Chapter 8

STUDY ON THE BEHAVIOURS OF COSINE IMPRECISE FUNCTIONS

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8.1. Introduction

A function of controllable form is known as imprecise function. Function, which is not controllable form, can also be transformed into controllable form with the assistance of multiplication of some suitable function. In this chapter, the function is multiplied by cosine function to get the controllable form. This resultant function is known as the cosine imprecise function. So, the cosine function is called the multiplication factor. Cosine imprecise function does not pass through the given origin. But sine imprecise function always passes through the given origin. Thus to represent the behavior of an object of the form of a function which is not passing through the origin of the axis is transformed into cosine imprecise function with the assistance of cosine function multiplication.

Here the algebraic polynomial function is multiplied by a cosine function of angle $lx; \forall\, l \in Z$. Resultant function is a cosine imprecise function. This phenomenon leads to divert the given algebraic polynomial function from a certain point to oscillation form. These points are left and right nearest value of the root of a given algebraic polynomial function. Right nearest value is the average value between the root of the given algebraic polynomial and nearest greater value for which the given multiplication factor, cosine function becomes zero. Left nearest value is the average value between the root of the given polynomial and a nearest smaller value for which the given multiplication factor, cosine function becomes zero. Later in the discussion, right and left the nearest value of the x-axis will be called conversion towards positive and negative x-axis respectively. Always our problem is not expressible in a form of algebraic function. In the most time, we obtained it in the form of imprecise function. For example, a receiver is getting a response of sound through the frequency of wave comes from the main place called server. This frequency of wave has an imprecise function character. When this activity is imprecise function form, then the given system will be controllable. If it is not an imprecise function, the system needs to modify. The phenomenon is called a controllable or non-controllable form of function. When the situation is uncontrollable, then it suggests to minimize the force produces from the device so that it can be transformed into a controllable form. This experiment is done available in the many fields of science and technology.

To identify an imprecise function of a particular place, the algebraic polynomial function is converting into an imprecise function with the assistance of the finite number of points. Then, with help of those points, we can identify a controllable function known as imprecise function. Coefficients of the variable are defined according to the rule of the elementary transformation of the matrix. This area of the imprecise function gives us information about the variation effect of impreciseness of any object occurred under various intervals. Where impreciseness is a membership value of indicator function obtained by the effect of the activity of any object for the definition of imprecise number.

8.2. Conversion of polynomial function into Cosine imprecise function

According to the information of graph, we found that most of the algebraic polynomials are freely floating without repeatedly coming back to the ground level. So it is called an uncontrolled function. For the example, first degree and second-degree algebraic polynomials are the functions which are meeting the ground level or real line at most one and twice.

These types of functions can also be transformable into oscillation form with the assistance of multiplication of the cosine function. Thus the resultant function is the cosine imprecise function. This function is possible to meet the level of axis repeatedly from a certain point. This point is known as conversion point. This phenomenon is occurring due to the multiplication of the other function. This needful multipliable function is known as a multiplication factor.

To discuss mathematically, let us consider

$p_1(x_1, y_1),\ p_2(x_2, y_2),\ p_3(x_3, y_3) \dots \dots \dots .. p_n(x_{n+1}, y_{n+1})$, be the (n+1)th collection of points which are above and below the given algebraic polynomial of degree n,

$y = c_0 + c_1x + c_2x^2 + c_3x^3 \dots \dots \dots \dots \dots \dots \dots ... + c_nx^n$ ……………………………(8.1)

Graphically this function is not a controllable function. To convert it into controllable form, we multiply this polynomial by a cosine function.

Thus, $y^* = (c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots \dots \dots .. + c_nx^n)\cos(lx);\ l\epsilon Z$ ………..…..(8.2)

is a cosine imprecise function.

To obtain standard cosine imprecise function passing very close to those points, this problem is expressed in the linear simultaneous equations with matrix form. Then we follow the rules of transpose of the matrices and the multiplication of matrices as follows.

By the law of roots given points $p_1, p_2, \dots \dots \dots \dots \dots \dots . p_{n+1}$ are satisfying the equation (8.2). Thus, we get (n+1)th set of simultaneous linear equations having arbitrary coefficients $c_0, c_1, c_2, \dots \dots \dots \dots \dots \dots \dots \dots ., c_n$ as follows.

$$y_1 = (c_0 + c_1x_1 + c_2{x_1}^2 + c_3{x_1}^3 \dots \dots \dots \dots \dots \dots \dots \dots \dots + c_n{x_1}^n)\cos(lx_1)$$

$$y_2 = (c_0 + c_1x_2 + c_2{x_2}^2 + c_3{x_2}^3 \dots \dots \dots \dots \dots \dots \dots \dots \dots + c_n{x_2}^n)\cos(lx_2)$$

$$y_3 = (c_0 + c_1x_3 + c_2{x_3}^2 + c_3{x_3}^3 \dots \dots \dots \dots \dots \dots \dots \dots \dots + c_3{x_3}^n)\cos(lx_3)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$y_{n+1} = (c_0 + c_1x_{n+1} + c_2{x_{n+1}}^2 + c_3{x_{n+!}}^3 \dots \dots \dots \dots \dots \dots \dots \dots \dots + c_n{x_{n+1}}^n)\cos(lx_{n+1})$$

The matrix form of equations is

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} \cos(lx_1) & x_1\cos(lx_1) & {x_1}^2\cos(lx_1) & \dots \dots . \dots & {x_1}^n\cos(lx_1) \\ \cos(lx_2) & x_2\cos(lx_2) & {x_2}^2\cos(lx_2) \dots \dots & \dots & {x_2}^n\cos(lx_2) \\ \cos(lx_3) & x_3\cos(lx_3) & {x_3}^2\cos(lx_3) & \dots \dots \dots \quad \dots & {x_3}^n\cos(lx_3) \\ & & \vdots & & \\ \cos(lx_{n+1}) & x_{n+1}\cos(lx_{n+1}) & {x_{n+1}}^2\cos(lx_{n+1}) & \dots \dots . \quad \dots & {x_{n+1}}^n\cos(lx_{n+1}) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$$

$$\cong AX = B \ \ (say) \dots \dots \dots \dots \dots \dots . . \dots \dots \dots \dots \dots \dots \dots \dots \dots . (8.3)$$

Where, $$A = \begin{bmatrix} \cos(lx_1) & x_1\cos(lx_1) & {x_1}^2\cos(lx_1) & \dots \dots . \dots & {x_1}^n\cos(lx_1) \\ \cos(lx_2) & x_2\cos(lx_2) & {x_2}^2\cos(lx_2) \dots \dots & \dots & {x_2}^n\cos(lx_2) \\ \cos(lx_3) & x_3\cos(lx_3) & {x_3}^2\cos(lx_3) & \dots \dots \dots \quad \dots & {x_3}^n\cos(lx_3) \\ & & \vdots & & \\ \cos(lx_{n+1}) & x_{n+1}\cos(lx_{n+1}) & {x_{n+1}}^2\cos(lx_{n+1}) & \dots \dots . \quad \dots & {x_{n+1}}^n\cos(lx_{n+1}) \end{bmatrix}$$

$$X = \begin{bmatrix} c_0 \\ c_1 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} \text{ and } B = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n+1} \end{bmatrix}$$

……………………………………….(8.4)

To obtain the solution of arbitrary constants, equation (8.3) is multiplied by a transpose of matrix A^T. Thus,

$$A^T(AX) = A^TB \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (8.5)$$

Where, $$A^TA = (a_{ij})_{(n+1)\times(n+1)}$$

$$A^TB = (b_i)_{(n+1)\times 1}$$

Such that,

$$a_{ij} = \sum_{k=1}^{m} [x_k^{i+j-2} \{\cos(lx_k)\}^2]$$

$$b_i = \sum_{k=1}^{m} [y_k x_k^i \cos(lx_k)]; 1 \le i, j \le (n+1)$$

Here, 'n' is the order of algebraic polynomial and 'm' is the number of the collection of points such that x_i and y_i are ordinates and abscissa of the given points.

Solution of arbitrary constants $c_0, c_1, c_2, \dots\dots\dots\dots\dots\dots\dots\dots, c_n$ can be obtained with the assistance of Gauss-elimination method. The rule of row elementary transformation occurred in this method is obtained in the following general form.

$R_i' \to R_i - \frac{R_j}{a_{(i-1)(i-1)}} \times a_{ij};\ (2 \le i \le n+1), (1 \le j \le n+1)$……………………(8.6)

Steps of transformation will be done as follows.

$$(2 \le i \le n+1\ when\ j = 1)$$

$$(3 \le i \le n+1\ when\ j = 2)$$

$$(j+1 \le i \le n+1\ \ when\ j = n)\ ; n \in N$$

Above operations help us to get upper triangular matrix. By backward substitution the solution of the arbitrary constants can be obtained. So, any polynomial is possible to transform into controllable form with the multiplication of cosine function. The resultant function is a cosine imprecise function.

8.2.1. Cosine imprecise function for polynomial of degree one

Here the polynomial of degree one is transformed into imprecise function with multiplied by a cosine function. This new function is called a cosine imprecise function for polynomial of degree one. And cosine function is called the multiplication factor of angle multiple one.

Example-1 (Cosine *imprecise function of angle x*)

Let, $y = c_0 + c_1 x$ …………………………………………………………..…..(8.7)

be a degree one algebraic polynomial.

In particular,

$y^* = (c_0 + c_1 x)\cos(x)\ ; for\ l = 1$……………………………………..(8.8)

is a cosine imprecise function for polynomial of degree one.

