CHAPTER 1

CHAPTER 1

INTRODUCTION, LITERATURE REVIEW AND PRELIMINARIES

1.1 Introduction

Topology is an umbrella term that includes several fields of study; that is, topological spaces show up naturally in almost every branch of mathematics. This has made topology one of the great unifying ideas of mathematics. These include point set topology, algebraic topology, and differential topology. Because of this, it is difficult to credit a single mathematician with introducing topology. By observing this, many authors studied branches of topological spaces. In general, topology is a special kind of geometry, a geometry that doesn't include a notion of distance. A topological group is a set that has both a topological structure and an algebraic structure. Topological groups have the algebraic structure of a group and the topological structure of a topological space, and they are linked by the requirement that multiplication and inversion be continuous functions. In this thesis, many interesting properties and examples of such objects will be explored based on the neutrosophic set.

1.2 Literature Review

In 1965, L. A. Zadeh [1] was first introduced the concept of fuzzy set. Fuzzy sets were established to provide a mathematical way to represent situations that result in ill-defined classes, i.e., there are no precise membership criteria for collections of objects. This type of collection has ambiguous or "fuzzy" boundaries; some objects are impossible to establish whether or not they belong to the collection. The classical mathematical theories of classical set theory and probability theory are used to express certain types of certainty. The set theory defines uncertainty as any given set of possible alternatives in situations where only one of the possibilities is likely to happen. The non-specificity inherent in each set causes uncertainty to be expressed in terms of sets of alternatives. Uncertainty is expressed in probability theory as a classical measure of subsets of a given set of alternatives. Zadeh introduced the concept of fuzzy set theory, which states that membership in a subset is a matter of degree rather than being completely in or completely out. As a result of using fuzzy set theory, one can obtain a logic in which statements can be true or false to various degrees rather than the bivalent situation of being true or false; as a result, many bivalent logic principles no longer apply.

After Zadeh [1] discovered the Fuzzy set, Chang [2] discussed the Fuzzy topology and proposed the concept of fuzzy topological spaces as an extension to classical topological spaces. Later that, many authors investigated topological characteristics in fuzzy environments and proposed various definitions for the same property, leading to various approaches. Some of the properties and theorems of the fuzzy topological space are also different from those of classical topology. Some basic concepts which play an important role in fuzzy topology are fuzzy interior, fuzzy closure, fuzzy point and fuzzy boundary. Fuzzy topology, fuzzy interior, fuzzy closure, fuzzy point and fuzzy boundary are some basic concepts which play an important role in fuzzy topology.

Fuzzy set theory and fuzzy topology were approached as generalizations of general set theory and general topology. The concept of Fuzzy topology was first introduced by Chang [2] in 1968 and, on a set X as a family $\mathfrak{T} \subset I^X$, where I = [0,1] with satisfying the well-known topological axioms, and an open set referred to each member of topological space. While analyzing his definition of a fuzzy topology some authors have noticed that fuzziness in the concept of openness of a fuzzy set has not been considered.

With the help of a fuzzy set, defined the concept of membership function and explained the idea of uncertainty. Atanassov [3] generalized the concept of fuzzy sets in 1986, introduced the degree of non-membership as an independent component, and proposed the intuitionistic fuzzy set. After that, many researchers defined various new concepts in a generalization of the fuzzy set. Coker [4] generalized the concept of fuzzy set to intuitionistic fuzzy topology. Observing the fuzzy set, Rosenfeld [5] introduced the concept of fuzzy groups, and Foster [6] also proposed the idea of fuzzy topological groups. Azad [7] discussed fuzzy semi-continuity, fuzzy almost continuity, and fuzzy weakly continuity. In 1963, Kelly [8] defined the study of bitopological spaces and also Kandil *et al.* [9] discussed fuzzy bitopological space. Lee *et al.* [10] discussed some properties of intuitionistic fuzzy bitopological spaces. Bhattacharya *et al.* [11, 12] investigated fuzzy independent topological spaces formed by fuzzy γ^*-open sets and their applications, as well as γ^* -Hyperconnectedness in fuzzy topological spaces. Bhattacharya *et al.* [13, 14] generalized γ-open set, contra continuity, and almost contra continuity in fuzzy topological spaces. Many researchers, such as Das *et al.* [15–17] and Paul *et al.* [18, 19], have examined many areas in fuzzy bitopological spaces. Garg *et al.* [20, 21] investigated at the use of spherical fuzzy soft topology in group decision-making problems, as well as experimental survey distance measurements for intuitionistic fuzzy sets based on various triangle centres of isosceles triangular fuzzy numbers and their applications. In the group decision-making process, Garg [22] explored novel exponential operation rules and operators for interval-valued q-rung orthopair fuzzy sets. Tripathy *et al.* [23–25] developed topological space based on a fuzzy set.

