Chapter 5

Some spherically symmetric R/W universe interacting with vacuum B-D scalar field

5.1 Introduction

The study of Brans-Dicke (B-D) theory (Brans and Dicke 1961) is somewhat classical in nature and for that reason it is expected to play a crucial role in the late-time evolution of the universe. It is also realized that most of the inflationary models based on B-D theory over many important elements about the evolution of the universe (Sahoo and Singh 2002, El-Nabulsi 2008). Earlier Brans-Dicke (Brans and Dicke 1961) obtained the vacuum solutions of B-D field equations followed by three more solutions for a spherically symmetric metric. Tabensky and Taub (1973) obtained B-D vacuum static solutions with plane symmetric self-gravitating fluids. Rao et al. (1974) discussed about cylindrically symmetric B-D fields. Vacuum solutions in the Brans-Dicke theory of gravitation were investigated by var-

This Chapter is published in **Ukrainian Journal of Physics, Ukrain in 2015** ious researchers (Tiwari and Nayak 1976, Rao and Tiwari 1979, Johri et al. 1983, Ram and Singh 1983, Singh and Singh 1984, Riazi and Askari 1993) considering different metric. Bhadra and Sarkar et al. (2005) obtained that only two classes are independent among the four classes of static spherically symmetric solutions of the vacuum Brans-Dicke theory of gravity. Adhav et al. (2009) obtained an exact solution of the vacuum Brans-Dicke field equations for the metric tensor which is spatially homogeneous and anisotropic model. Baykal et al. (2010) invetigated static as well as cylindrically symmetric B-D vacuum solutions with and without a cosmological constant. Rai et al. (2013) obtained an exact solution of the vacuum Brans-Dicke field equations for the metric tensor considering spatially homogeneous and anisotropic model. In this chapter, we studied the problem of B-D scalar field interacting with spherically symmetric Robertson-Walker metric.

5.2 Solutions of Field Equations

The vacuum Brans-Dicke field equations in the general form are given by

$$R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = -\frac{\omega}{\phi^2}\left[\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi^{,s}\phi_{,s}\right] - -\frac{1}{\phi}\left(\phi_{,ij} - g_{ij}\phi^{;s}_{;s}\right) \tag{5.1}$$

$$(3+2\omega)\phi^{;s}_{;s} = 4\Lambda \tag{5.2}$$

where ϕ is the scalar field and Λ is the cosmological constant.

The spherically symmetric Robertson-Walker metric is

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\varphi^2)\right] \tag{5.3}$$

where $R(t)$ is the scale factor and k is the curvature index which can take up the values $(-1,0,+1)$ for open, flat, closed model of the universe respectively.

For the metric (5.3), the Brans-Dicke field equation (5.1) becomes

$$\frac{k}{R^2} + \frac{\dot{R}^2}{R^2} + \frac{2\ddot{R}}{R} - \Lambda = -\frac{\omega}{2\phi^2}\left[\frac{(1-kr^2)}{R^2}\phi'^2 + \dot{\phi}^2\right] - \frac{1}{\phi}\left[-\frac{2(1-kr^2)}{R^2 r}\phi' + \frac{2\dot{R}\dot{\phi}}{R} + \ddot{\phi}\right] \tag{5.4}$$

$$\frac{k}{R^2}+\frac{\dot{R}^2}{R^2}+\frac{2\ddot{R}}{R}-\Lambda=-\frac{\omega}{2\phi^2}\left[-\frac{(1-kr^2)}{R^2}\phi'^2+\dot{\phi}^2\right]$$
$$-\frac{1}{\phi}\left[-\frac{(1-kr^2)}{R^2}\phi''+\frac{(2kr^2-1)}{R^2 r}\phi'+\frac{2\dot{R}\dot{\phi}}{R}+\ddot{\phi}\right] \tag{5.5}$$

$$3\left(\frac{k}{R^2}+\frac{\dot{R}^2}{R^2}\right)-\Lambda=\frac{\omega}{2\phi^2}\left[\dot{\phi}^2+\frac{(1-kr^2)}{R^2}\phi'^2\right]+\frac{1}{\phi}\left[\frac{(1-kr^2)}{R^2}\phi''-\frac{(3kr^2-2)}{R^2 r}\phi'-\frac{3\dot{R}\dot{\phi}}{R}\right] \tag{5.6}$$

and

$$\frac{\omega}{\phi^2}\phi'\dot{\phi}+\frac{\dot{\phi}'}{\phi}-\frac{\dot{R}\phi'}{R\phi}=0 \tag{5.7}$$

From equation (5.2), we get

$$(3+2\omega)\left[-\frac{(1-kr^2)}{R^2}\phi''+\frac{(3kr^2-2)}{R^2 r}\phi'+\frac{3\dot{R}\dot{\phi}}{R}+\ddot{\phi}\right]=4\Lambda \tag{5.8}$$

where a dot $(.)$ and dash $(')$ denotes differentiation with respect to time t and r.