To obtain the value of coefficients of the equation (8.8), let us consider (1,0.5), (2,-2.5),(3,2),(4,-4),(5,3.5),(6,-6),(7,7) be the points which are situated above and below the given imprecise function. So,

$$x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5, x_6 = 6, x_7 = 7$$

$$y_1 = 0.5, y_2 = -2.5, y_3 = 2, y_4 = -4, y_5 = 3.5, y_6 = -6, y_7 = 7$$

According to the laws of roots, these points always satisfy the equation (8.8) to have linear simultaneous equation with matrix form.

Form the equations (8.3), (8.4) and (8.5), we have

$$AX = B$$

$$=> (A^T A)X = A^T B$$

Where $A^T A = \left(a_{ij}\right)_{(n+1)\times(n+1)}$

$$A^T B = (b_i)_{(n+1)\times 1}$$

$$X = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

Such that,

$$a_{ij} = \sum_{k=1}^{m} x_k^{i+j-2}\{\cos(x_k)\}^2$$

$$b_i = \sum_{k=1}^{m} y_k x_k^i \cos(x_k)\,;\, 1 \le i,j \le (n+1)$$

Here, m=number of points=7 and n= degree of the polynomial=1

$$A^T A = \left(a_{ij}\right)_{2\times 2} \text{ and } A^T B = (b_i)_{2\times 1}$$

Where,

$$a_{ij} = \sum_{k=1}^{7} x_k^{i+j-2}\{\cos(x_k)\}^2$$

$$b_i = \sum_{k=1}^{7} y_k x_k^i \cos(x_k)\,;\, 1 \le i,j \le 2$$

$$a_{11} = \sum_{k=1}^{7} {x_k}^0 \{\cos(x_k)\}^2$$

$$= cos^2(x_1) + cos^2(x_2) + cos^2(x_3) + cos^2(x_4) + cos^2(x_5) + cos^2(x_6) + cos^2(x_7)$$

$$= 6.9574$$

$$a_{12} = a_{21} = \sum_{k=1}^{7} x_k \{\cos(x_k)\}^2$$

$$= x_1 cos^2(x_1) + x_2 cos^2(x_2) + x_3 cos^2(x_3) + x_4 cos^2(x_4) + x_5 cos^2(x_5) + x_6 cos^2(x_6) + x_7 cos^2(x_7)$$

$$= 27.76207$$

$$a_{22} = \sum_{k=1}^{7} {x_k}^2 \{\cos(x_k)\}^2$$

$$= {x_1}^2 cos^2(x_1) + {x_2}^2 cos^2(x_2) + {x_3}^2 cos^2(x_3) + {x_4}^2 cos^2(x_4) + {x_5}^2 cos^2(x_5) + {x_6}^2 cos^2(x_6) + {x_7}^2 cos^2(x_7)$$

$$= 138.58,$$

$$b_1 = \sum_{k=1}^{7} y_k x_k \cos(x_k)$$

$$= y_1 x_1 \cos(x_1) + y_2 x_2 \cos(x_2) + y_3 x_3 \cos(x_3) + y_4 x_4 \cos(x_4) + y_5 x_5 \cos(x_5) + y_6 x_6 \cos(x_6) + y_7 x_7 \cos(x_7)$$

$$= 15.79910$$

$$b_2 = \sum_{k=1}^{7} y_k {x_k}^2 \cos(x_k)$$

$$= y_1 {x_1}^2 \cos(x_1) + y_2 {x_2}^2 \cos(x_2) + y_3 {x_3}^2 \cos(x_3) + y_4 {x_4}^2 \cos(x_4) + y_5 {x_5}^2 \cos(x_5) + y_6 {x_6}^2 \cos(x_6) + y_7 {x_7}^2 \cos(x_7)$$

$$= 157.430$$

Thus $\begin{bmatrix} 6.95 & 27.76 \\ 27.76 & 138.58 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 15.79 \\ 157.43 \end{bmatrix}$

From the formula (8.6), we have

$$R_i' \to R_i - \frac{R_j}{a_{(i-1)(i-1)}} \times a_{ij};\ (2 \le i \le 2), (1 \le j \le 1)$$

$$\begin{bmatrix} 6.95 & 27.76 \\ 0 & 27.69 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 15.59 \\ 95.43 \end{bmatrix}$$

$$=> 6.95c_0 + 27.76c_1 = 15.59, 27.69c_1 = 95.43$$

So, $c_1 = 3.44,\ c_0 = -11.49$

So, $y^* = (-11.49 + 3.44x)\cos(\mathrm{x})$ … … … … … … … … … … . . … (8.9)

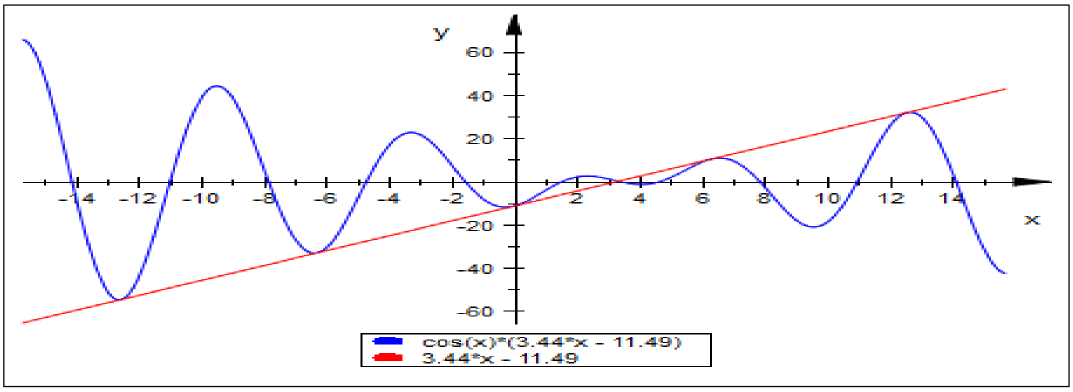


Fig.8.1. Graph of $y = (-11.49 + 3.44x)$ *and* $y^* = (-11.49 + 3.44x)cos(x)$

Here, cosine imprecise function (8.9) meet the x-axis only at the condition $y = 0$.

For this reason we have, $x = \frac{11.49}{3.44} = 3.340$ such that $\frac{\pi}{2} < 3.340 < \frac{3\pi}{2}$ and $x = \frac{\pi}{2}$ *and* $\frac{3\pi}{2}$ are the nearest two values of 3.340 for which $\cos(x) = 0$

So the graph start to oscillate from a point $x = \frac{3.340\times2+3\pi}{4} \simeq 4.067(approx.)$ towards the positive x-axis and $x = \frac{3.340\times2+\pi}{4} \simeq 2{,}456(approx)$ towards the negative x-axis which is shown in the Fig.8.1.

So, $\left(\frac{3.340\times2+3\pi}{4}, \left(-11.49 + 3.44\left(\frac{3.340\times2+3\pi}{4}\right)\right)\cos\left(\frac{3.340\times2+3\pi}{4}\right)\right)$ is called conversion point along the positive x-axis.

Thus the given function $y^* = (-11.49 + 3.44x)\cos(x)$ cuts the real line/X-axis repeatedly to meet at the ground level again and again. So, it is a cosine imprecise function.

Example-2 (Cosine *imprecise function multiple of angle 2x*)

Let $y = c_0 + c_1x$ be a polynomial of degree one.

In particular, For l = 2, $y = (c_0 + c_1x)\cos(2x)$…………………....….……..(8.10)

is an imprecise polynomial of degree one.

To define the imprecise function, example of (8.8) is used for the equation (8.10)

$$\text{So, } x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5, x_6 = 6, x_7 = 7$$

$$y_1 = 0.5, y_2 = -2.5, y_3 = 2, \quad y_4 = -4, y_5 = 3.5, y_6 = -6, y_7 = 7$$

By the law of roots this imprecise function satisfies the given collection of points to have linear simultaneous equation of the matrix as follows.

Form the equations (8.3), (8.4), (8.5), we have

$$AX = B$$

$$\Rightarrow (A^T A)X = A^T B$$

Where, $A^T A = \left(a_{ij}\right)_{(n+1)\times(n+1)}$

$$A^T B = (b_i)_{(n+1)\times 1}$$

$$X = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

Such that,

$$a_{ij} = \sum_{k=1}^{m} x_k^{i+j-2}\{\cos(2x_k)\}^2$$

$$b_i = \sum_{k=1}^{m} y_k x_k^i \cos(2x_k)\,;\, 1 \le i,j \le (n+1)$$

Here, m=number of points=7 and n= degree of the polynomial=1

$A^T A = \left(a_{ij}\right)_{2\times 2}$ and $A^T B = (b_i)_{2\times 1}$

Where,

$$a_{ij} = \sum_{k=1}^{7} x_k^{i+j-2}\{\cos(2x_k)\}^2$$

$$b_i = \sum_{k=1}^{7} y_k x_k^i \cos(2x_k)\,;\, 1 \le i,j \le 2$$

Now,

$$a_{11} = \sum_{k=1}^{7} {x_k}^0\{\cos(2x_k)\}^2$$

$$= cos^2(2x_1) + cos^2(2x_2) + cos^2(2x_3) + cos^2(2x_4) + cos^2(2x_5) + cos^2(2x_6) + cos^2(2x_7) = 4.85559$$

$$a_{12} = a_{21} = \sum_{k=1}^{7} x_k\{\cos(2x_k)\}^2$$

$$= x_1 cos^2(2x_1) + x_2 cos^2(2x_2) + x_3 cos^2(2x_3) + x_4 cos^2(2x_4) + x_5 cos^2(2x_5) + x_6 cos^2(2x_6) + x_7 cos^2(2x_7) = 27.05897$$