A Neutrosophic set is used to control uncertainty by using a truth membership function, an indeterminacy membership function, and a falsity membership function. Whereas a fuzzy set is used to control uncertainty by using membership function only. The Neutrosophic set is used indeterminacy as an independent measure of the membership and non-membership functions. As a result, the neutrosophic set is considered as a generalization of the fuzzy set and the intuitionistic fuzzy set and shows better results. In various problems, fuzzy sets and intuitionistic fuzzy sets cannot be completely assured due to their inconsistent characteristics. Therefore, the neutrosophic set shows a more rational way to design the membership function.

Smarandache [26, 27] established Neutrosophy as a new discipline of philosophy, explaining that a neutrosophic set is a generalization of an intuitionistic fuzzy set. The neutrosophic components T, I, and F were established by Smarandache to present membership, indeterminacy, and non-membership values, respectively, where $]^{-}0, 1^{+}[$ is nonstandard unit interval.

Let T, I, F be real standard or not standard subset of $]^{-}0, 1^{+}[$, with

$$\sup T = t\_sup, \quad \inf T = t\_inf,$$
$$\sup I = i\_sup, \quad \inf I = i\_inf,$$
$$\sup F = f\_sup, \quad \inf F = f\_inf,$$
$$n\_sup = t\_sup + i\_sup + f\_sup,$$
$$n\_inf = t\_inf + i\_inf + f\_inf.$$

Where T, I, and F are called neutrosophic components.

Smarandache [28–30] was introduced as an independent component of the degree of uncertainty and discovered the neutrosophic set. After the discovery of neutrosophic set, many researchers have developed the neutrosophic set theory for various branches of Science and Technology. Smarandache [31] extended neutrosophic set to neutrosophic overset, neutrosophic underset, and neutrosophic offset. Salama [32] used data mining technologies to investigate several topological characteristics of rough sets. The concept of rough neutrosophic sets was proposed by Broumi *et al.* [33]. The connectedness of $\alpha\omega$-closed sets in neutrosophic topological spaces was studied by Parimala *et al.* [34]. Garg *et al.* [35] investigated the trapezoidal bipolar neutrosophic number as a model for container inventory and also Garg [36–41] worked on a lot of research in many fields in neutrosophic sets. In neutrosophic bitopological spaces, Tripathy *et al.* [42] addressed paired neutrosophic b-continuous functions. Das *et al.* [43–45] did extensive research in topology using neutrosophic sets.

The neutrosophic set theory was used as a tool in a group discussion framework by Abdel-Basset *et al.* [46] and also, [47] investigated the use of the base-worst technique to solve chain problems using a novel plithogenic model. Abdel-Basset *et al.* [48] developed supplier selection with group decision-making under the type-2 neutrosophic number of TOPSIS technology. Abdel-Basset *et al.* [49, 50] studied the chain management practices of evaluation of the green supply and defined for achieving sustainable supplier selection of VIKOR method. Pamucar *et al.* [51] used single-valued neutrosophic sets to propose a projection-based multi-attributive border approximation area comparison method. Liu *et al.* [52] studied a new extension of the decision-making trial and evaluation laboratory method (DEMATEL). Guo *et al.* [53] extended the rough set model to a neutrosophic environment and used it to multi-attribute decision-making (MADM) problems. Nie *et al.* [54] studied the weighted aggregated sum product assessment (WASPAS) method in the context of interval neutrosophic sets. Ye [55] introduced interval neutrosophic hesitant fuzzy set (INHFS). Pamucar *et al.* [56, 57] studied the application of linguistic neutrosophic numbers. Karaaslan [58] introduced a type-2 single-valued neutrosophic set along with some distance measures. The neutrosophic number is used by Maiti *et al.* [59] to solve multi-objective linear programming problems.

Recently, Al-Omeri and Smarandache [60, 61] presented and investigated a number of definitions of neutrosophic closed sets, neutrosophic mapping, and got numerous preservations features as well as certain characterizations of neutrosophic connectedness continuity. In [62, 63] Abdel-Basset *et al.* have given how a new trend of neutrosophic theory is applicable in the field of medicine and multimedia with a novel and powerful model. The idea of neutrosophic topological space was introduced by Salama *et al.* [64], and Devi *et al.* [65] investigated at separation axioms in ordered neutrosophic bitopological space. Mwchahary *et al.* [66] worked in a bitopological neutrosophic space. Sumathi *et al.* [67, 68] established the fuzzy neutrosophic group, and also studied the topological group structure of the neutrosophic set. Imran *et al.* [69] addressed different types of neutrosophic topological groups in connection to neutrosophic alpha open sets. Broumi *et al.* [70] explored at neutrosophic graph extension and applications to robots. Hussain *et al.* [71] and Aparna *et al.* [72] investigated soft graphs and R-dynamic vertex colouring of graphs based on neutrosophic sets, respectively. Mehmood *et al.* [73, 74] generalized neutrosophic separation axioms and soft α-open sets in neutrosophic soft topological spaces, whereas Khattak *et al.* [75] studied soft b-separation axioms in neutrosophic soft topological structures. Edalatpanah [76] developed triangular neutrosophic numbers in data envelopment analysis. A novel method for solving multiobjective linear programming problems with triangular neutrosophic numbers was examined by Wang *et al.* [77]. The neutrosophic goal programming approach was used to solve the multiobjective fractional transportation problem by Veeramani *et al.* [78].