From equations (5.4) and (5.5), we obtain the relation

$$\frac{\phi''}{\phi'}+\omega\frac{\phi'}{\phi}=\frac{1}{r}+\frac{kr}{1-kr^2} \tag{5.9}$$

under the conditions $\phi'\neq 0, 1-kr^2\neq 0$.

Integrating equation (5.9),we get

$$\phi^{\omega+1}=B\sqrt{1-kr^2}+D \tag{5.10}$$

provided $k\neq 0$.

where B and D are arbitrary functions of time t.

Using (5.10) in (5.4) and (5.5), we obtain

$$\begin{aligned}\frac{k}{R^2}+\frac{\dot{R}^2}{R^2}+\frac{2\ddot{R}}{R}-\Lambda=&-\frac{\omega}{2R^2}\frac{1}{(1+\omega)^2}\frac{B^2k^2r^2}{(\phi^{1+\omega})^2}--\frac{\omega}{2(1+\omega)^2}\frac{\left(\dot{B}\sqrt{1-kr^2}+\dot{D}\right)^2}{(\phi^{1+\omega})^2}\\&-\frac{2Bk}{R^2(1+\omega)}\frac{\sqrt{1-kr^2}}{\phi^{1+\omega}}-\frac{2}{1+\omega}\frac{\dot{R}}{R}\frac{\left(\dot{B}\sqrt{1-kr^2}+\dot{D}\right)}{(\phi^{1+\omega})}\\&-\frac{\left(\ddot{B}\sqrt{1-kr^2}+\ddot{D}\right)}{(1+\omega)\phi^{1+\omega}}+\frac{\omega}{(1+\omega)^2}\frac{\left(\dot{B}\sqrt{1-kr^2}+\dot{D}\right)^2}{(\phi^{1+\omega})^2}\end{aligned}\tag{5.11}$$

Using (5.10) in (5.6), we obtain

$$\begin{aligned}3\left(\frac{k}{R^2}+\frac{\dot{R}^2}{R^2}\right)-\Lambda=&\frac{\omega}{2(1+\omega)}\frac{\left(\dot{B}\sqrt{1-kr^2}+\dot{D}\right)^2}{(\phi^{1+\omega})^2}-\frac{\omega}{2R^2(1+\omega)^2}\frac{B^2k^2r^2}{(\phi^{1+\omega})^2}\\&-\frac{3Bk\sqrt{1-kr^2}}{R^2(1+\omega)(\phi^{1+\omega})}-\frac{3}{1+\omega}\frac{\dot{R}}{R}\frac{\left(\dot{B}\sqrt{1-kr^2}+\dot{D}\right)}{(\phi^{1+\omega})}\end{aligned}\tag{5.12}$$

Using (5.10) in (5.8), we obtain

$$\begin{aligned}(3+2\omega)\Bigg[&\frac{3Bk\sqrt{1-kr^2}}{(1+\omega)R^2\phi^{1+\omega}}+\frac{\omega}{(1+\omega)^2}\frac{B^2k^2r^2}{R^2(\phi^{1+\omega})^2}+\frac{3}{1+\omega}\frac{\dot{R}}{R}\frac{(\dot{B}\sqrt{1-kr^2}+\dot{D})}{\phi^{1+\omega}}\\&-\frac{\omega}{(1+\omega)^2}\frac{(\dot{B}\sqrt{1-kr^2}+\dot{D})^2}{(\phi^{1+\omega})^2}+\frac{\ddot{B}\sqrt{1-kr^2}+\ddot{D}}{(1+\omega)\phi^{1+\omega}}\Bigg]=4\Lambda\end{aligned}\tag{5.13}$$

Using (5.10) in (5.7), we obtain

$$\frac{\dot{B}}{B}=\frac{\dot{R}}{R}\tag{5.14}$$

Now, we shall determine the values of the five unknowns B, ω, R, Λ and D by using the four equations (5.11), (5.12), (5.13)and (5.14). Now we try to solve the field equations under different physical situations.