$$a_{22} = \sum_{k=1}^{7} {x_k}^2\{\cos(2x_k)\}^2$$

$$= {x_1}^2 cos^2(2x_1) + {x_2}^2 cos^2(2x_2) + {x_3}^2 cos^2(2x_3) + {x_4}^2 cos^2(2x_4)$$

$$+{x_5}^2cos^2(2x_5) + {x_6}^2cos^2(2x_6) + {x_7}^2cos^2(2x_7) = 134.39326,$$

$$b_1 = \sum_{k=1}^{7} y_k x_k \cos(2x_k)$$

$$= y_1x_1\cos(2x_1) + y_2x_2\cos(2x_2) + y_3x_3\cos(2x_3) + y_4x_4\cos(2x_4)$$

$$+ y_5x_5\cos(2x_5) + y_6x_6\cos(2x_6) + y_7x_7\cos(2x_7) = 15.20003$$

$$b_2 = \sum_{k=1}^{7} y_k {x_k}^2\cos(2x_k) = y_1{x_1}^2\cos(2x_1) + y_2{x_2}^2\cos(2x_2) + y_3{x_3}^2\cos(2x_3)$$

$$+y_4{x_4}^2\cos(2x_4) + y_5{x_5}^2\cos(2x_5) + y_6{x_6}^2\cos(2x_6) + y_7{x_7}^2\cos(2x_7)$$

$$= 152.75052$$

Thus $\begin{bmatrix} 4.85 & 27.05 \\ 27.05 & 134.39 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 15.20 \\ 152.75 \end{bmatrix}$

From the formula (8.6), we have

$$R_i' \to R_i - \frac{R_j}{a_{(i-1)(i-1)}} \times a_{ij};\ (2 \le i \le 2), (1 \le j \le 1)$$

$$\begin{bmatrix} 4.85 & 27.05 \\ 0 & -16.47 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 15.20 \\ 67.97 \end{bmatrix}$$

$$=> 4.85c_0 + 27.05c_1 = 15.20, -16.47c_1 = 67.97$$

So, $c_1 = -\frac{67.97}{16.47} = -4.12$

$$c_0 = \frac{1}{4.85}(15.20 - 27.05c_1) = 27.69$$

So, $y^* = (27.69 - 4.12x)\cos(2x)$ … … … … … … … … … . … (8.11)

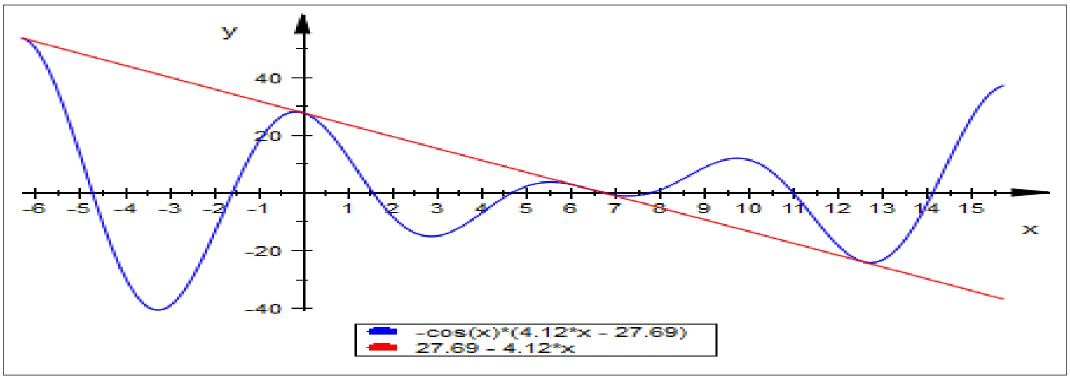


Fig.8.2. Graph of $y = (27.69 - 4.12x)$ *and* $y^* = (27.69 - 4.12x)\,cos(2x)$

Here, the given cosine imprecise function meet the x-axis only when $y = 0$.

For this condition we have, $x = \frac{27.69}{4.12} = 6.720$ such that $\frac{7\pi}{4} < 6.720 < \frac{9\pi}{4}$ and $x = \frac{7\pi}{4}$ and $\frac{9\pi}{4}$ are the nearest two values of 6.720 for which $\cos(2x) = 0$

So the graph start to oscillate from a point $x = \frac{6.720 \times 4 + 9\pi}{2\times 4} \simeq 6.8950(approx.)$ towards the positive x-axis and $x = \frac{6.720\times 4 + 7\pi}{2\times 4} \simeq 6.109(approx)$ towards the negative x-axis and is shown in Fig.8.2.

So, $\left(\frac{6.720 \times 4+9\pi}{2\times 4}, \left(27.69 + -4.12\left(\frac{6.720 \times 4+9\pi}{2\times 4}\right)\right)\cos\left(\frac{6.720 \times 4+9\pi}{4}\right)\right)$ is called conversion point of the positive x-axis.

Thus the given function $y^* = (27.69 - 4.12x)\cos(2x)$ cuts the real line or the X-axis repeatedly to meet level again and again. So, it is a cosine imprecise function.

In general for the cosine imprecise function, $y^* = (a + bx)\cos(lx)\,; l \in Z^+$ the conversion point is depending on the value of $x = -\frac{a}{b}$

8.2.2. Cosine imprecise functions for polynomial of degree two

Here the polynomial of degree two is transformed into imprecise function by the multiplication of cosine function of angle of $lx; l \in Z$. Resultant function will be known as cosine imprecise function for polynomial of degree two. And the cosine function is known as multiplication factor.

Example-1 (Cosine *imprecise function multiple of angle x*)

Let $y = c_0 + c_1x + c_2x^2$ ……………………………………………(8.12)

be a polynomial of degree two.

In particular

$y^* = (c_0 + c_1x + c_2x^2)\cos(x)\,; for\ l = 1$……..……………………..(8.13)

is an imprecise polynomial of degree two.

Let us consider (1,0.5), (2,-2.5),(3,2),(4,-4),(5,3.5),(6,-6),(7,7) be the data collection of points.

$$\text{So, } x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5, x_6 = 6, x_7 = 7$$

$$y_1 = 0.5, y_2 = -2.5, y_3 = 2, \quad y_4 = -4, y_5 = 3.5, y_6 = -6, y_7 = 7$$

Form the equations (8.3), (8.4) and (8.5), we have

$$AX = B$$

$$=> (A^TA)X = A^TB$$

$$\text{Where, } A^T A = \left(a_{ij}\right)_{(n+1)\times(n+1)}$$

$$A^T B = (b_i)_{(n+1)\times 1}$$

$$X = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

$$a_{ij} = \sum_{k=1}^{m} x_k^{i+j-2}\{\cos(x_k)\}^2$$

$$b_i = \sum_{k=1}^{m} y_k x_k^i \cos(x_k)\,;\, 1 \le i,j \le (n+1)$$

Here, m=number of points=7 and n= degree of the polynomial=2

Thus,

$$A^T A = \left(a_{ij}\right)_{3\times 3} \text{ and } A^T B = (b_i)_{3\times 1}$$

Where ,

$$a_{ij} = \sum_{k=1}^{7} x_k^{i+j-2}\{\cos(x_k)\}^2$$

$$b_i = \sum_{k=1}^{7} y_k x_k^i \cos(x_k)\,;\, 1 \le i,j \le 3$$

$$a_{11} = \sum_{k=1}^{7} {x_k}^0 \{\cos(x_k)\}^2$$

$$= cos^2(x_1) + cos^2(x_2) + cos^2(x_3) + cos^2(x_4) + cos^2(x_5) + cos^2(x_6) + cos^2(x_7) = 6.9574$$

$$a_{12} = a_{21} = \sum_{k=1}^{7} x_k \{\cos(x_k)\}^2$$

$$= x_1 cos^2(x_1) + x_2 cos^2(x_2) + x_3 cos^2(x_3) + x_4 cos^2(x_4) + x_5 cos^2(x_5) + x_6 cos^2(x_6) + x_7 cos^2(x_7) = 27.76207$$

$$a_{22} = a_{13} = a_{31} = \sum_{k=1}^{7} {x_k}^2\{\cos(x_k)\}^2$$

$$= {x_1}^2 cos^2(x_1) + {x_2}^2 cos^2(x_2) + {x_3}^2 cos^2(x_3) + {x_4}^2 cos^2(x_4) + {x_5}^2 cos^2(x_5) + {x_6}^2 cos^2(x_6) + {x_7}^2 cos^2(x_7) = 138.58$$

$$a_{23} = a_{32} = \sum_{k=1}^{7} x_k{}^3\{\cos(x_k)\}^2$$

$$= x_1{}^3 cos^2(x_1) + x_2{}^3 cos^2(x_2) + x_3{}^3 cos^2(x_3) + x_4{}^3 cos^2(x_4)$$
$$+ x_5{}^3 cos^2(x_5)$$
$$+x_6{}^3 cos^2(x_6) + x_7{}^3 cos^2(x_7) = 775.200717$$

$$a_{33} = \sum_{k=1}^{7} x_k{}^4\{\cos(x_k)\}^2$$

$$= x_1{}^4 cos^2(x_1) + x_2{}^4 cos^2(x_2) + x_3{}^4 cos^2(x_3) + x_4{}^4 cos^2(x_4)$$
$$+ x_5{}^4 cos^2(x_5) + x_6{}^4 cos^2(x_6) + x_7{}^4 cos^2(x_7) = 4619.94474$$

$$b_1 = \sum_{k=1}^{7} y_k x_k \cos(x_k)$$

$$= y_1 x_1 \cos(x_1) + y_2 x_2 \cos(x_2) + y_3 x_3 \cos(x_3) + y_4 x_4 \cos(x_4)$$
$$+ y_5 x_5 \cos(x_5)$$
$$+y_6 x_6 \cos(x_6) + y_7 x_7 \cos(x_7) = 15.79910$$

$$b_2 = \sum_{k=1}^{7} y_k x_k{}^2 \cos(x_k) = y_1 x_1{}^2 \cos(x_1) + y_2 x_2{}^2 \cos(x_2) + y_3 x_3{}^2 \cos(x_3)$$
$$+y_4 x_4{}^2 \cos(x_4) + y_5 x_5{}^2 \cos(x_5) + y_6 x_6{}^2 \cos(x_6) + y_7 x_7{}^2 \cos(x_7) = 157.430$$

$$b_3 = \sum_{k=1}^{7} y_k x_k{}^3 \cos(x_k) = y_1 x_1{}^3 \cos(x_1) + y_2 x_2{}^3 \cos(x_2) + y_3 x_3{}^3 \cos(x_3)$$
$$+y_4 x_4{}^3 \cos(x_4) + y_5 x_5{}^3 \cos(x_5) + y_6 x_6{}^3 \cos(x_6) + y_7 x_7{}^3 \cos(x_7) = 1309.09$$

