated to investigate different interesting forms of model universes containing Brans-Dicke field interacting with gravitational field, and specially their inter-relation with dark energy in the evolution of our universe. And from our study we get the evidence for the existence of dark energy, in one form or the other, in almost all the model universes obtained by us under different conditions, during the periods of their evolution, which gives the testimony to the present accelerated expansion of the universe. One peculiarity of some

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of the models we obtain is the existence of two forms of dark energy simultaneously in such models, one from cosmological constant and other due to Brans-Dicke scalar field. In one case there is possibility of our model universe to collapse and become a black hole. Interesting enough, yet in another case, one of our models is facing the fate of a Big Rip. And one of the model universes we obtain seems to behave like a cyclic model of the universe.

2.2 Solutions of Field Equations

Here, we consider the spherically symmetric Robertson-Walker metric

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\varphi^2)\right], \tag{2.1}$$

where k is the curvature index which can take values $-1, 0, 1$.
The action of the Brans-Dicke (B-D) theory of gravity is

$$S = \int d^4x\sqrt{|g|}\left[\frac{1}{16\pi}\left(\phi R - \frac{\omega}{\phi}g^{sl}\phi_{,l}\phi_{,s}\right) + L_m\right], \tag{2.2}$$

where R represents the curvature scalar associated with the 4D metric g_{ij}; g is the determinant of g_{ij}; ϕ is a scalar field; ω is a dimensionless coupling constant; L_m is the Lagrangian of the ordinary matter component.
The Einstein field equations in the most general form are given by

$$\begin{aligned} G_{ij} &\equiv R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} \\ &= -\frac{\kappa}{\phi}T_{ij} - \frac{\omega}{\phi^2}[\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi^{,s}\phi_{,s}] - \frac{1}{\phi}(\phi_{,ij} - g_{ij}\phi^{;s}_{;s}), \end{aligned} \tag{2.3}$$

$$(3+2\omega)\phi^{;s}_{;s} = \kappa T, \tag{2.4}$$

where $\kappa = 8\pi$, Λ is the cosmological constant, R_{ij} is Ricci-tensor, g_{ij} is metric tensor, $\Box\phi = \phi^{;s}_{;s}$, $\Box$ is the Laplace-Beltrami operator and $\phi_{,i}$ is the partial differentiation with respect to x^i coordinate.

The energy-momentum tensor for the perfect fluid distribution is

$$T_{ij} = (p+\rho)u_i u_j - p g_{ij}, \tag{2.5}$$

with u_i =Four velocity vector, p =Proper pressure and ρ =Proper rest mass density. Considering a co-moving system, we get $u_1 = u_2 = u_3 = 0; u_4 = 1$ and $g^{ij}u_i u_j = 1$.

Here a comma $(,)$or semicolon $(;)$followed by a subscript denotes partial differentiation or a covariant differentiation respectively. Also the velocity of light is assumed as unity.

For the metric (2.1) surviving field equations are

$$\begin{aligned} G_{11} &\equiv \frac{k}{R^2} + \frac{\dot{R}^2}{R^2} + \frac{2\ddot{R}}{R} - \Lambda \\ &= -\frac{\kappa p}{\phi} - \frac{\omega}{2\phi^2}\left[\frac{(1-kr^2)}{R^2}\phi'^2 + \dot{\phi}^2\right] - \frac{1}{\phi}\left[-\frac{2(1-kr^2)}{R^2 r}\phi' + \frac{2\dot{R}\dot{\phi}}{R} + \ddot{\phi}\right], \end{aligned} \tag{2.6}$$

$$\begin{aligned} G_{22} &\equiv \frac{k}{R^2} + \frac{\dot{R}^2}{R^2} + \frac{2\ddot{R}}{R} - \Lambda \\ &= -\frac{\kappa p}{\phi} - \frac{\omega}{2\phi^2}\left[-\frac{(1-kr^2)}{R^2}\phi'^2 + \dot{\phi}^2\right] - \frac{1}{\phi}\left[-\frac{(1-kr^2)}{R^2}\phi'' + \frac{(2kr^2-1)}{R^2 r}\phi' + \frac{2\dot{R}\dot{\phi}}{R} + \ddot{\phi}\right], \end{aligned} \tag{2.7}$$

$$G_{33} = G_{22}, \tag{2.8}$$

$$\begin{aligned} G_{44} &\equiv 3\left(\frac{k}{R^2} + \frac{\dot{R}^2}{R^2}\right) - \Lambda \\ &= \frac{\kappa\rho}{\phi} + \frac{\omega}{2\phi^2}\left[\dot{\phi}^2 + \frac{(1-kr^2)}{R^2}\phi'^2\right] + \frac{1}{\phi}\left[\frac{(1-kr^2)}{R^2}\phi'' - \frac{(3kr^2-2)}{R^2 r}\phi' - \frac{3\dot{R}\dot{\phi}}{R}\right], \end{aligned} \tag{2.9}$$

$$G_{14} \equiv \frac{\omega}{\phi^2}\phi'\dot{\phi} + \frac{\dot{\phi}'}{\phi} - \frac{\dot{R}\phi'}{R\phi} = 0, \tag{2.10}$$

