CHAPTER 3

CHAPTER 3

NEUTROSOPHIC BITOPOLOGICAL GROUP

In this chapter, the NBTG is introduced and the important properties of NBTGs are studied.

Definition 3.0.1

Let $\mathcal{G}$ be a NG on X, where X is a group. Let ${\daleth_1}^{\mathcal{G}},\ {\daleth_2}^{\mathcal{G}}$ be two NTs on $\mathcal{G}$, then $(\mathcal{G}, {\daleth_1}^{\mathcal{G}},\ {\daleth_2}^{\mathcal{G}})$ is said to be NBTG if the following conditions are satisfied:

(i) The mapping $g : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \times (\mathcal{G}, \daleth_i^{\mathcal{G}})$ to $(\mathcal{G}, \daleth_i^{\mathcal{G}})$ defined as $g(x, y) \mapsto xy,\ \forall\ x, y \in X$ for each $i = 1, 2;$ is relatively neutrosophic i -continuous.

(ii) The mapping $h : (\mathcal{G}, \daleth_i^{\mathcal{G}})$ to $(\mathcal{G}, \daleth_i^{\mathcal{G}})$ defined as $h(x) \mapsto x^{-1}$, $\forall\, x \in X$ for each $i = 1, 2;$ is relatively neutrosophic i -continuous.

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Definition 3.0.2

Let $\mathcal{G}$ be a NG of a group X. Then for fixed $a \in X$, the left translation $l_a : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \to (\mathcal{G}, \daleth_i^{\mathcal{G}})$ is defined as $l_a(x) \mapsto ax$, $\forall\ x \in X$, where $ax = \big\{\langle a, \mathcal{T}_i^{\mathcal{G}}(ax), \mathcal{I}_i^{\mathcal{G}}(ax), \mathcal{F}_i^{\mathcal{G}}(ax)\rangle : x \in X\big\}$ for each $i = 1, 2$. Similarly, the right translation $r_a : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \to (\mathcal{G}, \daleth_i^{\mathcal{G}})$ is defined as $r_a(x) \mapsto xa, \forall\ x \in X$, where $ax = \{\langle a, \mathcal{T}_i^{\mathcal{G}}(xa), \mathcal{I}_i^{\mathcal{G}}(xa), \mathcal{F}_i^{\mathcal{G}}(xa)\rangle : x \in X\}$ for each $i = 1, 2$.

Lemma 3.0.1

Let $\mathcal{G}$ be a NBTG in X with two NTs $\daleth_1^{\mathcal{G}}, \daleth_2^{\mathcal{G}}$, where X is a group. Then for each $a \in \mathcal{G}_e$, the left translation l_a and right translation r_a are relatively neutrosophic homeomorphism of $(\mathcal{G}, \daleth_1^{\mathcal{G}}, \daleth_2^{\mathcal{G}})$ into itself.

Proof

From Proposition 3.11 [67], we have $l_a[\mathcal{G}] = \mathcal{G}$ and $r_a[\mathcal{G}] = \mathcal{G}$, for all $a \in \mathcal{G}_e$ and let $h : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \to (\mathcal{G}, \daleth_i^{\mathcal{G}}) \times (\mathcal{G}, \daleth_i^{\mathcal{G}})$ defined as $h(x) \mapsto (a, x)$, for each $i = 1, 2$ and $x \in X$. Then $r_a : \psi \circ h$. Since $a \in \mathcal{G}_e$, $\mathcal{T}_i^{\mathcal{G}}(a) = \mathcal{T}_i^{\mathcal{G}}(e)$, $\mathcal{I}_i^{\mathcal{G}}(a) = \mathcal{I}_i^{\mathcal{G}}(e)$, and $\mathcal{F}_i^{\mathcal{G}}(a) = \mathcal{F}_i^{\mathcal{G}}(e)$, for each $i = 1, 2$. Thus, $\mathcal{T}_i^{\mathcal{G}}(a) \geq \mathcal{T}_i^{\mathcal{G}}(x)$, $\mathcal{I}_i^{\mathcal{G}}(a) \geq \mathcal{I}_i^{\mathcal{G}}(x)$, and $\mathcal{F}_i^{\mathcal{G}}(a) \leq \mathcal{F}_i^{\mathcal{G}}(x)$ for each $x \in X$. For each $i = 1, 2$ from proposition 3.34 [68] that $\phi : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \to (\mathcal{G}, \daleth_i^{\mathcal{G}}) \times (\mathcal{G}, \daleth_i^{\mathcal{G}})$ is relatively neutrosophic i-continuous. By the assumption, for each $i = 1, 2$; ψ is relatively neutrosophic i-continuous. So, for each $i = 1, 2$; r_a is relatively neutrosophic i-continuous. Moreover, $r_a^{-1} = r_{a^{-1}}$. Similarly, for each $i = 1, 2$, we have shown the relatively neutrosophic i-continuous of $l_a^{-1} = l_{a^{-1}}$.