Thus the simultaneous equation is converted into following matrix form.

$$\begin{bmatrix} 6.95 & 27.76 & 138.58 \\ 27.76 & 138.76 & 775.02 \\ 138.76 & 775.02 & 4619.94 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 15.97 \\ 157.43 \\ 1309.09 \end{bmatrix}$$

From the formula (6.6), we have

$$R_i' \to R_i - \frac{R_i}{a_{(i-1)(i-1)}} \times a_{ij}\, ; (2 \le i \le 3, 1 \le j \le 2)$$

If $j = 1$, then $2 \le i \le 3$ $\approx \begin{bmatrix} 6.95 & 27.76 & 138.58 \\ 0 & 27.69 & 221.67 \\ 0 & 221.67 & 1856.71 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 15.97 \\ 94.36 \\ 994.24 \end{bmatrix}$

If $j = 2$, then $3 \le i \le 3$ $\approx \begin{bmatrix} 6.95 & 27.76 & 138.58 \\ 0 & 27.69 & 221.67 \\ 0 & 0 & 82.12 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 15.97 \\ 94.36 \\ 238.84 \end{bmatrix}$

$$=> 6.95c_0 + 27.76c_1 + 138.58c_1 = 15.97$$

$$=> 27.69c_1 + 221.67c_2 = 94.36$$

$$82.12c_3 = 238.84$$

By Backward substitution, we get $c_0 = 9.11$, $c_1 = -16.19$, $c_2 = 2.90$

Thus $y^* = (9.11 - 16.19x + 2.901x^2)\cos(x)$ ……………….……………(8.14)

is a cosine imprecise function for polynomial of degree two.

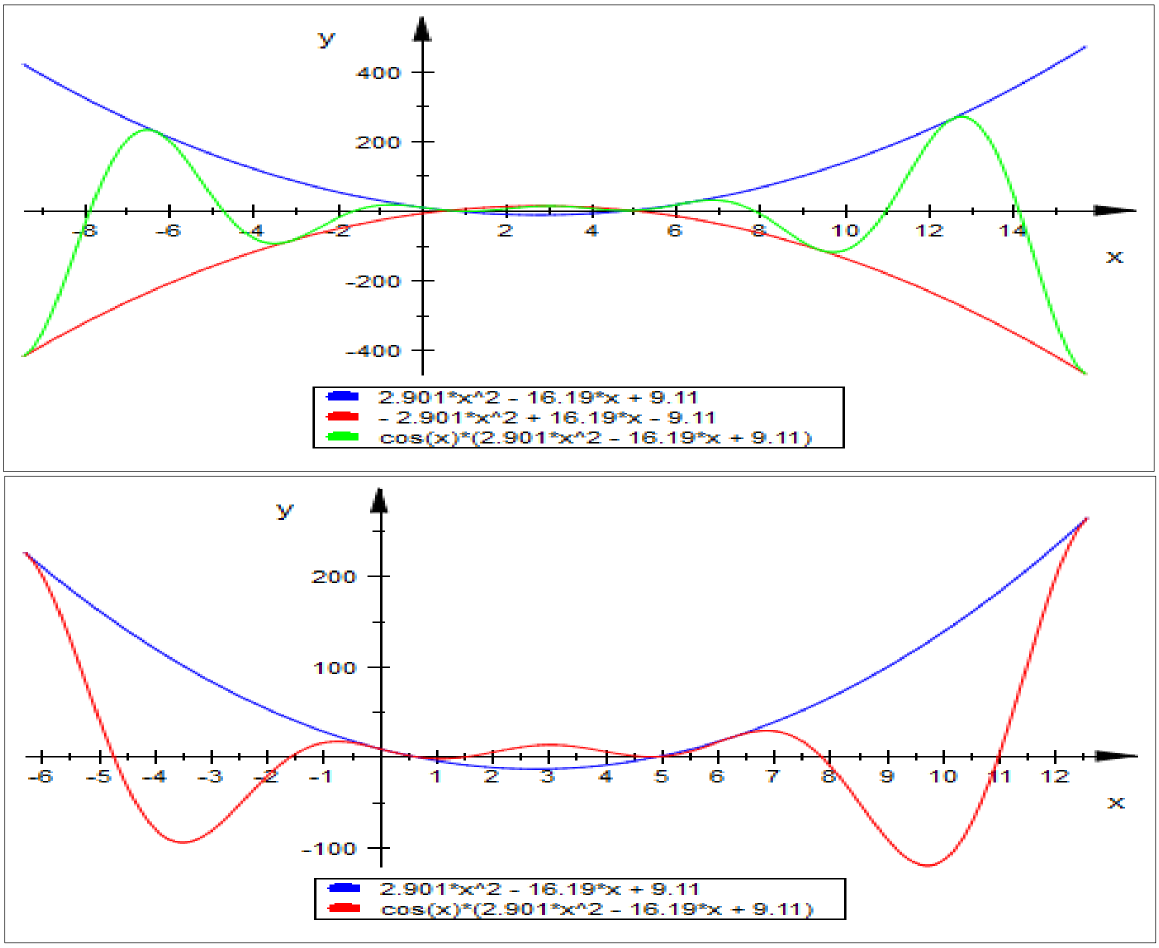


Fig.8.3. Graph of $y = \pm(9.11 - 16.19x + 2.901x^2)$ *and* $y^* = (9.11 - 16.19x + 2.901x^2)\cos(x)$

Here, the given cosine imprecise function meet the x-axis only when $y = 0$.

For this condition we take the solution of the given equation $9.11 - 16.19x + 2.901x^2 = 0$ as follows.

$$x = \frac{-(-16.19) \pm \sqrt{(-16.19)^2 + 4 \times 9.11 \times 2.901}}{2 \times 2.901}$$

$$= 4.94(approx.), 0.63(approx.)$$

We observed that $x = 4.94\ and\ 0.63$ such that $-\frac{\pi}{2} < 0.63\ and\ 4.94 < \frac{5\pi}{2}$. And $x = -\frac{\pi}{2}\ and\ \frac{5\pi}{2}$ are the nearest two values of $x = 4.94\ and\ 0.63$ for which $\cos(x) = 0$.

So, the graph start to oscillate from a point $x = \frac{4.94\times2+5\pi}{4} \simeq 6.40(approx)$ along the positive x-axis and $x = \frac{0.63\times2-\pi}{4} \simeq -0.468\ (approx)$ along the negative x-axis which is shown in the Fig.8.3.

So, $\left(\frac{4.94\times2+5\pi}{4}, (9.11 - 16.19\left(\frac{4.94\times2+5\pi}{4}\right) + 2.901\left(\frac{4.94\times2+5\pi}{4}\right)^2)\cos(\frac{4.94\times2+5\pi}{4})\right)$ is called conversion point towards the positive x-axis.

Thus the given function, $y^* = (9.11 - 16.19x + 2.901x^2)\cos(x)$ cuts the real line or X-axis repeatedly to meet the ground level again and again. So, it is a cosine imprecise function.

Example-2 (Cosine *imprecise function multiple of angle 2x*)

Let $y = c_0 + c_1x + c_1x^2$ be a degree two polynomial.

In particular, $y = (c_0 + c_1x + c_1x^2)\cos(2x)$; for l = 2 ………….……….(8.15)

is a cosine imprecise function for polynomial of degree two.

Let us consider, (1,0.5), (2,-2.5),(3,2),(4,-4),(5,3.5),(6,-6),(7,7) be the data collection points.