From equation (2.4), we get

$$(3+2\omega)\left[-\frac{(1-kr^2)}{R^2}\phi'' + \frac{(3kr^2-2)}{R^2 r}\phi' + \frac{3\dot{R}\dot{\phi}}{R} + \ddot{\phi}\right] = \kappa(\rho - 3p), \tag{2.11}$$

where a dot and dash denotes differentiation with respect to time t and r.

Equations (2.6) and (2.7) gives

$$0 = \frac{\phi'}{\phi}\left[\frac{1}{r} + \frac{kr}{1-kr^2} - \frac{\phi''}{\phi'} - \omega\frac{\phi'}{\phi}\right], \tag{2.12}$$

From equation (2.12), we get

$$\frac{\phi''}{\phi'} + \omega\frac{\phi'}{\phi} = \frac{1}{r} + \frac{kr}{1-kr^2}, \tag{2.13}$$

Integrating equation (2.13), we get

$$\frac{1}{\omega+1}\phi^{\omega+1} = -\frac{A\sqrt{1-kr^2}}{k} + B, \tag{2.14}$$

where A and B are functions of time.

Integrating equation (2.10), we get

$$\frac{1}{\omega+1}\phi^{\omega+1} = R(t)g(r) + Q(t), \tag{2.15}$$

Equation (2.12) gives

$$\frac{\phi'}{\phi}\frac{d}{dr}\left[I_n\phi'\phi^{\omega}r^{-1}(1-kr^2)^{\frac{1}{2}}\right] = 0, \tag{2.16}$$

Using equation (2.15) in equation (2.16), we get

$$\frac{\phi'}{\phi}\frac{d}{dr}\left[I_n r^{-1}(1-kr^2)^{\frac{1}{2}} + I_n g'(r)\right] = 0, \tag{2.17}$$

From which it is obvious that ϕ is a function of r only, i.e. $Q(t) = 0$ in equation (2.15) gives

$$\frac{1}{\omega+1}\phi^{\omega+1} = R(t)g(r), \tag{2.18}$$

Comparing equation (2.14) and (2.15), we get $Q(t) = B = 0$ From equation (2.14), we get

$$\frac{1}{\omega+1}\phi^{\omega+1} = -\frac{A\sqrt{1-kr^2}}{k}, \tag{2.19}$$

Equations (2.17) and (2.18) gives

$$\frac{\dot{R}}{R} = \frac{\dot{A}}{A}, \tag{2.20}$$

Integrating, we get

$$R = NA, \tag{2.21}$$

where N is integration constant.

Using equations (2.19) and (2.20) in equations (2.6), (2.7), (2.9) and (2.11), we get

$$\begin{aligned}\frac{\kappa p}{\phi} = &-\frac{k}{R^2} - \frac{2k}{R^2(\omega+1)} - \frac{\omega+3}{\omega+1}\frac{\dot{R}^2}{R^2} - \frac{2\omega+3}{\omega+1}\frac{\ddot{R}}{R} + \Lambda \\ &- \frac{\omega}{2}\left[\frac{k^2r^2}{R^2(\omega+1)^2(1-kr^2)} - \frac{\dot{R}^2}{(\omega+1)^2R^2}\right],\end{aligned} \tag{2.22}$$

$$\frac{\kappa\rho}{\phi} = \frac{3k}{R^2} + \frac{3k}{R^2(\omega+1)} + \frac{3\omega+6}{\omega+1}\frac{\dot{R}^2}{R^2} - \Lambda - \frac{\omega}{2}\left[\frac{\dot{R}^2}{(\omega+1)^2R^2} - \frac{k^2r^2}{R^2(\omega+1)^2(1-kr^2)^2}\right], \tag{2.23}$$

and

$$\begin{aligned}\frac{\kappa}{\phi}(\rho-3p) = (3+2\omega)[&\frac{3k}{R^2(\omega+1)} + \frac{\omega k^2r^2}{R^2(\omega+1)^2(1-kr^2)} + \frac{3\dot{R}^2}{(\omega+1)R^2} \\ &+ \frac{\ddot{R}}{(\omega+1)R} - \frac{\omega}{(\omega+1)^2}\frac{\dot{R}^2}{R^2}],\end{aligned} \tag{2.24}$$