Theorem 3.0.1

Let $\mathcal{G}$ be a NBTG on X with two NTs $\daleth_1^{\mathcal{G}}, \daleth_2^{\mathcal{G}}$. Let W be a NOS of $(\mathcal{G}, \daleth_i{}^{\mathcal{G}})$ for each $i = 1, 2$ and $x \in \mathcal{G}_e$, then xW and Wx are NOSs.

Proof

Since W is NOS of $\mathcal{G}$ and $x \in \mathcal{G}_e$, $l_x : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \to (\mathcal{G}, \daleth_i^{\mathcal{G}})$ is neutrosophic homomorphism for each $i = 1, 2$. This implies that $l_a(W) = xW$ is NOS in $\mathcal{G}$. Similarly, Wx is NOS in $\mathcal{G}$.

Lemma 3.0.2

Let $\mathcal{G}$ be NBTG on X, where X is a group. Then

(i) The inverse mapping $f : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \to (\mathcal{G}, \daleth_i^{\mathcal{G}})$ defined as $f(x) \mapsto x^{-1}$, $\forall\ x \in X$, for each i=1, 2; is relatively neutrosophic i-continuous homeomorphism.

(ii) The inner automorphism $h : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \to (\mathcal{G}, \daleth_i^{\mathcal{G}})$ defined by $h(g) = aga^{-1} = \{\langle g, \mathcal{T}_i^{\mathcal{G}}(aga^{-1}), \mathcal{I}_i^{\mathcal{G}}(aga^{-1}), \mathcal{F}_i^{\mathcal{G}}(aga^{-1})\rangle\}$, where $g \in X$ and $a \in \mathcal{G}_e$, for each $i = 1, 2$; is relatively neutrosophic homeomorphism.

Proof

(i) Clearly, f is one-to-one.
Since $f(\mathcal{G}) = \{\langle x, f(\mathcal{T}_i^{\mathcal{G}}(x)), f(\mathcal{I}_i^{\mathcal{G}}(x)), f(\mathcal{F}_i^{\mathcal{G}}(x))\rangle : x \in \mathcal{G}\}$ for each $i = 1, 2$ where

$$\begin{aligned} f(\mathcal{T}_i^{\mathcal{G}}(x)) &= \left\{\begin{matrix} \vee_{y\in f^{-1}(x)}\mathcal{T}_i^{\mathcal{G}}(y), & \text{if } f^{-1}(x) \neq 0 \\ 0, & \text{otherwise} \end{matrix}\right\} \\ &= \left\{\begin{matrix} \mathcal{T}_i^{\mathcal{G}}(x^{-1}), & \text{if } f^{-1}(x) \neq 0 \\ 0, & \text{otherwise} \end{matrix}\right\} \\ &= \left\{\begin{matrix} \mathcal{T}_i^{\mathcal{G}}(x), & \text{if } f^{-1}(x) \neq 0 \\ 0, & \text{otherwise} \end{matrix}\right\} \end{aligned}$$

Also, $f(\mathcal{I}_i^{\mathcal{G}}(x)) = \mathcal{I}_i^{\mathcal{G}}(x)$ and $\phi(\mathcal{F}_i^{\mathcal{G}}(x)) = \mathcal{F}_i^{\mathcal{G}}(x)$.
Thus, $f(\mathcal{G}) = \{\langle x, \mathcal{T}_i^{\mathcal{G}}(x), \mathcal{I}_i^{\mathcal{G}}(x), \mathcal{F}_i^{\mathcal{G}}(x)\rangle : x \in \mathcal{G}\}$, for each $i = 1, 2$. Also, f is neutrosophic i-continuous for each $i = 1, 2$

by definition because $(\mathcal{G}, \urcorner_1^{\mathcal{G}}, \urcorner_2^{\mathcal{G}})$ is NBTG. Since $f^{-1}(x) = x^{-1}$ is relatively neutrosophic i-continuous for each $i = 1, 2$. Hence, for every $x \in X$, f is relatively neutrosophic open. Thus, f is relatively neutrosophic homeomorphism.