1.3 Aims and Objectives

From the literature survey of neutrosophic set, it is noticed that exactly the properties of the neutrosophic topological group are not done. In this current decade, neutrosophic environments are mainly interested by different fields of researchers. In Mathematics also much theoretical research has been observed in the sense of neutrosophic environment. It will be necessry to carry out more theoritical research to establish a general framework for decision-making and to define patterns for the conception and implementation of complex networks. By observing this, the current research identify the following objective for the Ph.D. research work.

The main objectives of the research work are:

(i) To study the concept of neutrosophic semi-continuous and neutrosophic almost continuous mapping and also to investigate some examples and basic properties of such mapping.

(ii) To study the concept of neutrosophic bitopological group and also to study some of its characterization propositions and theorems.

(iii) To study the concept of neutrosophic almost topological group and neutrosophic almost bitopological group and their properties.

(iv) To study the concept of plithogenic neutrosophic hypersoft almost topological group.

(v) To study the concept of neutrosophic multi continuous and neutrosophic multi topological group and also to investigate some examples.

1.4 Preliminaries

In this section, the basic definitions for neutrosophic sets and its operations and properties are given.

Definition 1.4.1 *[27]*
Let X be a non-empty fixed set. A neutrosophic set A is an object having the form $A = \{\langle x, \mu_A, \sigma_A, \gamma_A \rangle : x \in X\}$, where $T, I, F : X \to [0, 1]$ and $0 \leq \mu_A + \sigma_A + \gamma_A \leq 3$ and $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x)$ which represents the degree of membership function, the degree indeterminacy and the degree of non-membership function respectively of each element $x \in X$ to the set A.

Remark 1.4.1 *[27]*

A neutrosophic set $A = \{\langle x, \mu_A, \sigma_A, \gamma_A \rangle : x \in X\}$ *can be identified to an ordered triple* $\langle \mu_A, \sigma_A, \gamma_A \rangle$ *in* $[0, 1]$ *on* X.

Remark 1.4.2 *[27]*

For the sake of simplicity, we shall use the symbol $A = \{\langle x, \mu_A, \sigma_A, \gamma_A \rangle\}$ *for the neutrosophic set* $A = \{\langle x, \mu_A, \sigma_A, \gamma_A \rangle : x \in X\}$.

Definition 1.4.2 *[27]*

Let X *be a non-empty set and* $A = \Big\{\langle x, T_A(x),\, I_A(x),\, F_A(x)\ \rangle : x \in X\Big\}$, $B = \Big\{\langle x, T_B(x),\, I_B(x), F_B(x)\rangle : x \in X\Big\}$*, are neutrosophic sets. Then*

(i) $A \wedge B = \Big\{\langle x, min\big(T_A(x), T_B(x)\big), min\big(I_A(x), I_B(x)\big), max\big(F_A(x), F_B(x)\big), \rangle : x \in X\Big\}$.

(ii) $A \vee B = \Big\{\langle x, max\big(T_A(x), T_B(x)\big), max\big(I_A(x), I_B(x)\big), min\big(F_A(x), F_B(x)\big), \rangle : x \in X\Big\}$.

(iii) $A \leq B$ *if for each* $x \in X, T_A(x) \leq T_B(x), I_A(x) \geq I_B(x), F_A(x) \geq F_B(x)$.

Definition 1.4.3 *[27]*

A neutrosophic set A *over the non-empty set* X *is said to be empty neutrosophic set if* $T_A(x) = 0,\ I_A(x) = 0,\ F_A(x) = 1$*, for all* $x \in X$.
It is denoted by 0_N.
A neutrosophic set A *over the non-empty set* X *is said to be universe neutrosophic set if* $T_A(x) = 1,\ I_A(x) = 1,\ F_A(x) = 0$*, for all* $x \in X$.
It is denoted by 1_N.

Definition 1.4.4 *[27]*

The complement of neutrosophic set A *is denoted by* A^c *and is defined* *as* $A^c(x) = \big\{\langle x, T_{A^c}(x) = F_A(x), I_{A^c}(x) = 1 - I_A(x), F_{A^c}(x) = T_A(x)\rangle : x \in X\big\}$.

Theorem 1.4.1 *[27]*
Let A and B be two neutrosophic sets on X then

1. $A \vee A = A$ *and* $A \wedge A = A$.
2. $A \vee B = B \vee A$ *and* $A \wedge B = B \wedge A$
3. $A \vee 0_N = A$ *and* $A \vee 1_N = 1_N$.
4. $A \wedge 0_N = 0_N$ *and* $A \wedge 1_N = A$
5. $A \vee (B \vee C) = (A \vee B) \vee C$ *and* $A \wedge (B \wedge C) = (A \wedge B) \wedge C$
6. $(A^c)^c = A$.