From equations (2.22) and (2.23), we get

$$\begin{aligned}\frac{\kappa}{\phi}(\rho-3p) = &\frac{6k}{R^2} + \frac{\rho k}{R^2(\omega+1)} + \frac{6\omega+15}{\omega+1}\frac{\dot{R}^2}{R^2} + \frac{6\omega+9}{\omega+1}\frac{\ddot{R}}{R} \\ &- 4\Lambda - \frac{\omega}{2}[\frac{4\dot{R}^2}{(\omega+1)^2R^2} - \frac{4k^2r^2}{R^2(\omega+1)^2(1-kr^2)}],\end{aligned} \tag{2.25}$$

Equations (2.24) and (2.25) gives

$$\frac{\rho k}{R^2(\omega+1)}+\frac{6\dot{R}^2}{R^2(\omega+1)}+\frac{4\omega+6}{\omega+1}\frac{\ddot{R}}{R}+\frac{(2\omega+1)\omega}{(\omega+1)^2}\frac{\dot{R}^2}{R^2}-\frac{(2\omega+1)k^2r^2\omega}{R^2(\omega+1)^2(1-kr^2)}-4\Lambda=0, \tag{2.26}$$

2.2.1 Case I: When $\omega = 0$

In this case, equations (2.22), (2.23) and (2.26) reduces to

$$\frac{\kappa p}{\phi}=-\frac{3k}{R^2}-\frac{3\dot{R}^2}{R^2}-\frac{3\ddot{R}}{R}+\Lambda, \tag{2.27}$$

$$\frac{\kappa\rho}{\phi}=\frac{6k}{R^2}+\frac{6\dot{R}^2}{R^2}-\Lambda, \tag{2.28}$$

$$\frac{6k}{R^2}+\frac{6\dot{R}^2}{R^2}+\frac{6\ddot{R}}{R}-4\Lambda=0, \tag{2.29}$$

Integrating equation (2.29), we get

$$R=\sqrt{\frac{3}{\Lambda}}cosh\left\{\sqrt{\frac{\Lambda}{3}}(t+D)\right\}, when \quad k=1, \tag{2.30}$$

$$R=\sqrt{\frac{3}{\Lambda}}sinh\left\{\sqrt{\frac{\Lambda}{3}}(t+D)\right\}, when \quad k=-1, \tag{2.31}$$

$$R=e^{\sqrt{\frac{\Lambda}{3}}(t+D)}, when \quad k=0, \tag{2.32}$$

where D is arbitrary constant of integration.

Case I(a): When $k=1$, we get

$$R=\sqrt{\frac{3}{\Lambda}}cosh\left\{\sqrt{\frac{\Lambda}{3}}(t+D)\right\}, \tag{2.33}$$

From equation (2.21), we get

$$A=\frac{1}{N}\sqrt{\frac{3}{\Lambda}}cosh\left\{\sqrt{\frac{\Lambda}{3}}(t+D)\right\}, \tag{2.34}$$

From equation (2.19), we get

$$\phi=-\frac{\sqrt{1-r^2}}{N}\sqrt{\frac{3}{\Lambda}}cosh\left\{\sqrt{\frac{\Lambda}{3}}(t+D)\right\}, \tag{2.35}$$

The gravitational variable is given by

$$G=-\sqrt{\frac{\Lambda}{3}}\frac{4N}{3\sqrt{1-r^2}}\frac{1}{cosh\left\{\sqrt{\frac{\Lambda}{3}}(t+D)\right\}}, \tag{2.36}$$

From equation (2.27) and (2.28), we get

$$p=-\frac{\sqrt{3\Lambda(1-r^2)}cosh\left\{\sqrt{\frac{\Lambda}{3}}(t+D)\right\}}{\kappa N}, \tag{2.37}$$

and,

$$\rho=-\frac{\sqrt{3\Lambda(1-r^2)}cosh\left\{\sqrt{\frac{\Lambda}{3}}(t+D)\right\}}{\kappa N}, \tag{2.38}$$

Hubble's parameter is given by

$$H=\sqrt{\frac{\Lambda}{3}}tanh\left\{\sqrt{\frac{\Lambda}{3}}(t+D)\right\}, \tag{2.39}$$

Scalar expansion is given by

$$\Theta=\sqrt{3\Lambda}tanh\left\{\sqrt{\frac{\Lambda}{3}}(t+D)\right\}, \tag{2.40}$$

In this model universe, it is seen that the gravitational variable G has a tendency to increase the pressure and decrease the density of the fluid whereas the Brans-Dicke scalar field has the tendency to decrease the pressure and increase the density of this universe. This model has a singularity at $r = 1$.