So, $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5, x_6 = 6, x_7 = 7$

$y_1 = 0.5, y_2 = -2.5, y_3 = 2,\ y_4 = -4, y_5 = 3.5, y_6 = -6, y_7 = 7$

Form the equations (8.3), (8.4), (8.5), we have

$$AX = B$$

$$=> (A^TA)X = A^TB$$

Where, $A^TA = \left(a_{ij}\right)_{(n+1)\times(n+1)}$

$$A^TB = (b_i)_{(n+1)\times1}$$

$X = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$. Now,

$$a_{ij} = \sum_{k=1}^{m} x_k^{i+j-2}\{\cos(2x_k)\}^2$$

$$b_i = \sum_{k=1}^{m} y_k x_k^i \cos(2x_k)\,;\, 1 \le i,j \le (n+1)$$

Here, m=number of points=7, n= degree of the polynomial=2

$$A^T A = \left(a_{ij}\right)_{3\times 3} \quad \text{and} \quad A^T B = \left(b_i\right)_{3\times 1}$$

Where,

$$a_{ij} = \sum_{k=1}^{7} x_k^{i+j-2}\{\cos(2x_k)\}^2$$

$$b_i = \sum_{k=1}^{7} y_k x_k^i \cos(2x_k)\,;\, 1 \le i,j \le 3$$

$$a_{11} = \sum_{k=1}^{7} {x_k}^0 \{\cos(2x_k)\}^2$$

$$= cos^2(2x_1) + cos^2(2x_2) + cos^2(2x_3) + cos^2(2x_4) + cos^2(2x_5) + cos^2(2x_6) + cos^2(2x_7)$$

$$= 4.85594$$

$$a_{12} = a_{21} = \sum_{k=1}^{7} x_k \{\cos(2x_k)\}^2$$

$$= x_1 cos^2(2x_1) + x_2 cos^2(2x_2) + x_3 cos^2(2x_3) + x_4 cos^2(2x_4) + x_5 cos^2(2x_5) + x_6 cos^2(2x_6) + x_7 cos^2(2x_7)$$

$$= 27.05897$$

$$a_{22} = a_{13} = a_{31} = \sum_{k=1}^{7} {x_k}^2 \{\cos(2x_k)\}^2$$

$$= {x_1}^2 cos^2(2x_1) + {x_2}^2 cos^2(2x_2) + {x_3}^2 cos^2(2x_3) + {x_4}^2 cos^2(2x_4) + {x_5}^2 cos^2(2x_5) + {x_6}^2 cos^2(2x_6) + {x_7}^2 cos^2(2x_7)$$

$$= 134.39326$$

$$a_{23} = a_{32} = \sum_{k=1}^{7} {x_k}^3 \{\cos(2x_k)\}^2$$

$$= {x_1}^3 cos^2(2x_1) + {x_2}^3 cos^2(2x_2) + {x_3}^3 cos^2(2x_3) + {x_4}^3 cos^2(2x_4) + {x_5}^3 cos^2(2x_5) + {x_6}^3 cos^2(2x_6) + {x_7}^3 cos^2(2x_7)$$

$$= 749.24443$$

$$a_{33} = \sum_{k=1}^{7} {x_k}^4 \{\cos(2x_k)\}^2$$

$$= {x_1}^4 cos^2(2x_1) + {x_2}^4 cos^2(2x_2) + {x_3}^4 cos^2(2x_3) + {x_4}^4 cos^2(2x_4) + {x_5}^4 cos^2(2x_5) + {x_6}^4 cos^2(2x_6) + {x_7}^4 cos^2(2x_7)$$

$$= 4454.68738$$

$$b_1 = \sum_{k=1}^{7} y_k x_k \cos(2x_k)$$

$$= y_1 x_1 \cos(2x_1) + y_2 x_2 \cos(2x_2) + y_3 x_3 \cos(2x_3) + y_4 x_4 \cos(2x_4) + y_5 x_5 \cos(2x_5) + y_6 x_6 \cos(2x_6) + y_7 x_7 \cos(2x_7)$$

$$= 15.2000$$

$$b_2 = \sum_{k=1}^{7} y_k {x_k}^2 \cos(2x_k)$$

$$= y_1 {x_1}^2 \cos(2x_1) + y_2 {x_2}^2 \cos(2x_2) + y_3 {x_3}^2 \cos(2x_3) + y_4 {x_4}^2 \cos(2x_4) + y_5 {x_5}^2 \cos(2x_5) + y_6 {x_6}^2 \cos(2x_6) + y_7 {x_7}^2 \cos(2x_7)$$

$$= 152.75052$$

$$b_3 = \sum_{k=1}^{7} y_k {x_k}^3 \cos(2x_k)$$

$$= y_1 {x_1}^3 \cos(2x_1) + y_2 {x_2}^3 \cos(2x_2) + y_3 {x_3}^3 \cos(2x_3) + y_4 {x_4}^3 \cos(2x_4) + y_5 {x_5}^3 \cos(2x_5) + y_6 {x_6}^3 \cos(2x_6) + y_7 {x_7}^3 \cos(2x_7)$$

$$= 1273.59811$$

Thus the simultaneous equation is expressible into the following matrix form.

$$\begin{bmatrix} 4.85 & 27.05 & 134.39 \\ 27.05 & 134.20 & 749.24 \\ 134.39 & 749.24 & 4454.68 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 15.20 \\ 152.75 \\ 1273.59 \end{bmatrix}$$

From the formula (8.6) ,we have

$$R_i' \to R_i - \frac{R_j}{a_{(i-1)(i-1)}} \times a_{ij} \, ; (2 \le i \le 3, 1 \le j \le 2)$$

If $j = 1$, then $2 \le i \le 3$ $\approx \begin{bmatrix} 4.85 & 27.05 & 134.39 \\ 0 & -16.47 & -0.29 \\ 0 & -0.29 & 730.83 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 15.20 \\ 67.97 \\ 852.40 \end{bmatrix}$

If $j = 2$, then $3 \le i \le 3$ $\approx \begin{bmatrix} 4.85 & 27.05 & 134.39 \\ 0 & -16.47 & -0.29 \\ 0 & 0 & 730.58 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 15.20 \\ 67.97 \\ 851.18 \end{bmatrix}$

$$=> 4.85c_0 + 27.05 + 134.39c_1 = 15.20$$

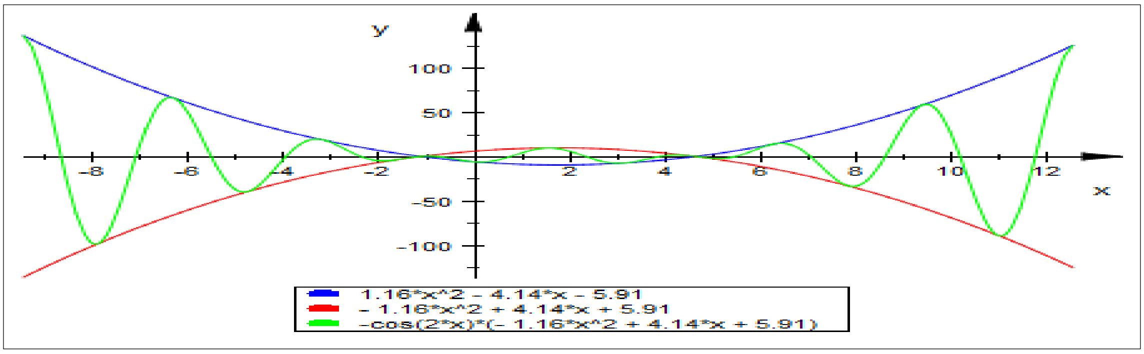
$$=> -16.81c_1 - 0.29c_2 = 67.97$$

$$730.58c_3 = 851.18$$

Thus, by backward substitution, we will get $c_0 = -5.91,\ c_1 = -4.14,\ c_2 = 1.16$

So, $y^* = (-5.91 - 4.14x + 1.16x^2)\cos(2x)$ …………………………......(8.16)

is a cosine imprecise function for polynomial of degree two.



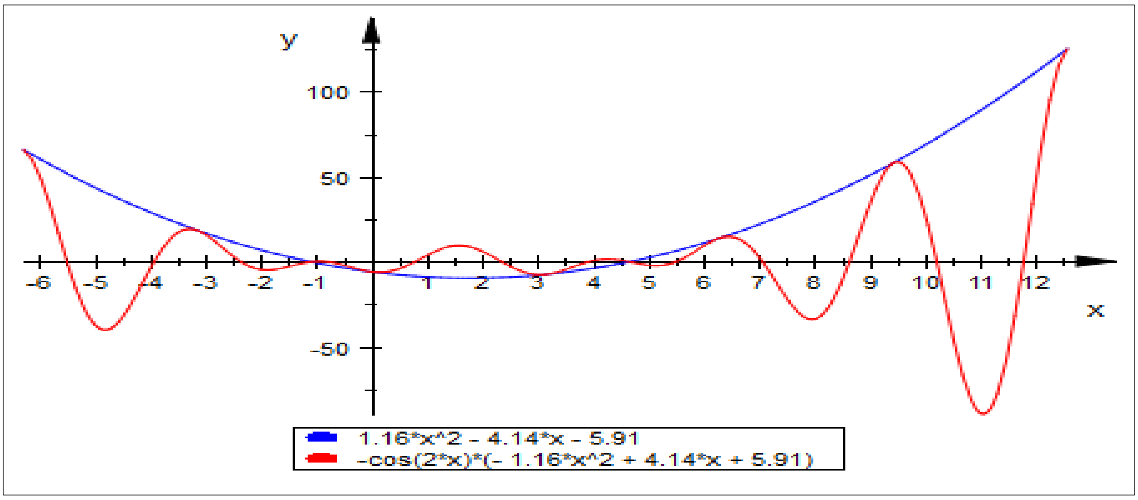


Fig.8.4. Graph of $y = \pm(-5.91 - 4.14x + 1.16x^2)$ *and* $y^* = (-5.91 - 4.14x + 1.16x^2)cos(2x)$

Here the given cosine imprecise function meet the x-axis only when $y = 0$. For this condition we take the solution of the equation $-5.91 - 4.14x + 1.16x^2 = 0$ as follow.

$$x = \frac{-(-4.147) \pm \sqrt{(-4.147)^2 + 4 \times 1.16 \times 5.918}}{2 \times 1.16}$$

$$= 4.66(approx.), -1.0918(approx.)$$

We observe that, $x = 4.66$ and -1.0918 such that $-\frac{3\pi}{4} < -1.0918$ and $4.66 < \frac{5\pi}{4}$. And $x = -\frac{3\pi}{4}$ *and* $\frac{5\pi}{4}$ are the nearest two values of 4.66 and -1.0918 for which $\cos(2x) = 0$.