Definition 1.4.5 *[67]*
Let X be a group and U, V be two neutrosophic sets in X. We define the product UV of neutrosophic set U, V and the inverse V^{-1} of V as follows:

$$T_{UV}(x) = sup\big\{min\{T_U(x_1), T_V(x_2)\}\big\}$$
$$I_{UV}(x) = sup\big\{min\{I_U(x_1), I_V(x_2)\}\big\}$$
$$F_{UV}(x) = sup\big\{min\{F_U(x_1), F_V(x_2)\}\big\}$$

where $x = x_1.x_2$ and for $V = \big\{\langle x, T_V(x), I_V(x), F_V(x)\rangle : x \in X\big\}$, we have $V^{-1} = \big\{\langle x, T_V(x^{-1}), I_V(x^{-1}), F_V(x^{-1})\rangle : x \in X\big\}$.

1.5 Neutrosophic Functions

Definition 1.5.1 *[79]*
Let X and Y be two non empty sets and consider f be a function from

a set X to a set Y. Let $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$, $B = \{\langle y, T_B(y), I_B(y), F_B(y)\rangle : y \in Y\}$ be neutrosophic set in X and Y. Then

(i) $f^{-1}(B)$, the pre-image of B under f is the neutrosophic set in X defined by

$$f^{-1}(B) = \{\langle x, f^{-1}(T_B)(x), f^{-1}(I_B)(x), f^{-1}(F_B)(x)\rangle : x \in X\}$$

where for all $x \in X$, $f^{-1}(T_B)(x) = T_B(f(x))$, $f^{-1}(I_B)(x) = I_B(f(x))$, $f^{-1}(F_B)(x) = F_B(f(x))$.

(ii) The image of A under f denoted by $f(A)$ is a neutrosophic set in Y defined by
$f(A) = (f(T_A), f(I_A), f(F_A))$, where for each $u \in Y$,

$$f(T_A)(u) = \left\{\begin{array}{cc} \bigvee_{x \in f^{-1}(u)} T_A(x), & \text{if } f^{-1}(u) \neq 0 \\ 0, & \text{otherwise} \end{array}\right\}$$

$$f(I_A)(u) = \left\{\begin{array}{cc} \bigvee_{x \in f^{-1}(u)} I_A(x), & \text{if } f^{-1}(u) \neq 0 \\ 0, & \text{otherwise} \end{array}\right\}$$

$$f(F_A)(u) = \left\{\begin{array}{cc} \bigvee_{x \in f^{-1}(u)} F_A(x), & \text{if } f^{-1}(u) \neq 0 \\ 0, & \text{otherwise} \end{array}\right\}.$$

Remark 1.5.1 *[79]*
Let $f : X \to Y$ be a function. Then the following statements hold.

(i) If A and B are neutrosophic subsets of X such that $A \leq B$ then $f(A) \leq f(B)$.

(ii) If A and B are neutrosophic subsets of Y such that $A \leq B$ then $f^{-1}(A) \leq f^{-1}(B)$.

Proposition 1.5.1 *[79]*
Let $A, A_i (i \in I)$ be neutrosophic sets in X and $B, B_j (j \in J)$ be neutrosophic sets in Y and let $f : X \to Y$ be a mapping, then

1. $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$
2. $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$
3. $A \subset f^{-1}\big(f(A)\big)$. *If f is injective, then* $A = f^{-1}\big(f(A)\big)$.
4. $f^{-1}\big(f(B)\big) \subset B$. *If f is surjective, then* $f^{-1}\big(f(B)\big) = B$.
5. $f^{-1}(\vee B_j) = \vee f^{-1}(B_j)$.
6. $f^{-1}(\wedge B_j) = \wedge f^{-1}(B_j)$.
7. $f(\vee A_i) = \vee f(A_i)$.
8. $f(\wedge A_i) \subset \wedge f(A_i)$. *If f is injective, then* $f(\wedge A_i) = \wedge f(A_i)$.
9. $f(1_N) = 1_N$, *if f is surjective and* $f(0_N) = 0_N$.
10. $f^{-1}(1_N) = 1_N$ *and* $f^{-1}(0_N) = 0_N$.
11. $[f(A)]^c \subset f(A^c)$ *iff f is surjective.*
12. $f^{-1}(B^c) = [f^{-1}(B)]^c$.

1.6 Soft Set, Hypersoft Set and Neutrosophic Hypersoft Set

Definition 1.6.1 *[80]*
Let $\mathcal{U}$ be a universal set, let $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and E be the set of attributes values. Then the ordered pair $(F,\mathcal{U})$ is said to be SS over $\mathcal{U}$, where $F : E \to \mathcal{P}(\mathcal{U})$.

Definition 1.6.2 *[80, 82]*

Let $\mathcal{U}$ be a universal set and $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$. Let $a_1, a_2, a_3, \ldots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attributes value are respectively the sets $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_n$, with $\mathcal{A}_i \cap \mathcal{A}_j = \phi$, for $i \neq j$ and $i, j \in 1, 2, 3, \ldots, n$. Let $E_\alpha = \mathcal{A}_1 \times \mathcal{A}_2 \times \ldots \times \mathcal{A}_n$. Then the ordered pair (F, E_α) is called HS of $\mathcal{U}$, where $F : E_\alpha \to \mathcal{P}(\mathcal{U})$.