Case I(b): When $k = -1$, we get

$$R = \sqrt{\frac{3}{\Lambda}} sinh \left\{ \sqrt{\frac{\Lambda}{3}} (t + D) \right\}, \tag{2.41}$$

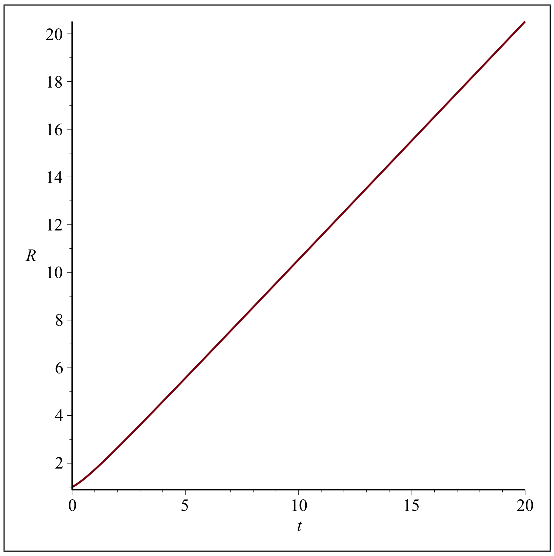

Figure 2.1: Graph of R vs. t according to (2.41)

From equation (2.21), we get

$$A = \frac{1}{N} \sqrt{\frac{3}{\Lambda}} sinh \left\{ \sqrt{\frac{\Lambda}{3}} (t + D) \right\}, \tag{2.42}$$

From equation (2.19), we get

$$\phi = \frac{\sqrt{1 + r^2}}{N} \sqrt{\frac{3}{\Lambda}} sinh \left\{ \sqrt{\frac{\Lambda}{3}} (t + D) \right\}, \tag{2.43}$$

which is both function of r and t.

When $t \to \infty$, both R and A tends to ∞. And when $r \to \infty$ and $t \to \infty$, the B-D scalar ϕ tends to ∞.

Therefore, we conclude that for $k = -1$ the B-D scalar ϕ is an increasing function of both r and t.

The gravitational variable is given by

$$G = \sqrt{\frac{\Lambda}{3}} \frac{4N}{3\sqrt{1+r^2}} \frac{1}{sinh\left\{\sqrt{\frac{\Lambda}{3}}(t+D)\right\}}, \tag{2.44}$$

which shows that gravitational variable G decreases as r and t increases and tends to zero when either $r \to \infty$ or $t \to \infty$.

From equation (2.27) and (2.28), we get

$$p = -\frac{\sqrt{3A(1+r^2)}}{\kappa N} sinh\left\{\sqrt{\frac{\Lambda}{3}}(t+D)\right\}, \tag{2.45}$$

and

$$\rho = \frac{\sqrt{3\Lambda(1+r^2)}}{\kappa N} sinh\left\{\sqrt{\frac{\Lambda}{3}}(t+D)\right\}, \tag{2.46}$$

Hubble's parameter is given by

$$H = \sqrt{\frac{\Lambda}{3}} coth\left\{\sqrt{\frac{\Lambda}{3}}(t+D)\right\}, \tag{2.47}$$

Scalar expansion is given by

$$\Theta = \sqrt{3\Lambda} coth\left\{\sqrt{\frac{\Lambda}{3}}(t+D)\right\}, \tag{2.48}$$

For this model universe, the scalar field helps in the expansion of the universe. Also the expansion factor R increases with time thus making testimony to the expansion of the universe. Here in this type of model universe it is seen that pressure is negative and the equation of state $\omega_1 = \frac{p}{\rho} = -1$. Thus this universe seems to be a universe containing dark energy due to cosmological constant Λ. Again, here the scalar field ϕ also contributes to the expansion of this universe. Thus some part of the dark energy contained may be taken as quintessence form of dark energy which is in agreement with the present day observational data, as according to the present observations, equation of state $\omega_1 \simeq -1$.

Case I(c): When $k = 0$, we get

$$R = e^{\sqrt{\frac{\Lambda}{3}}(t+D)}, \tag{2.49}$$

and

$$A = \frac{1}{N} e^{\sqrt{\frac{\Lambda}{3}}(t+D)}, \tag{2.50}$$

From equation (2.13), we get

$$\phi = \frac{1}{2N} r^2 e^{\sqrt{\frac{\Lambda}{3}}(t+D)}, \tag{2.51}$$

which is both functions of r and t.

When $t \to \infty, R \to \infty$.

And either $r \to \infty$ or $t \to \infty$, the B-D scalar ϕ tends to infinity.

The gravitational variable is given by

$$G = \frac{8N}{3r^2 e^{\sqrt{\frac{\Lambda}{3}}(t+D)}}, \tag{2.52}$$

which shows that gravitational variable G decreases as r and t increases and tends to zero as $r \to \infty$ or $t \to \infty$.