Here, graph starts to oscillate from the value $x = \frac{4.66\times4+5\pi}{8} \simeq 4.29(approx)$ along the positive x-axis and $x = \frac{-1.0918\times4-3\pi}{8} \simeq -1.72\ (approx)$ along the negative x-axis which is shown in the Fig.8.4.

So, $\left(\frac{4.66\times4+5\pi}{8}, (-5.91 - 4.14\left(\frac{4.66\times4+5\pi}{8}\right) + 1.16\left(\frac{4.66\times4+5\pi}{8}\right)^2)\cos(\frac{4.66\times4+5\pi}{4})\right)$ is called conversion point along the positive x-axis.

Thus the given function $y^* = (-5.91 - 4.14x + 1.16x^2)\cos(2x)$ cuts the real line or X-axis repeatedly to meet the ground level again and again. So, it is a cosine imprecise function.

In general, cosine imprecise function $y = (c_0 + c_1x + c_1x^2)\cos(2x)\,; l \in Z$ has the conversion point which is depending on the value of $x = \frac{-b\pm\sqrt{(-b)^2-4ac}}{2a}$.

From the same point, cosine imprecise function will return back to the original polynomial function. So this point is also known as a diversion point of the imprecise function with respect to the multiplication factor cosine function.

Thus, from the above calculations, we have observed that if the angle of trigonometry function is increased, then conversion point become come closer to the origin of co-ordinate axis system. It means that considered experiment starts to oscillate within the short time period to have shorter wavelength and larger energy in the system.

For example, if the wings of a generator moves more angle then speed is increased and the energy output will be larger due to shorter wave length and the closer conversion point.

8.3. Area bounded by Cosine Imprecise Function for Polynomial of degree one

For a cosine imprecise function $y^* = (c_0 + c_1x)\cos(x), l = 1$, we have the imprecise numbers, $\left[\frac{\pi}{2}; \pi; \frac{3\pi}{2}\right]$, $\left[\frac{3\pi}{2}; 2\pi; \frac{5\pi}{2}\right]$, $\left[\frac{5\pi}{2}; 3\pi; \frac{7\pi}{2}\right]$, $\left[\frac{7\pi}{2}; 4\pi; \frac{9\pi}{2}\right]$……………………… etc. within the interval $\left[\frac{\pi}{2}, \frac{(2n+1)\pi}{2}\right]$, having indicator function as follows.

$$\rho_{N_C}(x) = \begin{cases} \rho_1(x); & \frac{(2n-1)\pi}{2} \le x \le n\pi \\ \rho_2(x); & n\pi \le x \le \frac{(2n+1)\pi}{2} \end{cases} \quad \text{……………………………….(8.17)}$$

such that $\rho_1\left(\frac{(2n-1)\pi}{2}\right) = \rho_2\left(\frac{(2n+1)\pi}{2}\right) = 0$ and $\rho_1(n\pi) = \rho_2(n\pi)$

Here, function has maximum level at the point $x = n\pi$ and the minimum at $x = \frac{(2n-1)\pi}{2}$ and $x = \frac{(2n+1)\pi}{2}$.

So, $\left[\frac{\pi}{2}; \pi, ; \frac{3\pi}{2}\right]$, $\left[\frac{3\pi}{2}; 2\pi; \frac{5\pi}{2}\right]$, $\left[\frac{5\pi}{2}; 3\pi; \frac{7\pi}{2}\right]$, $\left[\frac{7\pi}{2}; 4\pi; \frac{9\pi}{2}\right]$ ………….……….etc. are imprecise numbers.

Similarly, $y = (c_0 + c_1x)\cos(2x)$; for $l = 2$, we have the imprecise numbers, $\left[\frac{\pi}{4}; \frac{\pi}{2}; \frac{3\pi}{4}\right], \left[\frac{3\pi}{4}; \pi; \frac{5\pi}{4}\right], \left[\frac{5\pi}{4}; \frac{3\pi}{2}; \frac{7\pi}{4}\right], \left[\frac{7\pi}{4}; 2\pi; \frac{9\pi}{4}\right]$……………………………………….. etc. within the interval $\left[\frac{\pi}{4}, \frac{(2n+1)\pi}{4}\right]$ having indicator function as follows.

$$\rho_{N_C}(x) = \begin{cases} \rho_1(x); & \frac{(2n-1)\pi}{4} \le x \le \frac{n\pi}{2} \\ \rho_2(x); & \frac{n\pi}{2} \le x \le \frac{(2n+1)\pi}{4} \end{cases} \quad \text{.........................(8.18)}$$

such that $\rho_1\left(\frac{(2n-1)\pi}{4}\right) = \rho_2\left(\frac{(2n+1)\pi}{4}\right) = 0$ and $\rho_1\left(\frac{n\pi}{2}\right) = \rho_2\left(\frac{n\pi}{2}\right)$

Here, function has maximum level at the point, $x = \frac{n\pi}{4}$ and minimum value at $x = \frac{(2n-1)\pi}{4}$ and $x = \frac{(2n+1)\pi}{4}$.

So, $\left[\frac{\pi}{4}; \frac{\pi}{2}; \frac{3\pi}{4}\right]$, $\left[\frac{3\pi}{4}; \pi; \frac{5\pi}{4}\right]$, $\left[\frac{5\pi}{4}; \frac{3\pi}{2}; \frac{7\pi}{4}\right]$, $\left[\frac{7\pi}{4}; 2\pi; \frac{9\pi}{4}\right]$ ……………………. etc. are imprecise numbers.

In general, for $y = (c_0 + c_1x)$; $l \epsilon Z$,we have the imprecise numbers, $\left[\frac{\pi}{2l}; \frac{\pi}{l}, ; \frac{3\pi}{2l}\right], \left[\frac{3\pi}{2l}; \frac{2\pi}{l}; \frac{5\pi}{2l}\right], \left[\frac{5\pi}{2l}; \frac{3\pi}{l}; \frac{7\pi}{2l}\right], \left[\frac{7\pi}{2l}; \frac{4\pi}{l}; \frac{9\pi}{2l}\right]$ ……………………………………. etc. within the interval $\left[\frac{\pi}{2l}, \frac{(2n-1)\pi}{2l}\right]$ having indicator function as follows.

$$\rho_{N_C}(x) = \begin{cases} \rho_1(x); & \frac{(2n-1)\pi}{2l} \le x \le \frac{n\pi}{l} \\ \rho_2(x); & \frac{n\pi}{l} \le x \le \frac{(2n+1)\pi}{2l} \end{cases} \quad \text{...........................(8.19)}$$

such that $\rho_1\left(\frac{(2n-1)\pi}{2l}\right) = \rho_2\left(\frac{(2n+1)\pi}{2l}\right) = 0$ *and* $\rho_1\left(\frac{n\pi}{l}\right) = \rho_2\left(\frac{n\pi}{l}\right)$

Here, function has maximum level at the point $x = \frac{n\pi}{l}$ and minimum value at $x = \frac{(2n-1)\pi}{l}$ and $= \frac{(2n+1)\pi}{l}$.

So, $\left[\frac{\pi}{2l}; \frac{\pi}{l}, ; \frac{3\pi}{l}\right], \left[\frac{3\pi}{2l}; \frac{2\pi}{l}; \frac{5\pi}{2l}\right]$, $\left[\frac{5\pi}{2l}; \frac{3\pi}{l}; \frac{7\pi}{2l}\right], \left[\frac{7\pi}{2l}; \frac{4\pi}{l}; \frac{9\pi}{2l}\right]$ ……………………. etc. are the imprecise numbers.

Indicator function of imprecise number has the membership function and the reference function. Thus the area of cosine imprecise function will be measured separately for the respective imprecise numbers. After that, we will collect all the areas of imprecise numbers for the same function to obtain the total area of cosine imprecise function.

Thus the area of the given cosine imprecise function $y = (c_0 + c_1x)\cos(lx)\,; l \in Z$ is

$$I = \int_0^{\frac{n\pi}{l}} (c_0 + c_1x)\cos(lx)\,dx = \frac{2nc_0}{l} + \frac{c_1}{l^2}\sum_{k=1}^{n}[(2k-1)\pi];\forall\, n \in N\,, l \in Z$$

……………………………….(8.20)

Here, cosine imprecise function has maximum value at $x = \frac{n\pi}{l}$

To prove the equation (8.20) let, l=1,2,3,…………………….....so on.