(ii) Since r_a and l_a are relatively neutrosophic homeomorphism and $r_a^{-1} = r_{a^{-1}}$. The inner automorphism h is a composition $r_{a^{-1}}$ and l_a. Hence, h is a relatively neutrosophic homeomorphism.

Theorem 3.0.2

Let $\mathcal{G}$ be a NBTG in a group X and e be the identity of X. If $a \in \mathcal{G}_e$ and N is a nbhd of e such that $\mathcal{T}_i^N(e) = 1$, $\mathcal{I}_i^N(e) = 1$, $\mathcal{F}_i^N(e) = 0$ for each $i = 1, 2$ then aN is a nbhd of a such that $aN(a) = 1_N$.

Proof

Since N is a nbhd of e such that $\mathcal{T}_i^N = 1$, $\mathcal{I}_i^N = 1$, $\mathcal{F}_i^N = 0$ for each $i = 1, 2$; $\exists$ a NOS U such that $U \subseteq N$ and $\mathcal{T}_i^U(e) = \mathcal{T}_i^N(e) = 1$, $\mathcal{I}_i^U(e) = \mathcal{I}_i^N(e) = 1$, $\mathcal{F}_i^U(e) = \mathcal{F}_i^N(e) = 0$, for each $i = 1, 2$. Consider $l_a : (\mathcal{G}, \urcorner_i^{\mathcal{G}}) \to (\mathcal{G}, \urcorner_i^{\mathcal{G}})$ be a left translation defined as $l_a(g) \mapsto ag$, for each $g \in X$ and $i = 1, 2$. Then l_a is neutrosophic homeomorphism. Then aU is a NOS.

Now,

$$\begin{aligned}
aU(a) &= \{\langle a, \mathcal{T}_i^{aU}(a), \mathcal{I}_i^{aU}(a), \mathcal{F}_i^{aU}(a)\rangle\}, \text{for each } i = 1, 2. \\
&= \{\langle a, \mathcal{T}_i^{U}(aa^{-1}), \mathcal{I}_i^{U}(aa^{-1}), \mathcal{F}_i^{U}(aa^{-1})\rangle\} \\
&= \{\langle a, \mathcal{T}_i^{U}(e), \mathcal{I}_i^{U}(e), \mathcal{F}_i^{U}(e)\rangle\} \\
&= \{\langle a, 1, 1, 0\rangle\} \\
\text{Also, } aN(x) &= \{\langle x, \mathcal{T}_i^{aN}(x), \mathcal{I}_i^{aN}(x), \mathcal{F}_i^{aN}(x)\rangle : x \in X\}, \\
&\quad \text{for each } i = 1, 2. \\
&= \{\langle x, \mathcal{T}_i^{N}(a^{-1}x), \mathcal{I}_i^{N}(a^{-1}x), \mathcal{F}_i^{N}(a^{-1}x)\rangle : x \in X\}
\end{aligned}$$

$$
\begin{aligned}
&\geq \{\langle x, \mathcal{T}_i^U(a^{-1}x), \mathcal{I}_i^U(a^{-1}x), \mathcal{F}_i^U(a^{-1}x)\rangle : x \in X\} \\
&= \{\langle x, \mathcal{T}_i^{aU}(x), \mathcal{I}_i^{aU}(x), \mathcal{F}_i^{aU}(x)\rangle\} \\
&= aU(x) \\
aN(x) &\geq aU(x); \text{for each } x \in X. \\
\text{and } aN(a) &= \{\langle a, \mathcal{T}_i^{aN}(a), \mathcal{I}_i^{aN}(a), \mathcal{F}_i^{aN}(a)\rangle\}, \text{for each } i = 1, 2. \\
&= \{\langle a, \mathcal{T}_i^N(aa^{-1}), \mathcal{I}_i^N(aa^{-1}), \mathcal{F}_i^N(aa^{-1})\rangle\} \\
&= \{\langle a, \mathcal{T}_i^N(e), \mathcal{I}_i^N(e), \mathcal{F}_i^N(e)\rangle\} \\
&= \{\langle a, 1, 1, 0\rangle\} \\
\Rightarrow aN(a) &= \{\langle a, 1, 1, 0\rangle\}
\end{aligned}
$$

Thus, $\exists$ a NOS aU such that $aU \subseteq aN$ and $aU(a) = aN(a) = \{\langle a, 1, 1, 0\rangle\}$.