From equations (2.27) and (2.28), we get

$$p = -\frac{\Lambda r^2}{2\kappa N} e^{\sqrt{\frac{\Lambda}{3}}(t+D)}, \tag{2.53}$$

and

$$\rho = \frac{\Lambda r^2}{2\kappa N} e^{\sqrt{\frac{\Lambda}{3}}(t+D)}, \tag{2.54}$$

Hubble's parameter is given by

$$H = \sqrt{\frac{\Lambda}{3}}, \tag{2.55}$$

Scalar expansion is given by

$$\Theta = \sqrt{3\Lambda}, \tag{2.56}$$

Again for the solution in this case, it is found that the Brans-Dicke scalar field ϕ is singular at the origin. However, on the other hand, at the origin, the gravitational force is very strong. As time t increases, the pressure decreases whereas the density increases. Thus there is possibility that the model universe in this case contracts gradually and at some stage the density will be very high, thereby it is possible that the universe becomes a black hole in course of time. Or, on the other hand, here the equation of state is $\omega_1 = \frac{p}{\rho} = -1$ whereas the pressure is negative. This implies that our model universe is an expanding universe containing dark energy due to the cosmological constant which is in agreement with the present observational data, namely, $\frac{p}{\rho} \simeq -1$.

2.2.2 Case II: When $\omega = 0$ and $\Lambda = 0$

From equation (2.29), we get

$$\frac{6k}{R^2} + \frac{6\dot{R}^2}{R^2} + \frac{6\ddot{R}}{R} = 0, \tag{2.57}$$

Integrating, we get

$$R = \sqrt{-kt^2 + 2at + 2b}, \tag{2.58}$$

where a and b are integration constants.
From equation (2.21), we get

$$A = \frac{1}{N}\sqrt{-kt^2 + 2at + 2b}, \tag{2.59}$$

Case II(a): When $k=1$.

From equation (2.58) and (2.59), we get

$$R=\sqrt{-t^2+2at+2b}, \tag{2.60}$$

and

$$A=\frac{1}{N}\sqrt{-t^2+2at+2b}, \tag{2.61}$$

From equation (2.19), we get

$$\phi=-\frac{1}{N}\sqrt{-t^2+2at+2b}\sqrt{1-r^2}, \tag{2.62}$$

which is function of both r and t.

The gravitational variable is given by

$$G=-\frac{4N}{3\sqrt{-t^2+2at+2b}\sqrt{1-r^2}}, \tag{2.63}$$

where $N<0$.

From equations (2.60), (2.61), (2.62) and (2.63), we see that the reality condition for R, A, ϕ and k is $(a^2+2b)>(t-a)^2$ and $r^2<1$.

From equations (2.27) and (2.28), we get

$$p=0, \tag{2.64}$$

and

$$\rho=-\frac{6(a^2+2b)\sqrt{(1-r^2)}}{\kappa N(-t^2+2at+2b)^{\frac{3}{2}}}, \tag{2.65}$$

which is function of both r and t. The reality condition is same as above.

Hubble's parameter is given by

$$H=\frac{t-a}{t^2-2at-2b}, \tag{2.66}$$

Scalar expansion is given by

$$\Theta = \frac{3(t-a)}{t^2-2at-2b}, \tag{2.67}$$

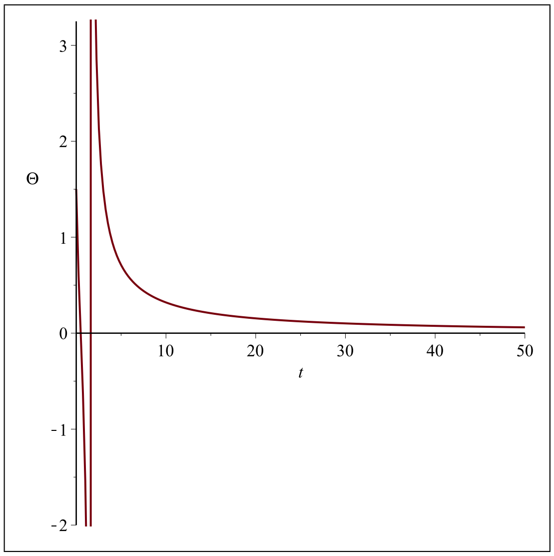

Figure 2.2: Graph of Θ *vs*. t according to (2.67)

For this model universe, it is seen that at time t given by $t^2 - 2at - 2b = 0$ there may be a gravitational collapse. Since, in this case, the energy density is negative there is possibility that this universe contains phantom form of dark energy. But there is a doubt in this case as here the pressure is zero and this universe is closed, since dark energy is assumed to help in the accelerated expansion of the universe. Thus, when $k = 1$, $\omega = 0$ and $\Lambda = 0$, the problem reduces to the case of dust distribution.