Case I: Taking the arbitrary constant $D=0$ and using equation (5.14) in (5.11), (5.12) and

(5.13), we obtain the relations

$$\left(\frac{\dot{R}}{R}\right)^2\left[\frac{2(1+\omega)^2+4(1+\omega)-\omega}{2(1+\omega)^2}\right]+\frac{3+2\omega}{1+\omega}\frac{\ddot{R}}{R}-\Lambda = -\frac{k}{R^2}\left[\frac{\omega k r^2}{2(1+\omega)^2(1-kr^2)}+\frac{2}{1+\omega}+1\right] \tag{5.15}$$

$$\left(\frac{\dot{R}}{R}\right)^2\left[3-\frac{\omega}{2(1+\omega)^2}+\frac{3}{1+\omega}\right]-\Lambda=-\frac{k}{R}\left[3+\frac{\omega k r^2}{2(1+\omega)^2(1-kr^2)}+\frac{3}{1+\omega}\right] \tag{5.16}$$

$$(3+2\omega)\left[\frac{3k}{R^2(1+\omega)}+\frac{\omega}{(1+\omega)^2}.\frac{k^2r^2}{R^2(1-kr^2)}+\frac{3+2\omega}{(1+\omega)^2}\left(\frac{\dot{R}}{R}\right)^2+\frac{\ddot{R}}{R(1+\omega)}\right]=4\Lambda \tag{5.17}$$

To obtain the exact solutions from equations (5.15), (5.16) and (5.17), we consider a case where the coupling constant, $\omega = 0$. Then the equations (5.15), (5.16) and (5.17) reduce to the following forms respectively.

$$3\left(\frac{\dot{R}}{R}\right)^2+3\frac{\ddot{R}}{R}-\Lambda=-\frac{3k}{R^2} \tag{5.18}$$

$$3\left(\frac{\dot{R}}{R}\right)^2-\frac{\Lambda}{2}=-\frac{3k}{R^2} \tag{5.19}$$

$$3\left(\frac{\dot{R}}{R}\right)^2+\frac{\ddot{R}}{R}+\frac{3k}{R^2}=4\Lambda \tag{5.20}$$

Corresponding to $k=-1$, we find from equations (5.18), (5.19) and (5.20) that $\Lambda = 0$ and $R=t$.

In this case, the values of ϕ from equations (5.10) is given by

$$\phi = t\sqrt{1+r^2} \tag{5.21}$$

From equations (5.14) and (5.21) we observe that the expansion parameter is purely a function of time t while B-D scalar ϕ is a function of both r and t. Where $r\to\infty, \phi\to\infty$,while R remains finite. However,when $t\to\infty$ both ϕ and R tends to ∞.We can further conclude that corresponding to $k = -1$ and $\omega = 0$, the B-D scalar ϕ is an increasing function of both r and t.

Since, the B-D scalar ϕ and the gravitational variable G are related by the relation

$$G = \frac{1}{\phi}\left(\frac{4+2\omega}{3+2\omega}\right) \tag{5.22}$$

So, the gravitational variable

$$G \propto \frac{1}{\phi} \tag{5.23}$$

i.e.G decreases as t (or r) increases. From equation (5.14), we further observe that at the initial stage i.e. when $t = 0$, the radius of the universe is zero thereby showing that the universe was concentrated to a mass point and expands gradually till it becomes infinitely large which supports the present finding for accelerated expansion of the universe. This is in conformity with the steady state theory of the cosmological universe.The corresponding deceleration parameter is zero.

Case II:Using the well known Hubble's parameter

$$\frac{\dot{R}}{R} = H \tag{5.24}$$

where H is the Hubble's constant.

Considering the arbitrary function $D = 0$, from equations (5.11), (5.12) and (5.13), we obtain the relations

$$\frac{k}{R^2} + 3H^2 - \Lambda = -\frac{\omega}{2R^2}\frac{1}{(1+\omega)^2}\frac{k^2r^2}{1-kr^2} + +\frac{\omega H^2}{2(1+\omega)^2} - \frac{2k}{R^2(1+\omega)} - \frac{3H^2}{(1+\omega)} \tag{5.25}$$

$$\frac{3k}{R^2} + 3H^2 - \Lambda = -\frac{\omega}{2R^2}\frac{1}{(1+\omega)^2}\frac{k^2r^2}{1-kr^2} - -\frac{3k}{R^2(1+\omega)} - \frac{6+5\omega}{2(1+\omega)^2}H^2 \tag{5.26}$$

and

$$(3+2\omega)\left[\frac{3k}{R^2(1+\omega)} + \frac{\omega k^2r^2}{R^2(1+\omega)^2(1-kr^2)} + +\frac{4+3\omega}{(1+\omega)^2}H^2\right] = 4\Lambda \tag{5.27}$$

To find a relation between the constants, we consider a case where $k = 0$.