Proposition 3.0.1

Let $\mathcal{G}$ be a NBTG on X with two NTs $\daleth_1^{\mathcal{G}}, \daleth_2^{\mathcal{G}}$, where X is a group. Consider $\lambda : X \times X \to X$ be the mapping defined as $\lambda(g, h) = gh^{-1}$ for any $g, h \in X$. Then $\mathcal{G}$ is a NBTG in X iff the mapping $\lambda : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \times (\mathcal{G}, \daleth_i^{\mathcal{G}}) \to (\mathcal{G}, \daleth_i^{\mathcal{G}})$ is relatively neutrosophic i-continuous for each $i = 1, 2$.

Proof

The mapping $\gamma : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \times (\mathcal{G}, \daleth_i^{\mathcal{G}}) \to (\mathcal{G}, \daleth_i^{\mathcal{G}}) \times (\mathcal{G}, \daleth_i^{\mathcal{G}})$ is relatively neutrosophic i-continuous for each $i = 1, 2$; by the corollary to Proposition 3.28 [68]. Also, since $\mathcal{G}$ is a NBTG in X by the definition 3.0.1, we have $\psi : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \times (\mathcal{G}, \daleth_i^{\mathcal{G}}) \to (\mathcal{G}, \daleth_i^{\mathcal{G}})$ is relatively neutrosophic i-continuous for each $i = 1, 2$. Then $\beta : \psi \circ \gamma : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \times (\mathcal{G}, \daleth_i^{\mathcal{G}}) \to (\mathcal{G}, \daleth_i^{\mathcal{G}})$ is relatively neutrosophic i-continuous for each $i = 1, 2$.

Conversely, let $\lambda : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \times (\mathcal{G}, \daleth_i^{\mathcal{G}}) \to (\mathcal{G}, \daleth_i^{\mathcal{G}})$ is RN i-continuous for each $i = 1, 2$. If e is the identity element of X, then $\daleth_i^{\mathcal{G}}(e) \geq \mathcal{T}_i^{\mathcal{G}}(g)$, $\mathcal{I}_i^{\mathcal{G}}(e) \geq \mathcal{I}_i^{\mathcal{G}}(g)$ and $\mathcal{F}_i^{\mathcal{G}}(e) \leq \mathcal{F}_i^{\mathcal{G}}(g)$ for all $g \in X$. By the

Proposition 3.34 [68], the mapping $\pi : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \to (\mathcal{G}, \daleth_i^{\mathcal{G}}) \times (\mathcal{G}, \daleth_i^{\mathcal{G}})$ defined by $\pi(h) = (e, h)$ is relatively neutrosophic i-continuous for each $i = 1, 2$. Thus, the mapping $\mu = \lambda \circ \pi : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \to (\mathcal{G}, \daleth_i^{\mathcal{G}})$ is relatively neutrosophic i-continuous for each $i = 1, 2$. The mapping $\gamma : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \times (\mathcal{G}, \daleth_i^{\mathcal{G}}) \to (\mathcal{G}, \daleth_i^{\mathcal{G}}) \times (\mathcal{G}, \daleth_i^{\mathcal{G}})$ is relatively neutrosophic i-continuous for each $i = 1, 2$ by the corollary to Proposition 3.28 [68]. Thus, $\psi = \lambda \circ \gamma : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \times (\mathcal{G}, \daleth_i^{\mathcal{G}}) \to (\mathcal{G}, \daleth_i^{\mathcal{G}})$ is relatively neutrosophic i-continuous for each $i = 1, 2$. Therefore, $\mathcal{G}$ is a NBTG in X.

Proposition 3.0.2

Let $f : X \to Y$ be a group homomorphism and $\daleth_1^{\mathcal{G}}, \daleth_2^{\mathcal{G}}$ and $\mathcal{U}_1^{\mathcal{G}}, \mathcal{U}_2^{\mathcal{G}}$ be the NTs on X and Y respectively, where $\daleth_i^{\mathcal{G}}$ is the inverse image of $\mathcal{U}_i^{\mathcal{G}}$ under f and let $\mathcal{G}$ be a NBTG in Y. Then the inverse image $f^{-1}(\mathcal{G})$ of $\mathcal{G}$ is a NBTG in X.