8.3.1. Cosine imprecise function of angle multiple one

$$I_0 = \int_0^{\frac{\pi}{2}} (c_0 + c_1x)\cos(x)\,dx = c_0 + \frac{c_1(\pi-2)}{2}$$

$$I_1 = \int_{\frac{\pi}{2}}^{\pi} (c_0 + c_1x)\cos(x)\,dx = -c_0 - \frac{c_1(\pi+2)}{2}$$

$$I_2 = \int_{\pi}^{\frac{3\pi}{2}} (c_0 + c_1x)\cos(\mathrm{x})\,dx = -c_0 - \frac{c_1(3\pi-2)}{2}$$

$$I_3 = \int_{\frac{3\pi}{2}}^{2\pi} (c_0 + c_1x)\cos(\mathrm{x})dx = c_0 + \frac{c_1(3\pi+2)}{2}$$

$$I_4 = \int_{2\pi}^{\frac{5\pi}{2}} (c_0 + c_1x)\cos(x)\,dx = c_0 + \frac{c_1(5\pi-2)}{2}$$

$$I_5 = \int_{\frac{5\pi}{2}}^{3\pi} (c_0 + c_1x)\cos(x)\,dx = -c_0 - \frac{c_1(5\pi+2)}{2}$$

$$I_6 = \int_{3\pi}^{\frac{7\pi}{2}} (c_0 + c_1 x)\cos(x)\,dx = -c_0 - \frac{c_1(7\pi - 2)}{2}$$

$$I_7 = \int_{\frac{7\pi}{2}}^{4\pi} (c_0 + c_1 x)\cos(x)\,dx = c_0 + \frac{c_1(7\pi + 2)}{2}$$

$$I_8 = \int_{4\pi}^{\frac{9\pi}{2}} (c_0 + c_1 x)\cos(x)\,dx = c_0 + \frac{c_1(9\pi - 2)}{2}$$

so on.

Here, negative sign is showing lower part area bounded by cosine imprecise function.

Area of membership of the given cosine imprecise function is $I_M = I_1 + |I_3| + I_5 + |I_7|$

$$= (4)c_0 + \frac{c_1}{2}\sum_{k=1}^{4}[(2k-1)\pi + 2(4)]$$

Area of the reference function of cosine imprecise function is $I_R = I_0 + |I_2| + I_4 + |I_6|$

$$= (4)c_0 + \frac{c_1}{2}\sum_{k=1}^{4}[(2k-1)\pi - 2(4)]$$

Area of this cosine imprecise function in an interval $[0, 4\pi]$ is $I = I_M + I_R$

$$= \int_{0}^{4\pi} (c_0 + c_1 x)\cos(x)\,dx = c_0(8) + c_1\frac{\pi}{2}\sum_{k=1}^{4} 2.\,[(2k-1)]$$

$$= c_0(8) + c_1\left[\pi\sum_{k=1}^{4}(2k-1)\right]$$

Here, cosine imprecise function has a maximum value at point $x = \pi$

Thus the area of cosine imprecise function in an interval $[0, n\pi]$ is

$$\int_{0}^{\frac{n\pi}{1}} (c_0 + c_1 x)\cos(x)\,dx = c_0(2n) + c_1\pi\left[\sum_{k=1}^{n}(2k-1)\right];\forall\, n \in N$$

………………………..(8.21)

8.3.2. Cosine imprecise function of angle multiple two

$$I_0 = \int_{0}^{\frac{\pi}{4}} (c_0 + c_1 x)\cos(2x)\,dx = \frac{c_0}{2} + \frac{c_1(\pi - 2)}{8}$$

$$I_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (c_0 + c_1 x)\cos(2x)\,dx = -\frac{c_0}{2} - \frac{c_1(\pi + 2)}{8}$$

$$I_2 = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} (c_0 + c_1 x)\cos(2\text{x})\,dx = -\frac{c_0}{2} - \frac{c_1(3\pi - 2)}{8}$$

$$I_3 = \int_{\frac{3\pi}{4}}^{\pi} (c_0 + c_1 x)\cos(2\text{x})dx = \frac{c_0}{2} + \frac{c_1(3\pi + 2)}{8}$$

$$I_4 = \int_{\pi}^{\frac{5\pi}{4}} (c_0 + c_1 x)\cos(2x)\,dx = \frac{c_0}{2} + \frac{c_1(5\pi - 2)}{8}$$

$$I_5 = \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} (c_0 + c_1 x)\cos(2x)\,dx \ = -\frac{c_0}{2} - \frac{c_1(5\pi + 2)}{8}$$

$$I_6 = \int_{\frac{3\pi}{2}}^{\frac{7\pi}{4}} (c_0 + c_1 x)\cos(2x)\,dx = -\frac{c_0}{2} - \frac{c_1(7\pi - 2)}{8}$$

$$I_7 = \int_{\frac{7\pi}{4}}^{2\pi} (c_0 + c_1 x)\cos(2x)\,dx = \frac{c_0}{2} + \frac{c_1 7\pi + 2)}{8}$$

$$I_8 = \int_{2\pi}^{\frac{9\pi}{4}} (c_0 + c_1 x)\cos(2x)\,dx = \frac{c_0}{2} + \frac{c_1(9\pi - 2)}{8})$$

so on.

Here, negative sign is the sign of the lower part area bounded by the given cosine imprecise function.

Area of membership function of the given cosine imprecise function is $I_M = |I_1| + I_3 + |I_5| + I_7$

$$= (4)\frac{c_0}{2} + \frac{c_1}{8}\sum_{k=1}^{4}[(2k-1)\pi + 2(4)]$$

Area of the reference function of cosine imprecise function is $I_R = I_0 + |I_2| + I_4 + |I_6|$

$$= (4)\frac{c_0}{2} + \frac{c_1}{8}\sum_{k=1}^{4}[(2k-1)\pi - 2(4)]$$

Area of the given cosine imprecise function for the interval $\left[0, \frac{4\pi}{2}\right]$ is $I = I_M + I_R$

$$= \int_0^{\frac{4\pi}{2}} (c_0 + c_1 x)\cos(2x)\, dx = \frac{c_0}{2}(8) + c_1 \frac{\pi}{8}\sum_{k=1}^{4} 2.[(2k-1)]$$

$$= \frac{c_0}{2}(8) + c_1 \frac{\pi}{2^2}\sum_{k=1}^{4}[(2k-1)]$$

Here cosine imprecise function has maximum value at $x = \frac{\pi}{2}$

Thus the area bounded by the cosine imprecise function in an interval $\left[0, \frac{n\pi}{2}\right]$ is

$$\int_0^{\frac{n\pi}{2}} (c_0 + c_1 x)\cos(x)\, dx = \frac{c_0}{2}(2n) + c_1 \frac{\pi}{2^2}\sum_{k=1}^{n}[(2k-1)]\ ;\ for\ l = 2$$

Thus, area bounded by a cosine imprecise function of degree one algebraic polynomial $y = (c_0 + c_1 x)\cos(lx)\,; l \in Z$ is

$$I = \int_0^{\frac{n\pi}{l}} (c_0 + c_1 x)\cos(lx)\, dx = \frac{2nc_0}{l} + \frac{c_1}{l^2}\sum_{k=1}^{n}[(2k-1)\pi];\ \ \forall\, n \in N\ and\ l \in Z$$

……………………………….(8.22)

Here, cosine imprecise function has maximum value at $x = \frac{n\pi}{l}$

8.4. Area Bounded by Cosine Imprecise Function for Polynomial of Degree Two

Here, we will discuss how to obtain the area of cosine imprecise function for polynomial of degree two in the summation form**.** In general area bounded by cosine imprecise function $y^* = (c_0 + c_1 x + c_0 x^2)\cos(lx)\,; l \in Z$ is

$$I = \int_{0}^{\frac{n\pi}{l}} (c_0 + c_1 x + c_0 x^2)\cos(lx)\,dx$$

$$= \frac{2nc_0}{l} + \frac{c_1}{l^2}\sum_{k=1}^{n}[(2k-1)\pi] + \frac{c_0}{l^3}\left[\frac{\pi^2}{2}\left\{\sum_{k=1}^{n}(2k-1)^2\right\} + 2n(\pi-2)\right];$$

$\forall\, n \in N, l \in Z$

…………………………………(8.23)

Here, cosine imprecise function has maximum value at the point $x = \frac{n\pi}{l}$.

To prove the equation (8.23), we have considered *l*=1,2,3,4,…………..…….so on

8.4.1. Cosine imprecise function of angle multiple one

$$I_0 = \int_{0}^{\frac{\pi}{2}} (c_0 + c_1 x + c_2 x^2)\cos(x)dx = c_0 + c_1(\frac{\pi}{2} - 1) + c_2\left(\frac{\pi^2}{4} - 2\right)$$

$$I_1 = \int_{\frac{\pi}{2}}^{\pi} (c_0 + c_1 x + c_2 x^2)\cos(x)\,dx = -c_0 - c_1(\frac{\pi}{2} + 1) + c_2\left(\frac{\pi^2}{4} + 2\pi - 2\right)$$

$$I_2 = \int_{\pi}^{\frac{3\pi}{2}} (c_0 + c_1 x + c_2 x^2)\cos(\mathrm{x})dx = -c_0 - c_1\left(\frac{3\pi}{2} - 1\right) - c_2\left(\frac{9\pi^2}{4} - 2\pi - 2\right)$$

$$I_3 = \int_{\frac{3\pi}{2}}^{2\pi} (c_0 + c_1 x + c_2 x^2)\cos(\mathrm{x})dx = c_0 + c_1\left(\frac{3\pi}{2} + 1\right) + c_2\left(\frac{9\pi^2}{4} + 4\pi - 2\right)$$

$$I_4 = \int_{2\pi}^{\frac{5\pi}{2}} (c_0 + c_1 x + c_2 x^2)\cos(x)\,dx = c_0 + c_1\left(\frac{5\pi}{2} - 1\right) + c_2\left(\frac{25\pi^2}{4} - 4\pi - 2\right)$$

$$I_5 = \int_{\frac{5\pi}{2}}^{3\pi} (c_0 + c_1 x + c_2 x^2)\cos(x)\,dx = -c_0 - c_1\left(\frac{5\pi}{2} + 1\right) - c_2\left(\frac{25\pi^2}{4} + 6\pi - 2\right)$$

$$I_6 = \int_{3\pi}^{\frac{7\pi}{2}} (c_0 + c_1 x + c_2 x^2)\cos(x)\,dx = -c_0 - c_1\left(\frac{7\pi}{2} - 1\right) - c_2\left(\frac{49\pi^2}{4} - 6\pi - 2\right)$$

$$I_7 = \int_{\frac{7\pi}{2}}^{4\pi} (c_0 + c_1 x + c_2 x^2)\cos(x)\,dx = c_0 + c_1\left(\frac{7\pi}{2} + 1\right) + c_2\left(\frac{49\pi^2}{4} + 8\pi - 2\right)$$

$$I_8 = \int_{4\pi}^{\frac{9\pi}{2}} (c_0 + c_1 x + c_2 x^2)\cos(x)\,dx = c_0 + c_1\left(\frac{9\pi}{2} - 1\right) + c_2\left(\frac{81\pi^2}{4} - 8\pi - 2\right)$$

so on.