Definition 1.6.3 *[80, 82]*

Let $\mathcal{U}$ be a universal set and $P \subseteq \mathcal{U}$. A plithogenic set is denoted by $P_r = (P, \alpha, E_\alpha, p, q)$ where α be an attribute, E_α is the respective range of attributes values, $p : P \times E_\alpha \to [0, 1]^r$ is the degree of appurtenance function (DAF) and $q : E_\alpha \times E_\alpha \to [0, 1]^s$ is the corresponding degree of contradiction function (DCF), where $r, s \in \{1, 2, 3\}$.

Definition 1.6.4 *[82, 83]*

Let $\mathcal{U}_\mathcal{N}$ be the universal set termed as a neutrosophic universal set if $\forall x \in \mathcal{U}_\mathcal{N}, x$ has truth belongingness, indeterminacy belongingness, and falsity belongingness to $\mathcal{U}_\mathcal{N}$, i.e., membership of x belonging to $[0, 1] \times [0, 1] \times [0, 1]$.

Definition 1.6.5 *[80, 82]*

Let $\mathcal{U}_\mathcal{N}$ be a neutrosophic universal set and $\alpha = a_1, a_2, a_3, \ldots, a_n$ be a set of attributes with attribute value sets respectively as $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_n$, with $\mathcal{A}_i \cap \mathcal{A}_j = \phi$, for $i \neq j$ and $i, j \in \{1, 2, 3, \ldots, n\}$. Also, let $E_\alpha = \mathcal{A}_1 \times \mathcal{A}_2 \times \ldots \times \mathcal{A}_n$. Then (F, E_α), where $F : E_\alpha \to \mathcal{P}(\mathcal{U}_\mathcal{N})$ is said to be a NHS over $\mathcal{U}_\mathcal{N}$.

Definition 1.6.6 *[82]*

Let $\mathcal{U}_P$ be a plithogenic universal set and $\alpha = a_1, a_2, a_3, \ldots, a_n$ be a set of attributes with attribute value sets respectively as $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots,$

$\mathcal{A}_n$, with $\mathcal{A}_i \cap \mathcal{A}_j = \phi$, for $i \neq j$ and $i, j \in \{1, 2, 3, \ldots, n\}$. Also, let $E_\alpha = \mathcal{A}_1 \times \mathcal{A}_2 \times \ldots \times \mathcal{A}_n$. Then (F, E_α), where $F : E_\alpha \to \mathcal{P}(\mathcal{U}_P)$ is said to be PHS over $\mathcal{U}_P$.

Definition 1.6.7 *[82]*

The ordered pair (F, E_α) is said to be a PNHS if for all $B \in$ range (F) and for all $i \in \{1, 2, \ldots, n\}$, there exists $f_{\mathcal{N}_i} : B \times R_i \to [0,1] \times [0,1] \times [0,1]$ such that for all $(b, r) \in B \times R_i$, $f_{\mathcal{N}_i}(b, r) \in [0,1] \times [0,1] \times [0,1]$. A set of all the PNHSs over a set $\mathcal{U}$ is denoted by $PNHS(\mathcal{U})$.

Definition 1.6.8 *[82]*

Let the ordered pair (F, E_α) be a PNHS of a crisp group $\mathcal{U}$, where $E_\alpha = \mathcal{A}_1 \times \mathcal{A}_2, \times \ldots \times \mathcal{A}_n$ and $\forall\ i \in \{1, 2, \ldots, n\}$, R_i are crisp groups. Then (F, E_α) is said to be a PNHSG of $\mathcal{U}$ if and only if $\forall\ B \in$ range (F); $\forall\ (b_1, r_1), (b_2, r_2) \in B \times R_i$ and $\forall\ f_{\mathcal{N}_i} : B \times R_i \to [0,1] \times [0,1] \times [0,1]$; with $f_{\mathcal{N}_i}(b, r) = \Big\{ \langle (b, R), f^T_{\mathcal{N}_i}(b, r), f^I_{\mathcal{N}_i}(b, r), f^F_{\mathcal{N}_i}(b, r) \rangle : (b, r) \in B \times R_i \Big\}$, the following subsequent conditions are satisfied:

1. $f^T_{\mathcal{N}_i}\big((b_1, r_1).(b_2, r_2)^{-1}\big) \geq \min\big\{f^T_{\mathcal{N}_i}(b_1, r_1), f^T_{\mathcal{N}_i}(b_2, r_2)\big\}$.
2. $f^T_{\mathcal{N}_i}(b_1, r_1)^{-1} \geq f^T_{\mathcal{N}_i}(b_1, r_1)$.
3. $f^I_{\mathcal{N}_i}\big((b_1, r_1).(b_2, r_2)^{-1}\big) \geq \min\big\{f^I_{\mathcal{N}_i}(b_1, r_1), f^I_{\mathcal{N}_i}(b_2, r_2)\big\}$.
4. $f^I_{\mathcal{N}_i}(b_1, r_1)^{-1} \geq f^I_{\mathcal{N}_i}(b_1, r_1)$.
5. $f^F_{\mathcal{N}_i}\big((b_1, r_1).(b_2, r_2)^{-1}\big) \leq \max\big\{f^F_{\mathcal{N}_i}(b_1, r_1), f^F_{\mathcal{N}_i}(b_2, r_2)\big\}$.
6. $f^F_{\mathcal{N}_i}(b_1, r_1)^{-1} \leq f^F_{\mathcal{N}_i}(b_1, r_1)$.