Proof

Consider the mapping $\alpha : X \times X \to X$ defined by $\alpha(g_1, g_2) = g_1 g_2^{-1}$ for any $g_1, g_2 \in X$. We are to show that the mapping $\alpha : \Big(f^{-1}(\mathcal{G}), \daleth_i^{f^{-1}(\mathcal{G})}\Big) \times \Big(f^{-1}(\mathcal{G}), \daleth_i^{f^{-1}(\mathcal{G})}\Big) \to \Big(f^{-1}(\mathcal{G}), \daleth_i^{f^{-1}(\mathcal{G})}\Big)$ is relatively neutrosophic i-continuous for each $i = 1, 2$. Since $\daleth_i^{\mathcal{G}}$ is the inverse image of $\mathcal{U}_i^{\mathcal{G}}$ under f, $f : (X, \daleth_i^{\mathcal{G}}) \to (X, \mathcal{U}_i^{\mathcal{G}})$ is relatively neutrosophic i-continuous for each $i = 1, 2$. Also, $f\big(f^{-1}(\mathcal{G})\big) \subset \mathcal{G}$. By Proposition 3.9 [68], $f : \Big(f^{-1}(\mathcal{G}), \daleth_i^{f^{-1}(\mathcal{G})}\Big) \to \Big(\mathcal{G}, \mathcal{U}_i^{\mathcal{G}}\Big)$ is relatively neutrosophic i-continuous for each $i = 1, 2$.

Let $U = \daleth_i^{f^{-1}(\mathcal{G})}$. Then $\exists$ a $V = \mathcal{U}_i^{\mathcal{G}}$ such that $f^{-1}(V) = U$.

Let $(g_1, g_2) \in X \times X$. Then

$$\mathcal{T}_i^{\alpha^{-1}(U)}(g_1, g_2) = \alpha^{-1}\Big(\mathcal{T}_i^{U}\Big)(g_1, g_2) = \mathcal{T}_i^{U}\Big(\alpha(g_1, g_2)\Big)$$
$$= \mathcal{T}_i^{U}\Big(g_1, g_2^{-1}\Big), \text{for each } i = 1, 2.$$

$$= \mathcal{T}_i^{f^{-1}(V)}(g_1, g_2^{-1})$$
$$= f^{-1}_{(\mathcal{T}_i^V)}(g_1, g_2^{-1})$$
$$= \mathcal{T}_i^V\Big(f(g_1, g_2^{-1})\Big)$$
$$= \mathcal{T}_i^V\Big(f(g_1), f(g_2^{-1})\Big)$$
$$= \mathcal{T}_i^V\Big(f(g_1), \big(f(g_2)\big)^{-1}\Big)$$

Thus, $\mathcal{T}_i^{\alpha^{-1}(U)}(g_1, g_2) = \mathcal{T}_i^V\Big(f(g_1), \big(f(g_2)\big)^{-1}\Big)$.

Similarly, we have

$$\mathcal{I}_i^{\alpha^{-1}(U)}(g_1, g_2) = \mathcal{I}_i^V\Big(f(g_1), \big(f(g_2)\big)^{-1}\Big) \text{and}$$
$$\mathcal{F}_i^{\alpha^{-1}(U)}(g_1, g_2) = \mathcal{F}_i^V\Big(f(g_1), \big(f(g_2)\big)^{-1}\Big) \text{for each } i = 1, 2.$$

By the assumption, the mapping $\beta : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \times (\mathcal{G}, \daleth_i^{\mathcal{G}}) \to (\mathcal{G}, \daleth_i^{\mathcal{G}})$ given by $\beta(h_1, h_2) = h_1 h_2^{-1}$ for any $h_1, h_2 \in Y$ is relatively neutrosophic i-continuous for each $i = 1, 2$. By corollary to the Proposition 3.28 [68] the product mapping $f \times f : \Big(f^{-1}(\mathcal{G}), \daleth_i^{f^{-1}(\mathcal{G})}\Big) \times \Big(f^{-1}(\mathcal{G}), \daleth_i^{f^{-1}(\mathcal{G})}\Big) \to (\mathcal{G}, \daleth_i^{\mathcal{G}})$ is the neutrosophic i-continuous for each $i = 1, 2$.
Now, let $(g_1, g_2) \in X \times X$. Then

$$\mathcal{T}_i^V\Big(f(g_1), \big(f(g_2)\big)^{-1}\Big) = \mathcal{T}_i^{\beta^{-1}(V)}\big(f(g_1), f(g_2)\big)$$
$$= \mathcal{T}_i^{(f\times f)^{-1}(\beta^{-1}(V))}\big(g_1, g_2\big),$$
$$\mathcal{I}_i^V\Big(f(g_1), \big(f(g_2)\big)^{-1}\Big) = \mathcal{I}_i^{\beta^{-1}(V)}\big(f(g_1), f(g_2)\big)$$
$$= \mathcal{I}_i^{(f\times f)^{-1}(\beta^{-1}(V))}\big(g_1, g_2\big)$$
$$\text{and } \mathcal{F}_i^V\Big(f(g_1), \big(f(g_2)\big)^{-1}\Big) = \mathcal{F}_i^{\beta^{-1}(V)}\big(f(g_1), f(g_2)\big)$$
$$= \mathcal{F}_i^{(f\times f)^{-1}(\beta^{-1}(V))}\big(g_1, g_2\big)$$
$$\text{for each } i = 1, 2.$$