Case II(b):When $k = -1$.

From equations (2.58) and (2.59), we get

$$R = \sqrt{t^2+2at+2b}, \tag{2.68}$$

and

$$A = \frac{1}{N}\sqrt{t^2+2at+2b}, \tag{2.69}$$

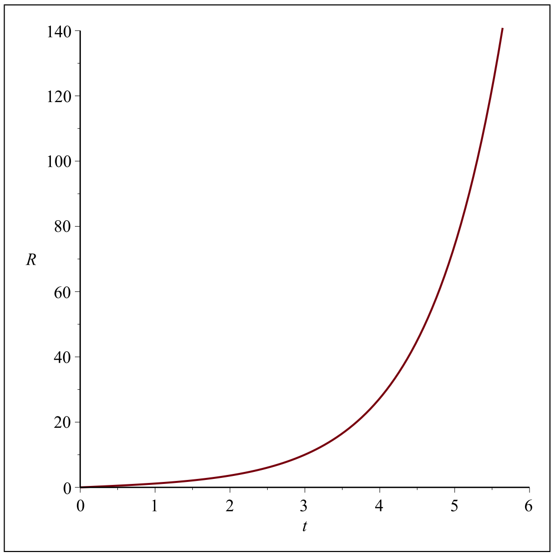

Figure 2.3: Graph of R $vs.$ t according to (2.68)

From equation (2.19), we get

$$\phi = \frac{1}{N}\sqrt{t^2+2at+2b}\sqrt{1-r^2}, \tag{2.70}$$

which is function of both r ad t.

When $t \to \infty$, the radius of universe R tends to infinity.

And the B-D scalar ϕ tends to infinity either when $r \to \infty$ or $t \to \infty$.

The gravitational variable is given by

$$G = \frac{4N}{3\sqrt{t^2+2at+2b}\sqrt{1-r^2}}, \tag{2.71}$$

From equation (2.71), we see that the gravitational variable G decreases when t and r increase and tends to zero when $r \to \infty$ or $t \to \infty$.

From equation (2.27) and (2.28), we get

$$p = 0, \tag{2.72}$$

and

$$\rho = \frac{6(a^2 - 2b)\sqrt{(1+r^2)}}{\kappa N(t^2 + 2at + 2b)^{\frac{3}{2}}}, \tag{2.73}$$

which is real where $a^2 - 2b > 0$.

Hubble's parameter is given by

$$H = \frac{t+a}{t^2 + 2at + 2b}, \tag{2.74}$$

Scalar expansion is given by

$$\Theta = \frac{3(t+a)}{t^2 + 2at + 2b}, \tag{2.75}$$

In the solution for this case, it is obtained that as time t increases the radius of our (model) universe increases, that is our universe is an expanding one which is the sign of being a realistic one. But here it is seen that this universe expands initially at a high rate and gradually the expansion slows down until it stops at infinitely large time preparing for contraction. In this model universe the Brans- Dicke field has its influence in the area given by $r = 1$, and is inversely proportional to the gravitational potential due to G. Thus, when $k = -1$, $\omega = 0$ and $\Lambda = 0$, the problem reduces to the case of dust distribution.

Case II(c):When $k = 0$.

From equations (2.58) and (2.59), we get

$$R = \sqrt{2at + 2b}, \tag{2.76}$$

and

$$A = \frac{1}{N}\sqrt{2at + 2b}, \tag{2.77}$$

From equation (2.76), we know that that radius of the universe R tends to infinity when t tends to infinity.

From equation (2.13), we get

$$\phi = \frac{r^2\sqrt{2at + 2b}}{2N}, \tag{2.78}$$

which is function of both r and t.

When either $r \to \infty$ or $t \to \infty$, the B-D scalar ϕ tends to infinity.

The gravitational variable is given by

$$G = \frac{8N}{3r^2\sqrt{2at+2b}}, \tag{2.79}$$

which show that the gravitational variable G is decreases when r and t increase and tends to zero when either $r \to \infty$ or $t \to \infty$.

From equation (2.27) and (2.28), we get

$$p = 0, \tag{2.80}$$

and

$$\rho = \frac{3a^2r^2}{\kappa N(2at+2b)^{\frac{3}{2}}}, \tag{2.81}$$

Hubble's parameter is given by

$$H = \frac{t}{2(at+b)}, \tag{2.82}$$

Scalar expansion is given by

$$\Theta = \frac{3t}{2(at+b)}, \tag{2.83}$$

From equation (2.81), we see that ρ decreases where r is fixed and t increases and ρ increases when r increases and t decreases.