A set of all the PNHSG of a crisp group $\mathcal{U}$ is denoted by PNHSG$(\mathcal{U})$.

1.7 Neutrosophic Group

Definition 1.7.1 *[67]*

Let $(X, \circ)$ be a group and let A be a NG in X. Then A is said to be a NG in X if it satisfies the following conditions:

(i) $T_A(xy) \geq T_A(x) \wedge T_A(y)$, $I_A(xy) \geq I_A(x) \wedge I_A(y)$ *and* $F_A(xy) \leq F_A(x) \vee F_A(y)$,

(ii) $T_A(x^{-1}) \geq T_A(x)$, $I_A(x^{-1}) \geq I_A(x)$, *and* $F_A(x^{-1}) \leq F_A(x)$.

Definition 1.7.2 *[67]*

Let X be a group and let $\mathbb{G}$ be NG in X and e be the identity of X. We define the NS $\mathbb{G}_e$ by

$$\mathbb{G}_e = \left\{x \in X : T_{\mathbb{G}}(x) = T_{\mathbb{G}}(e), I_{\mathbb{G}}(x) = I_{\mathbb{G}}(e), F_{\mathbb{G}}(x) = F_{\mathbb{G}}(e)\right\}.$$

We note for a NG $\mathbb{G}$ in a group X, for every $x \in X : T_{\mathbb{G}}(x^{-1}) = T_{\mathbb{G}}(x)$, $I_{\mathbb{G}}(x^{-1}) = I_{\mathbb{G}}(x)$ and $F_{\mathbb{G}}(x^{-1}) = F_{\mathbb{G}}(x)$. Also for the identity e of the group $X : T_{\mathbb{G}}(e) \geq T_{\mathbb{G}}(x), I_{\mathbb{G}}(e) \geq I_{\mathbb{G}}(x)$, and $F_{\mathbb{G}}(e) \leq F_{\mathbb{G}}(x)$.

Proposition 1.7.1 *[67]*

Let $\mathbb{G}$ be a NG in a group X then for all $x, y \in X$,

(i) $T_{\mathbb{G}}(xy^{-1}) = T_{\mathbb{G}}(e) \Rightarrow T_{\mathbb{G}}(x) = T_{\mathbb{G}}(y)$

(ii) $I_{\mathbb{G}}(xy^{-1}) = I_{\mathbb{G}}(e) \Rightarrow I_{\mathbb{G}}(x) = I_{\mathbb{G}}(y)$

(iii) $F_{\mathbb{G}}(xy^{-1}) = F_{\mathbb{G}}(e) \Rightarrow F_{\mathbb{G}}(x) = F_{\mathbb{G}}(y)$.

Proposition 1.7.2 *[67]*

Let X be a group. Then the following statements are equivalent:

(i) $\mathbb{G}$ *is a NG in* X.

(ii) For all $x, y \in X, T_{\mathbb{G}}(xy^{-1}) \geq T_{\mathbb{G}}(x) \wedge T_{\mathbb{G}}(y)$, $I_{\mathbb{G}}(xy^{-1}) \geq I_{\mathbb{G}}(x) \wedge I_{\mathbb{G}}(y)$, $F_{\mathbb{G}}(xy^{-1}) \leq F_{\mathbb{G}}(x) \vee F_{\mathbb{G}}(y)$.

Definition 1.7.3 *[67]*
Let $f : X \to Y$ be a group homomorphism and let A be a NG in a group X. Then A is said to be neutrosophic-invariant if for any $x, y \in X$, $T_A(x) = T_A(y), I_A(x) = I_A(y)$ and $F_A(x) = F_A(y)$. It is clear that if A is neutrosophic-invariant then $f(A) \in$ neutrosophic group (Y). For each $A \in$ neutrosophic group (X), let $X_A = \{x \in X : T_A(x) = T_A(e), I_A(x) = I_A(e), F_A(x) = F_A(e)\}$. Then it is clear that X_A is a subgroup of X. For each $a \in X$, consider $r_a : X \to X$ and $l_a : X \to X$ be the right and left translations of X into itself, defined by $r_a(x) = xa$ and $l_a(x) = ax$, respectively for each $x \in X$.

1.8 Neutrosophic Topology

The theory of neutrosophic topological spaces was introduced and investigated by A. A. Salama and S. A. Alblowi [64] in 2012. Since then, a number of mathematicians have published numerous papers in this field. In the neutrosophic setting, numerous results in ordinary topological spaces have been posed, and various departures have been found. Researchers have investigated the topology of neutrosophic sets in depth, such as Smarandache [27] and Lupiáñez [86–89].