Thus, $\alpha^{-1}(U) \cap \Big(f^{-1}(\mathcal{G}) \times f^{-1}(\mathcal{G})\Big)$

$$= (f \times f)^{-1}\big(\beta^{-1}(V)\big) \cap \Big(f^{-1}(\mathcal{G}) \times f^{-1}(\mathcal{G})\Big)$$
$$= [\beta \circ (f \times f)]^{-1}(V) \cap \Big(f^{-1}(\mathcal{G}) \times f^{-1}(\mathcal{G})\Big).$$

So, $\alpha^{-1}(U) \cap \Big(f^{-1}(\mathcal{G}) \times f^{-1}(\mathcal{G})\Big) \in \daleth_i^{f^{-1}(\mathcal{G})} \times \daleth_i^{f^{-1}(\mathcal{G})}$, i.e., $\alpha : \Big(f^{-1}(\mathcal{G}), \daleth_i^{f^{-1}(\mathcal{G})}\Big) \times \Big(f^{-1}(\mathcal{G}), \daleth_i^{f^{-1}(\mathcal{G})}\Big) \to \Big(f^{-1}(\mathcal{G}), \daleth_i^{f^{-1}(\mathcal{G})}\Big)$ is a relatively neutrosophic i-continuous for each $i = 1, 2$. By Result 3.9 [67], $f^{-1}(\mathcal{G})$ is NG in X. Hence, by Proposition 3.0.1, $f^{-1}(\mathcal{G})$ is NBTG in X.

Proposition 3.0.3

Let $f : X \to Y$ be a group homomorphism. Let $\daleth_1^{\mathcal{G}}, \daleth_2^{\mathcal{G}}$ and $\mathcal{U}_1^{\mathcal{G}}, \mathcal{U}_2^{\mathcal{G}}$ be the NTs on X and Y respectively, where $\mathcal{U}_i^{\mathcal{G}}$ is the image under f and $\daleth_i^{\mathcal{G}}$, for each $i = 1, 2$; and let $\mathcal{G}$ be a NBTG in X. If $\mathcal{G}$ is the neutrosophic invariant, then the image $f(\mathcal{G})$ of $\mathcal{G}$ is a NBTG in Y.

Proof

Consider the mapping $\beta : Y \to Y$ defined by $\beta(h_1, h_2) = h_1 h_2^{-1}$ for any $h_1, h_2 \in Y$. We are to show that the mapping $\beta : \Big(f(\mathcal{G}), \mathcal{U}_i^{f(\mathcal{G})}\Big) \times \Big(f(\mathcal{G}), \mathcal{U}_i^{f(\mathcal{G})}\Big) \to \Big(f(\mathcal{G}), \mathcal{U}_i^{f(\mathcal{G})}\Big)$ is a relatively neutrosophic i- continuous for each $i = 1, 2$. Let $\mathcal{G}$ is a neutrosophic invariant. By the Definition 3.0.2, $f(\mathcal{G})$ is a NG in Y. Let $U \in \daleth_i^{\mathcal{G}}$. Also, $U \subset f^{-1}\big(f(U)\big)$. Then $\exists$ a family $\{U_\lambda\}_{\lambda \in \wedge} \subset \daleth_i^{\mathcal{G}}$ such that $f^{-1}\big(f(U)\big) = \bigcup_{\alpha \in \wedge} U_\alpha$. So, $f^{-1}\big(f(U)\big) \in \daleth_i^{\mathcal{G}}$. Since $\mathcal{U}_i$ is the image of $\daleth_i^{\mathcal{G}}$ under f, $f(U) \in \mathcal{U}_i^{\mathcal{G}}$, for each $i = 1, 2$. So, f is neutrosophic i-open. Now, let $U \in \daleth_i^{\mathcal{G}}$. Then $\exists$ a $U = U_1 \cap \mathcal{G}$. Since $\mathcal{G}$ is neutrosophic invariant, by Proposition 3.12 [67], $f(U) = f(U_1) \cap f(\mathcal{G})$. Since f is neutrosophic i-open, $f(U_1) = \daleth_i^{\mathcal{G}}$, for each $i = 1, 2$. Then $f(U) \in \mathcal{U}_i^{f(\mathcal{G})}$, for each $i = 1, 2$. Thus, $f : (\mathcal{G}, \daleth_i^{\mathcal{G}}) \to \Big(f(\mathcal{G}), \mathcal{U}_i^{f(\mathcal{G})}\Big)$ is relatively neutrosophic