Regarding our model universe in this case, we have seen, from the expressions of R and ϕ, that the scalar field has a tendency to increase the radius of the universe, thereby helping in the expansion of the universe. The density of this universe is also seen to decrease with time which is the sign of a realistic universe. The expansion factor here is found to increase with time, thereby implying our universe to be an expanding one which is in testimony with the present universe. Thus, when $k = 0$, $\omega = 0$ and $\Lambda = 0$, the problem reduces to the case of dust distribution.

2.2.3 Case III: When $\omega \neq 0$ and $\Lambda = 0$

Since R is function of t. So, we consider the only case $k = 0$. Then, equation (2.26) reduces to

$$\frac{6\dot{R}^2}{R^2(\omega+1)} + \frac{4\omega+6}{\omega+1}\frac{\ddot{R}}{R} + \frac{(2\omega+1)\omega}{(\omega+1)^2}\frac{\dot{R}^2}{R^2} = 0, \tag{2.84}$$

Integrating, we get

$$R = \left[\frac{(4+3\omega)(at+b)}{(2+2\omega)}\right]^{\frac{2+2\omega}{4+3\omega}}, \tag{2.85}$$

where a and b are arbitrary constant of integration.

From equation (2.21), we get

$$A = \frac{1}{N}\left[\frac{(4+3\omega)(at+b)}{(2+2\omega)}\right]^{\frac{2+2\omega}{4+3\omega}}, \tag{2.86}$$

If $\omega > 0$, the radius of the universe increases as t increases and tends to infinity as t tends to infinity.

From equation (2.13), we get

$$\phi = \left\{\frac{(\omega+1)r^2}{2N}\right\}^{\frac{1}{\omega+1}} \left\{\frac{(4+3\omega)(at+b)}{2+2\omega}\right\}^{\frac{2}{4+3\omega}}, \tag{2.87}$$

which is function of both r and t.

If $\omega > 0$, the B-D scalar ϕ tends to infinity either when $r \to \infty$ or $t \to \infty$.

The gravitational variable is given by

$$G = \frac{4+2\omega}{3+2\omega}\left\{\frac{2N}{(\omega+1)r^2}\right\}^{\frac{1}{\omega+1}} \left\{\frac{2+2\omega}{(4+3\omega)(at+b)}\right\}^{\frac{2}{4+2\omega}}, \tag{2.88}$$

If $\omega > 0$, G decreases as r and t increase and tends to zero when either $r \to \infty$ or $t \to \infty$. From equations (2.22) and (2.23), we get

$$p = -\frac{4a^2(2\omega+3)^2}{\kappa(4+3\omega)^2(at+b)^2}\left\{\frac{(\omega+1)r^2}{2N}\right\}^{\frac{1}{\omega+1}}\left\{\frac{(4+3\omega)(at+b)}{2+2\omega}\right\}^{\frac{2}{4+3\omega}}, \tag{2.89}$$

and

$$\rho = \frac{2a^2(2\omega+3)}{\kappa(3\omega+4)(at+b)^2}\left\{\frac{(\omega+1)r^2}{2N}\right\}^{\frac{1}{\omega+1}}\left\{\frac{(4+3\omega)(at+b)}{2+2\omega}\right\}^{\frac{2}{4+3\omega}}, \tag{2.90}$$

Hubble's parameter is given by

$$H = \frac{2a(\omega+1)}{(4+3\omega)(at+b)}, \tag{2.91}$$

Scalar expansion is given by

$$\Theta = \frac{6a(\omega+1)}{(4+3\omega)(at+b)}, \tag{2.92}$$

Regarding the solution obtained in this case, the gravitational variable G is found to vary inversely with the scalar field ϕ. Thus in this case the Brans-Dicke scalar field has a tendency to decrease the gravitational potential. For this universe it is seen that the equation of state $\omega_1 < -1$,namely, $\omega_1 = \frac{p}{\rho} = -\frac{2(2\omega+3)}{4+3\omega} = -1 - \frac{\omega+2}{4+3\omega} < -1$. Thus the dark energy contained in this universe may be taken as the k-essence form of energy. Here we see that for the k-essence energy, with $\omega_1 < -1$, the scalar field grows in the future. And since the k-essence fields are similarly uniform on small scale, the abundance of k-essence energy within a bound object grows with time, thereby expecting a growing influence on the internal dynamics. Ultimately, there is possibility that the repulsive k-essence energy will overcome the forces holding this model together and rips apart this universe in a Big rip. Thus, when $k = 0$, $\omega \neq 0$ and $\Lambda = 0$, the problem reduces to the case of dust distribution.

2.2.4 Case IV: When $\omega \neq 0$ and $\Lambda \neq 0$

Since R is function of t only. Therefore, we consider the only case $k = 0$.