The equations (5.25) and (5.26) reduce to a single relation

$$\Lambda = \frac{(3+2\omega)(4+3\omega)}{2(1+\omega)^2} H^2 \tag{5.28}$$

Also, the equation (5.27) reduce to

$$\Lambda = \frac{(3+2\omega)(4+3\omega)}{4(1+\omega)^2} H^2 \tag{5.29}$$

The equations (5.28) and (5.29) are possible only when $\Lambda = 0$.

Since, $H^2 \neq 0$. The equation (5.28) and (5.29) reduces to the relation

$$(3+2\omega)(4+3\omega) = 0 \tag{5.30}$$

From equation (5.30), we have seen that

$$(3+2\omega) = 0 \quad or \quad (4+3\omega) = 0 \tag{5.31}$$

When $\Lambda = 0$, using $\omega = -\frac{3}{2}$ in equation (5.10), we obtain

$$\phi = \frac{1}{N^2} e^{-2Ht} \tag{5.32}$$

Since from (5.14), $B = Ne^{Ht}$ where N is an arbitrary constant.

Using the value of $\omega = -\frac{4}{3}$ in (5.10), we get

$$\phi = \frac{1}{N^3} e^{-3Ht} \tag{5.33}$$

From equations (5.32) and (5.33), we have seen that corresponding to $\omega = -\frac{3}{2}$ the B-D scalar ϕ is always positive while corresponding $\omega = -\frac{4}{3}$ the B-D scalar ϕ may be positive or negative according to N is positive or negative. Since the values of H corresponds to an expanding model, we find that in either case $\phi \to 0$ as $t \to \infty$ while the universe continues to expand exponentially.

Also, from (5.22) we have

$$G = \frac{1}{\phi}\left(\frac{4+2\omega}{3+2\omega}\right) \tag{5.34}$$

Therefore, corresponding to the value of $\omega = -\frac{3}{2}$, we find that the gravitational variable G remains infinite while the B-D scalar ϕ continues to decrease as t increases. Corresponding to the value of $\omega = -\frac{4}{3}$, we find that

$$G \propto \frac{1}{\phi} \tag{5.35}$$

which implies that ϕ and G will remain finite for all finite values of time and G will be an exponentially increasing function of time.

The line element in either case becomes

$$ds^2 = dt^2 - A^2 e^{2Ht}\left[dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\right] \tag{5.36}$$

where A is an arbitrary constant.

It represents an expanding model which has no singularity at any epoch. Further, we observe that R/W universe corresponding to the curvature index $k = 0$, the B-D scalar ϕ continues to decreases as the universe expands gradually with the lapse of time and the cosmological term has no contribution towards the B-D scalar interactions in the model (5.36).

The deceleration parameter is found to be

$$q = -\frac{\ddot{R}R}{\dot{R}^2} = -1 \tag{5.37}$$

Case III: Taking $\phi' = 0$ and $3+2\omega \neq 0$ in the field equations, we obtain

$$3\frac{\dot{R}}{R}\dot{\phi} + \ddot{\phi} = \frac{4\Lambda}{3+2\omega} \tag{5.38}$$

$$\frac{k}{R^2} + \left(\frac{\dot{R}}{R}\right)^2 + 2\frac{\ddot{R}}{R} - \Lambda = -\frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} - 2\frac{\dot{R}}{R}\frac{\dot{\phi}}{\phi} - \frac{\ddot{\phi}}{\phi} \tag{5.39}$$

and

$$\frac{3k}{R^2}+3\left(\frac{\dot{R}}{R}\right)^2-\Lambda=\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2-3\frac{\dot{R}}{R}\frac{\dot{\phi}}{\phi} \tag{5.40}$$