i-open for each $i = 1, 2$. By Proposition 3.31 [68], the product mapping $(f \times f) : (\mathcal{G}, \urcorner_i^{\mathcal{G}}) \times (\mathcal{G}, \urcorner_i^{\mathcal{G}}) \to \Big(f(\mathcal{G}), \mathcal{U}_i^{f(\mathcal{G})}\Big)$ for each $i = 1, 2$; is relatively neutrosophic i-open.
Let $V \in \mathcal{U}_i^{f(\mathcal{G})}$ and let $(g_1, g_2) \in X \times X$. Then

$$\begin{aligned}
\mathcal{T}_i^{\beta\circ(f\times f)^{-1}(V)}(g_1, g_2) &= \big[\beta \circ (f \times f)\big]^{-1}\Big(\mathcal{T}_i^{V}\Big)(g_1, g_2), \text{for each } i = 1, 2. \\
&= \mathcal{T}_i^{V}\big[\beta \circ (f \times f)\big](g_1, g_2) \\
&= \mathcal{T}_i^{V}\Big(f(g_1), f(g_2)\Big) \\
&= \mathcal{T}_i^{V}\Big(f(g_1), \big(f(g_2)^{-1}\big)\Big) \\
&= \mathcal{T}_i^{V}\Big(f(g_1), f(g_2^{-1})\Big) \ \ [\text{Since} f \text{ is homomorphism}] \\
&= \mathcal{T}_i^{V}\Big(f(g_1 g_2^{-1})\Big) \\
&= \mathcal{T}_i^{V} f\Big(\alpha(g_1, g_2)\Big) \\
&= \mathcal{T}_i^{V}\Big(f \circ \alpha(g_1, g_2)\Big) \\
&= (f \circ \alpha)^{-1}\Big(\mathcal{T}_i^{V}(g_1, g_2)\Big) \\
&= \mathcal{T}_i^{(f\circ\alpha)^{-1}(V)}(g_1, g_2),
\end{aligned}$$

where $\alpha : X \times X \to X$ is the mapping given by $\alpha(g_1, g_2) = g_1 g_2^{-1}$ for each $(g_1, g_2) \in X \times X$.
Thus, $\mathcal{T}_i^{[\beta\circ(f\times f)]^{-1}(V)} = \mathcal{T}_i^{(f\circ\alpha)^{-1}(V)}, \mathcal{T}_i^{(f\times f)^{-1}[\beta^{-1}(V)]} = \mathcal{T}_i^{\alpha^{-1}(f^{-1}(V))}$.
And $\mathcal{I}_i^{(f\times f)^{-1}[\beta^{-1}(V)]} = \mathcal{I}_i^{\alpha^{-1}(f^{-1}(V))}$; $\mathcal{F}_i^{(f\times f)^{-1}[\beta^{-1}(V)]} = \mathcal{F}_i^{\alpha^{-1}(f^{-1}(V))}$.
So, $(f \times f)^{-1}[\beta^{-1}(V)] = \alpha^{-1}(f^{-1}(V))$. Since $\mathcal{G}$ is NBTG in X, $\alpha : (\mathcal{G}, \urcorner_i^{\mathcal{G}}) \times (\mathcal{G}, \urcorner_i^{\mathcal{G}}) \to (\mathcal{G}, \urcorner_i^{\mathcal{G}})$ is relatively neutrosophic i- continuous for each $i = 1, 2$. Since $\mathcal{U}_i^{\mathcal{G}}$ is the image of $\urcorner_i^{\mathcal{G}}$ under f, $f : (\mathcal{G}, \urcorner_i^{\mathcal{G}}) \to (\mathcal{G}, \urcorner_i^{\mathcal{G}}), \mathfrak{T}_i^{f(\mathcal{G})}$ is relatively neutrosophic i- continuous for each $i = 1, 2$. Then $(f \times f) : (\mathcal{G}, \urcorner_i^{\mathcal{G}}) \times (\mathcal{G}, \urcorner_i^{\mathcal{G}}) \to \Big(f(\mathcal{G}), \mathcal{U}_i^{f(\mathcal{G})}\Big) \times \Big(f(\mathcal{G}), \mathcal{U}_i^{f(\mathcal{G})}\Big)$ is relatively neutrosophic i- continuous for each $i = 1, 2$. Thus, $(f \times f) \circ \beta : (\mathcal{G}, \urcorner_i^{\mathcal{G}}) \times (\mathcal{G}, \urcorner_i^{\mathcal{G}}) \to \Big(f(\mathcal{G}), \mathcal{U}_i^{f(\mathcal{G})}\Big)$ is relatively neutrosophic