Then, equation (2.26) reduces to

$$\frac{6\dot{R}^2}{R^2(\omega+1)} + \frac{4\omega+6}{\omega+1}\frac{\ddot{R}}{R} + \frac{(2\omega+1)\omega}{(\omega+1)^2}\frac{\dot{R}^2}{R^2} - 4\Lambda = 0, \tag{2.93}$$

Integrating, we get

$$R = e^{\frac{2(\omega+1)\sqrt{\Lambda}}{\sqrt{(2\omega+3)(3\omega+4)}}t}, \tag{2.94}$$

and

$$A = \frac{1}{N}e^{\frac{2(\omega+1)\sqrt{\Lambda}}{\sqrt{(2\omega+3)(3\omega+4)}}t}, \tag{2.95}$$

If $\omega > 0$, the radius of the universe R tends to infinity as t tends to infinity.

From equation (2.13), we get

$$\phi = \left\{\frac{(\omega+1)r^2}{2N}\right\}^{\frac{1}{\omega+1}} e^{\frac{2\sqrt{\Lambda}}{\sqrt{(2\omega+3)(3\omega+4)}}t}, \tag{2.96}$$

which is function of both r and t.

When either $r \to \infty$ or $t \to \infty$, the B-D scalar ϕ tends to infinity.

The gravitational variable is given by

$$G = \frac{4+2\omega}{3+2\omega}\left\{\frac{2N}{(\omega+1)r^2}\right\}^{\frac{1}{\omega+1}} e^{-\frac{2\sqrt{\Lambda}}{\sqrt{(2\omega+3)(3\omega+4)}}t}, \tag{2.97}$$

which is function of both r and t.

From equation (2.97), we see that the gravitational variable G decreases when r and t increase and tends to zero when either $r \to \infty$ or $t \to \infty$.

From equations (2.22) and (2.23), we get

$$p = \frac{\Lambda}{\kappa(4+3\omega)}\left\{\omega - \frac{8(\omega+1)^2\sqrt{\Lambda}}{\sqrt{(2\omega+3)(3\omega+4)}}\right\}\left\{\frac{(\omega+1)r^2}{2N}\right\}^{\frac{1}{\omega+1}} e^{\frac{2\sqrt{\Lambda}}{\sqrt{(2\omega+3)(3\omega+4)}}t}, \tag{2.98}$$

and

$$\rho = \frac{\Lambda}{\kappa}\left\{\frac{(\omega+1)r^2}{2N}\right\}^{\frac{1}{\omega+1}} e^{\frac{2\sqrt{\Lambda}}{\sqrt{(2\omega+3)(3\omega+4)}}t}, \tag{2.99}$$

Hubble's parameter is given by

$$H = \frac{2(\omega+1)\sqrt{\Lambda}}{\sqrt{(2\omega+3)(3\omega+4)}}, \tag{2.100}$$

Scalar expansion is given by

$$\Theta = \frac{6(\omega+1)\sqrt{\Lambda}}{\sqrt{(2\omega+3)(3\omega+4)}}, \tag{2.101}$$

In this model universe, the scalar field is seen to have a tendency to increase the expansion of the universe, thereby flattening the universe. Here, also the Brans -Dicke field has a tendency to decrease the gravitational potential. And the gravitational variable G tends to decrease the pressure and the density of the universe. Since here, as $t \to \infty$, it is found that $R \to \infty$ as well as$\rho \to \infty$, there is possibility of a bounce at some point of time, thereby remembering us of this universe to behave as a cyclic universe. If $8\sqrt{\Lambda} > \frac{\omega\sqrt{(2\omega+3)(3\omega+4)}}{(\omega+1)^2}$, then this model universe will have an accelerated expansion instigated by the negative pressure. And in this model the vacuum energy due to the cosmological constant may be taken as the dark energy part causing the accelerated expansion of the universe.

2.3 Conclusion

The universes we have investigated are found to behave in different ways and to show different manifestations under different conditions. Some of them show signs containing cosmological constant form and quintessence form of dark energy, whereas some others seem to contain fluids behaving like phantom and k-essence forms of dark energy, which can explain the present day accelerated expansion of the universe. Thus the model universes we obtain in these cases may be taken as realistic models of our universe, and many more unknown properties of the universe and of dark energy may be realized and known from further studies of these models, which we will perform and report elsewhere afterwards. Further, one model of ours seems to undergo a gravitational collapse leading to a black hole; whereas another model surprisingly seems to face the fate of a Big Rip. Another new finding in some of our models is that they contain simultaneously two forms of dark energy, one due to cosmological constant and another due to Brans-Dicke scalar field. And, interestingly enough, one of our models seems to behave like a universe obeying the newly proposed cyclic theory of the universe.