Definition 1.8.1 *[86]*
Let X be a non empty set and NT is a family $\mathfrak{T}$ of neutrosophic subsets of X satisfying the following axioms:

(i) $0_X, 1_X \in \mathfrak{T}$

(ii) $G_1 \cap G_2 \in \mathfrak{T}$ *for any* $G_1, G_2 \in \mathfrak{T}$

(iii) $\bigcup G_i \in \mathfrak{T}, \ \forall \{G_i : i \in J\} \subseteq \mathfrak{T}$

In this case the pair $(X, \mathfrak{T})$ is called a NTS and any NS in $\mathfrak{T}$ is known as NOS. The element of $\mathfrak{T}$ are known as NOS, a neutrosophic set F is a NCoS if and only if F^c is a NOS.

Definition 1.8.2 *[64]*
Let $(X, \mathfrak{T})$ be a NTS and A be a NS in X. Then the induced neutrosophic topology on A is the collection of NSs in A which are the intersection of NOSs in X with A. Then the pair $(A, \mathfrak{T}_A)$ is called a neutrosophic subspace of $(X, \mathfrak{T})$. The induced neutrosophic topology is denoted by $\mathfrak{T}_A$.

Definition 1.8.3 *[64]*
Let $(X, \mathfrak{T})$ be NTS and A be a NS in X. Then the neutrosophic closure of A is defined by

$$\mathcal{N} \sim Cl(A) = \bigcap \big\{K : K \text{ is a NCoS in } X \text{and } A \subseteq K\big\}.$$

Definition 1.8.4 *[64]*
Let $(X, \mathfrak{T})$ be NTS and A be a NS in X. Then the neutrosophic interior of A is defined by

$$\mathcal{N} \sim Int(A) = \bigcup \big\{G : G \text{ is a NOS in } X \text{and } G \subseteq A\big\}.$$

It can be also shown that $\mathcal{N} \sim Cl(A)$ is NCoS, $\mathcal{N} \sim Int(A)$ is a NOS in X.

(a) *A is NOS iff $\mathcal{N} \sim Int(A) = A$.*

(b) *A is NCoS iff $\mathcal{N} \sim Cl(A) = A$.*

Remark 1.8.1 *[64]*
For any NS A in $(X, \mathfrak{T})$ we have

(i) $\mathcal{N} \sim Cl(A^c) = \big(\mathcal{N} \sim Int(A)\big)^c$.

(ii) $\mathcal{N} \sim Int(A^c) = \big(\mathcal{N} \sim Cl(A)\big)^c$.

Remark 1.8.2 *[64]*
Let $(X, \mathfrak{T})$ *be a NTS and* A, B *be two NSs in* X*, we have the following properties are holds:*

(a) $\mathcal{N} \sim Int(A) \leq A$

(b) $A \leq \mathcal{N} \sim Cl(A)$

(c) $A \leq B \Rightarrow \mathcal{N} \sim Int(A) \leq \mathcal{N} \sim Int(B)$

(d) $A \leq B \Rightarrow \mathcal{N} \sim Cl(A) \leq \mathcal{N} \sim Cl(B)$

(e) $\mathcal{N} \sim Int\big(\mathcal{N} \sim Int(A)\big) = \mathcal{N} \sim Int(A)$

(f) $\mathcal{N} \sim Cl\big(\mathcal{N} \sim Cl(A)\big) = \mathcal{N} \sim Cl(A)$

(g) $\mathcal{N} \sim Int(A \wedge B) = \mathcal{N} \sim Int(A) \wedge \mathcal{N} \sim Int(B)$

(h) $\mathcal{N} \sim Cl(A \vee B) = \mathcal{N} \sim Cl(A) \vee \mathcal{N} \sim Cl(B)$

(i) $\mathcal{N} \sim Int(0_N) = 0_N$

(j) $\mathcal{N} \sim Int(1_N) = 1_N$

(k) $\mathcal{N} \sim Cl(0_N) = 0_N$

(l) $\mathcal{N} \sim Cl(1_N) = 1_N$

(m) $A \leq B \Rightarrow B^c \leq A^c$

(n) $\mathcal{N} \sim Cl(A \wedge B) \leq \mathcal{N} \sim Cl(A) \wedge \mathcal{N} \sim Cl(B)$

(o) $\mathcal{N} \sim Int(A \vee B) \geq \mathcal{N} \sim Int(A) \vee \mathcal{N} \sim Int(B)$.