i- continuous for each $i = 1, 2$. Since $\mathcal{G}$ is neutrosophic invariant, $(f \times f)^{-1}\big[\beta^{-1}(V) \cap \big(f(\mathcal{G}) \times f(\mathcal{G})\big)\big] = (f \times f)^{-1}\big[\beta^{-1}(V)\big] \cap (\mathcal{G} \times \mathcal{G})$. So, $(f \times f)^{-1}\big[\beta^{-1}(V) \cap \big(f(\mathcal{G}) \times f(\mathcal{G})\big)\big] \in \daleth_i^{\mathcal{G}} \times \daleth_i^{\mathcal{G}}$. Since $(f \times f)$ is relatively neutrosophic i-open for each $i = 1, 2$; $(f \times f)(f \times f)^{-1}\big[\beta^{-1}(V) \cap \big(f(\mathcal{G}) \times f(\mathcal{G})\big)\big] \in \mathcal{U}_i^{f(\mathcal{G})} \times \mathcal{U}_i^{f(\mathcal{G})}$ for each $i = 1, 2$. But $(f \times f)(f \times f)^{-1}\big[\beta^{-1}(V) \cap \big(f(\mathcal{G}) \times f(\mathcal{G})\big)\big] = \beta^{-1}(V) \cap \big(f(\mathcal{G}) \times f(\mathcal{G})\big)$. So, $\beta^{-1}(V) \cap \big(f(\mathcal{G}) \times f(\mathcal{G})\big) \in \mathcal{U}_i^{f(\mathcal{G})} \times \mathcal{U}_i^{f(\mathcal{G})}$ for each $i = 1, 2$. Hence, $f(\mathcal{G})$ is a NBTG in Y.

Proposition 3.0.4

Let $\mathcal{G}$ be a NBTG in a group X with two NTs $\daleth_1^{\mathcal{G}}, \daleth_2^{\mathcal{G}}$. Let N be a normal subgroup of X and let f be the canonical homomorphism of X onto the quotient group X/N. If $\mathcal{G}$ is constant on N, then $\mathcal{G}$ is f invariant.

Proof

For any $x_1, x_2 \in N$, let $f(x_1) = f(x_2)$, then $x_1N = x_2N$. Thus, $\exists\ k_1, k_2 \in N$ such that $x_1k_1 = x_2k_2$. Since $\mathcal{G}$ is a constant on N, $\mathcal{T}_i^{\mathcal{G}}(x) = \mathcal{T}_i^{\mathcal{G}}(e), \mathcal{I}_i^{\mathcal{G}}(x) = \mathcal{I}_i^{\mathcal{G}}(e)$ and $\mathcal{F}_i^{\mathcal{G}}(x) = \mathcal{F}_i^{\mathcal{G}}(e)$ for each $i = 1, 2$ and $x \in X$. Then

$$
\begin{aligned}
\mathcal{T}_i^{\mathcal{G}}(x_1) &= \mathcal{T}_i^{\mathcal{G}}(x_2k_2k_1^{-1}) \\
&\geq \mathcal{T}_i^{\mathcal{G}}(x_2) \wedge \mathcal{T}_i^{\mathcal{G}}(k_2k_1^{-1}) \\
&= \mathcal{T}_i^{\mathcal{G}}(x_2) \wedge \mathcal{T}_i^{\mathcal{G}}(e)(k_2k_1^{-1} \in N) \\
&= \mathcal{T}_i^{\mathcal{G}}(x_2) \\
i.e., \mathcal{T}_i^{\mathcal{G}}(x_1) &\geq \mathcal{T}_i^{\mathcal{G}}(x_2)
\end{aligned}
$$

Similarly, we get $\mathcal{T}_i^{\mathcal{G}}(x_2) \geq \mathcal{T}_i^{\mathcal{G}}(x_1)$.
Thus, $\mathcal{T}_i^{\mathcal{G}}(x_1) = \mathcal{T}_i^{\mathcal{G}}(x_2)$.
Similarly, we can show that $\mathcal{I}_i^{\mathcal{G}}(x_1) = \mathcal{I}_i^{\mathcal{G}}(x_2)$ and $\mathcal{F}_i^{\mathcal{G}}(x_1) = \mathcal{F}_i^{\mathcal{G}}(x_2)$.
Hence, $\mathcal{G}$ is f invariant.