Under the conditions $\Lambda = 0$ and $k = 0$, (5.38)-(5.40) becomes

$$\frac{\dot{R}}{R}=-\frac{1}{3}\frac{\ddot{\phi}}{\dot{\phi}} \tag{5.41}$$

$$\left(\frac{\dot{R}}{R}\right)^2+2\frac{\ddot{R}}{R}=-\frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2}-2\frac{\dot{R}}{R}\frac{\dot{\phi}}{\phi}-\frac{\ddot{\phi}}{\phi} \tag{5.42}$$

$$3\left(\frac{\dot{R}}{R}\right)^2=\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2-3\frac{\dot{R}}{R}\frac{\dot{\phi}}{\phi} \tag{5.43}$$

Adding (5.42) and (5.43), we get

$$4\left(\frac{\dot{R}}{R}\right)^2+2\frac{\ddot{R}}{R}=-5\frac{\dot{R}}{R}\frac{\dot{\phi}}{\phi}-\frac{\ddot{\phi}}{\phi} \tag{5.44}$$

Integrating (5.41) and (5.44), we get

$$R^3\dot{\phi}=a=constant \tag{5.45}$$

$$\phi\frac{d}{dt}(R^3)=b=constant \tag{5.46}$$

The sum of equations (5.45) and (5.46) becomes

$$\frac{d}{dt}(\phi R^3)=a+b=c=constant \tag{5.47}$$

Integrating, we get

$$\phi=\frac{ct+l}{R^3} \tag{5.48}$$

Also, from (5.45) and (5.46),we get

$$\frac{\dot{\phi}}{\phi}=3\nu\frac{\dot{R}}{R} \tag{5.49}$$

where $\nu = \frac{a}{b} = constant$.

Using (5.49) in (5.43), we get

$$\left(1 + 3\nu - \frac{3}{2}\omega\nu^2\right)\left(\frac{\dot{R}}{R}\right)^2 = 0 \quad (5.50)$$

Since $\frac{\dot{R}}{R} \neq 0$, then equation (5.50) becomes

$$\omega = \frac{2}{3}\left(\frac{1+3\nu}{\nu^2}\right) \quad (5.51)$$

For $\nu = -\frac{1}{3}, \omega = 0$ and for $\nu = -1, \nu = -\frac{1}{2}, \omega = -\frac{4}{3}$.

Corresponding to the values of $\omega = 0, \omega = -\frac{4}{3}$ from equation (5.22), we find that

$$G \propto \frac{1}{\phi} \quad (5.52)$$

which implies that ϕ and G will remain finite for all finite values of time and G will be an increasing function of time. Also, variation of ω for different values of ν according to (5.51)

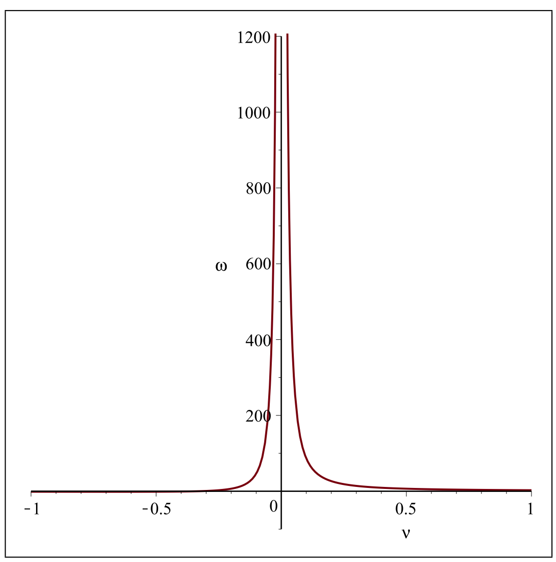

Figure 5.1: Variation of ω for different values of ν according to (5.51)

has been shown in $Fig.1$.

5.3 Conclusion

Since, we assume the validity of Hubble's principle for all the solutions of the universe corresponding to $k = \pm 1$.The B-D scalar ϕ does not exist in the flat model of the universe.

Here, we have seen that the role played by the scalar ϕ relating to the expansion and contraction of the universe is that the B-D scalar ϕ which is a negative, decreasing function of time may be treated as something reflecting the expansion of the universe while the B-D scalar ϕ which is a positive increasing function of time can be treated as something reflecting the contraction of the universe.

For $\omega < -\frac{4}{3}$ but not equal to zero, ϕ is found to be a negative decreasing function of time thereby causing the expansion of the universe till it becomes infinitely large. Corresponding to a case where $\omega > -\frac{4}{3}$, ϕ is found to be a positive increasing function of time thereby causing the universe to contract and the universe becomes concentrated to a point when ϕ becomes positively infinite.