1.9 Neutrosophic Bitopological Space

Definition 1.9.1 *[84]*
Let $(X, \mathfrak{T}_1)$ *and* $(X, \mathfrak{T}_2)$ *be the two NTs on X. Then* $(X, \mathfrak{T}_1, \mathfrak{T}_2)$ *is called a neutrosophic bitopological space (NBTS).*

Definition 1.9.2 *[84]*
Let $(X, \mathfrak{T}_1, \mathfrak{T}_2)$ *be a NBTS. A NS* $A = \big\{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\big\}$ *over X is said to a pair wise NOS in* $(X, \mathfrak{T}_1, \mathfrak{T}_2)$ *if* $\exists$ *a NS* $A_1 = \big\{\langle x, T_{A_1}(x), I_{A_1}(x), F_{A_1}(x)\rangle : x \in X\big\}$ *in* $\mathfrak{T}_1$ *and a NS* $A_2 = \big\{\langle x, T_{A_2}(x), I_{A_2}(x), F_{A_2}(x)\rangle : x \in X\big\}$ *in* $\mathfrak{T}_2$ *such that* $A = A_1 \cup A_2 = \big\{\langle x, min\big(T_{A_1}(x), T_{A_2}(x)\big), min\big(I_{A_1}(x), I_{A_2}(x)\big), max\big(F_{A_1}(x), F_{A_2}(x)\big)\rangle : x \in X\big\}$.

Definition 1.9.3 *[66]*
Let $(X, {\mathfrak{T}_i}^X, {\mathfrak{T}_j}^X)$ *be a NBTSs. Then for a set* $A = \big\{\langle x, \alpha_{ij}, \beta_{ij}, \gamma_{ij}\rangle : x \in X\big\}$*, neutrosophic* $({\mathfrak{T}_i}^X, {\mathfrak{T}_j}^X)N-$ *interior of* A *is the union of all* $({\mathfrak{T}_i}^X, {\mathfrak{T}_j}^X)N-$ *open sets of* X *contained in* A *and can be defined as follows:*

$$({\mathfrak{T}_i}^X, {\mathfrak{T}_j}^X)N \sim Int(A) =$$
$$\Big\{\langle x, \vee_{{\mathfrak{T}_i}^X} \vee_{{\mathfrak{T}_j}^X}\, \alpha_{ij}, \vee_{{\mathfrak{T}_i}^X} \vee_{{\mathfrak{T}_j}^X}\, \beta_{ij}, \wedge_{{\mathfrak{T}_i}^X}, \wedge_{{\mathfrak{T}_j}^X} \gamma_{ij}\rangle : x \in X\Big\}.$$

Definition 1.9.4 *[66]*
Let $(X, {\mathfrak{T}_i}^X, {\mathfrak{T}_j}^X)$ *be a NBTSs. Then for a set* $A = \big\{\langle x, \alpha_{ij}, \beta_{ij}, \gamma_{ij}\rangle : x \in X\big\}$*, neutrosophic* $({\mathfrak{T}_i}^X, {\mathfrak{T}_j}^X)N-$ *closure of* A *is the intersection of all* $({\mathfrak{T}_i}^X, {\mathfrak{T}_j}^X)N-$ *closed sets of* X *contained in* A *and can be defined as follows:*

$$({\mathfrak{T}_i}^X, {\mathfrak{T}_j}^X)N \sim Cl(A) =$$
$$\Big\{\langle x, \wedge_{{\mathfrak{T}_i}^X} \wedge_{{\mathfrak{T}_j}^X}\, \alpha_{ij}, \wedge_{{\mathfrak{T}_i}^X} \wedge_{{\mathfrak{T}_j}^X}\, \beta_{ij}, \vee_{{\mathfrak{T}_i}^X}, \vee_{{\mathfrak{T}_j}^X} \gamma_{ij}\rangle : x \in X\Big\}.$$

Theorem 1.9.1 *[68]*
Let $(X, \mathfrak{T}_X)$ *and* $(Y, \mathfrak{T}_Y)$ *be two NTGs and* $f : (X, \mathfrak{T}_X) \to (Y, \mathfrak{T}_Y)$ *be a mapping , then* f *is neutrosophic continuous if and only if* f *is neutrosophic continuous at neutrosophic point* $x_{(\alpha,\beta,\gamma)}$*, for each* $x \in X$.

Definition 1.9.5 *[68]*
Let $f : (X, \mathfrak{T}_X) \to (Y, \mathfrak{T}_Y)$ *is neutrosophic continuous if the pre-image of each NOS in* Y *is NOS in* X.

1.10 Neutrosophic Topological Group

Definition 1.10.1 *[68]*
Let X *be a group and* $\mathbb{G}$ *be a NG on* X*. Let* $\mathfrak{T}^{\mathbb{G}}$ *be a NT on* $\mathbb{G}$ *then* $(\mathbb{G}, \mathfrak{T}^{\mathbb{G}})$ *is said to be NTG if the following conditions are satisfied:*

(i) The mapping $\psi : (\mathbb{G}, \mathfrak{T}^{\mathbb{G}}) \times (\mathbb{G}, \mathfrak{T}^{\mathbb{G}}) \to (\mathbb{G}, \mathfrak{T}^{\mathbb{G}})$ *defined by* $\psi(x, y) = xy$*, for all* $x, y \in X$*, is relatively neutrosophic continuous.*

(ii) The mapping $\mu : (\mathbb{G}, \mathfrak{T}^{\mathbb{G}}) \to (\mathbb{G}, \mathfrak{T}^{\mathbb{G}})$ *defined by* $\mu(x) = x^{-1}$*, for all* $x \in X$*, is relatively neutrosophic continuous.*