Here, negative sign is the sign of lower part area bounded by cosine imprecise function.

Membership function area of the cosine imprecise function is given by

$$I_M = |I_1| + I_3 + |I_5| + I_7$$

$$= c_0(4) + c_1\left[\frac{\pi}{2}\sum_{k=1}^{4}(2k-1) + 4\right] + c_2\left[\frac{\pi^2}{4}\sum_{k=1}^{4}(2k-1)^2 + 2\pi\sum_{k=1}^{4} k - 2(4)\right]$$

Reference function area of the cosine imprecise function is given by

$$I_R = I_0 + |\,I_2| + I_4 + |I_6|$$

$$= c_0(4) + c_1\left[\frac{\pi}{2}\sum_{k=1}^{4}(2k-1) - 4\right] + c_2\left[\frac{\pi^2}{4}\sum_{k=1}^{4}(2k-1)^2 - 2\pi\sum_{k=1}^{4}(k-1) - 2(4)\right]$$

Area bounded by a cosine imprecise function in an interval $[0, 4\pi]$ is $I = I_M + I_R$

$$\int_{0}^{4\pi} \left((c_0 + c_1 x + c_2 x^2)\cos(x)\right)dx$$

$$= c_0(8) + c_1\left[\frac{\pi}{2}\sum_{k=1}^{4}\{(2k-1) + (2k-1)\} - 4 + 4\right]$$

$$+c_2\left[\frac{\pi^2}{4}\sum_{k=1}^{4}\{(2k-1)^2 + (2k-1)^2\} + 2\pi\sum_{k=1}^{4}(k-k+1) - 2(4+4))\right]$$

$$= c_0(8) + c_1\pi\sum_{k=1}^{8}(2k-1) + c_2\left[\frac{\pi^2}{2}\sum_{k=1}^{4}(2k-1)^2 + 8\pi - 2(8)\right]$$

$$= c_0(8) + c_1\pi\sum_{k=1}^{8}(2k-1) + c_2\left[\frac{\pi^2}{2}\left\{\sum_{k=1}^{4}(2k-1)^2\right\} + 8(\pi - 2)\right]$$

Here, maximum value of the cosine imprecise function is found at $x = n\pi;\ \forall\, n \in N$

Thus, area bounded by a cosine imprecise function in a given interval $[0, n\pi]$ is

$$\int_{0}^{n\pi} \big((c_0 + c_1 x + c_2 x^2)\cos(x)\big)dx$$

$$= c_0(2n) + c_1\pi\left\{\sum_{k=1}^{\frac{n}{2}}(2k-1)\right\} + c_2\left[\frac{\pi^2}{2}\left\{\sum_{k=1}^{\frac{n}{2}}(2k-1)^2\right\} + 2n(\pi-2)\right]$$

$;\forall\ n \in N$

…………………………….(8.24)

8.4.2. Cosine imprecise function angle multiple two

$$I_0 = \int_{0}^{\frac{\pi}{4}} (c_0 + c_1 x + c_2 x^2)\cos(2x)dx = \frac{c_0}{2} + c_1\left(\frac{\pi}{8} - \frac{1}{4}\right) + c_1\left(\frac{\pi^2}{32} - \frac{1}{4}\right)$$

$$I_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (c_0 + c_1 x + c_2 x^2)\cos(2x)\,dx = -\frac{c_0}{2} - c_1\left(\frac{\pi}{8} + \frac{1}{4}\right) - c_1\left(\frac{\pi^2}{32} + \frac{\pi}{4} - \frac{1}{4}\right)$$

$$I_2 = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} (c_0 + c_1 x + c_2 x^2)\cos(2\mathrm{x})dx = -\frac{c_0}{2} - c_1\left(\frac{3\pi}{8} - \frac{1}{4}\right) - c_1\left(\frac{9\pi^2}{32} - \frac{\pi}{4} - \frac{1}{4}\right)$$

$$I_3 = \int_{\frac{3\pi}{4}}^{\pi} (c_0 + c_1 x + c_2 x^2)\cos(2\mathrm{x})dx = \frac{c_0}{2} + c_1\left(\frac{3\pi}{8} + \frac{1}{4}\right) + c_1\left(\frac{9\pi^2}{32} + \frac{2\pi}{4} - \frac{1}{4}\right)$$

$$I_4 = \int_{\pi}^{\frac{5\pi}{4}} (c_0 + c_1 x + c_2 x^2)\cos(2x)\,dx = \frac{c_0}{2} + c_1\left(\frac{5\pi}{8} - \frac{1}{4}\right) + c_1\left(\frac{25\pi^2}{32} - \frac{2\pi}{4} - \frac{1}{4}\right)$$

$$I_5 = \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} (c_0 + c_1 x + c_2 x^2)\cos(2x)\,dx = -\frac{c_0}{2} - c_1\left(\frac{5\pi}{8} + \frac{1}{4}\right) - c_1\left(\frac{25\pi^2}{32} + \frac{3\pi}{4} - \frac{1}{4}\right)$$

$$I_6 = \int_{\frac{3\pi}{2}}^{\frac{7\pi}{4}} (c_0 + c_1 x + c_2 x^2) \cos(2x)\, dx = -\frac{c_0}{2} - c_1 \left(\frac{7\pi}{8} - \frac{1}{4}\right) - c_1 (\frac{49\pi^2}{32} - \frac{3\pi}{4} - \frac{1}{4})$$

$$I_7 = \int_{\frac{7\pi}{4}}^{2\pi} (c_0 + c_1 x + c_2 x^2) \cos(2x)\, dx = \frac{c_0}{2} + c_1 \left(\frac{7\pi}{8} + \frac{1}{4}\right) + c_1 (\frac{49\pi^2}{32} + \frac{4\pi}{4} - \frac{1}{4})$$

$$I_8 = \int_{2\pi}^{\frac{9\pi}{4}} (c_0 + c_1 x + c_2 x^2) \cos(2x)\, dx = \frac{c_0}{2} + c_1 \left(\frac{9\pi}{8} - \frac{1}{4}\right) + c_1 (\frac{81\pi^2}{32} - \frac{4\pi}{4} - \frac{1}{4})$$

so on.

Here, negative sign is showing lower part area bounded by the cosine imprecise function

Summation of the membership function of a cosine imprecise function in a given interval is $|I_1| + I_3 + |I_5| + I_7$

$$= \frac{c_0}{2}(4) + \frac{c_1}{4}\left[\frac{\pi}{2}\sum_{k=1}^{4}(2k-1) + 4\right] + \frac{c_2}{8}\left[\frac{\pi^2}{4}\sum_{k=1}^{4}(2k-1)^2 + 2\pi\sum_{k=1}^{4} k - 2(4)\right]$$

Summation of the reference functions of a cosine imprecise function in a given interval is $I_R = I_0 + |\, I_2| + I_4 + |I_6| + I_8$

$$= \frac{c_0}{2}(4) + \frac{c_1}{4}\left[\frac{\pi}{2}\sum_{k=1}^{4}(2k-1) - 4\right] + \frac{c_2}{8}\left[\frac{\pi^2}{4}\sum_{k=1}^{4}(2k-1)^2 - 2\pi\sum_{k=1}^{4}(k-1) - 2(4)\right]$$

Area obtained for the cosine imprecise function in an interval $\left[0, \frac{4\pi}{2}\right]$ is $I = I_M + I_R$

$$\int_0^{\frac{4\pi}{2}} \left((c_0 + c_1 x + c_2 x^2) \cos(2x)\right) dx$$

$$= \frac{c_0}{2}(8) + \frac{c_1}{4}\left[\frac{\pi}{2}\sum_{k=1}^{4}\{(2k-1) + (2k-1)\} - 4 + 4\right]$$

$$+ \frac{c_2}{8}\left[\frac{\pi^2}{4}\sum_{k=1}^{4}\{(2k-1)^2 + (2k-1)^2 + 2\pi\sum_{k=1}^{4}(-k-1+k) - 2(4+4))\right]$$

$$= \frac{c_0}{2}(8) + \frac{c_1}{4}\pi\sum_{k=1}^{4}(2k-1) + \frac{c_2}{8}\left[\frac{\pi^2}{4}\sum_{k=1}^{4} 2.\,(2k-1)^2 + 8\pi - 2(8)\right]$$

$$= \frac{c_0}{2}(8) + \frac{c_1}{2^2}\pi\left\{\sum_{k=1}^{4}(2k-1)\right\} + \frac{c_2}{2^3}\left[\frac{\pi^2}{2}\left\{\sum_{k=1}^{4}(2k-1)^2\right\} + 8(\pi-2)\right];$$

$$for\ n \in Z$$

Here the given cosine imprecise function has a maximum value at the point, $x = \frac{n\pi}{2}$

Thus, the area obtained for the cosine imprecise function in a given interval $\left[0, \frac{n\pi}{2}\right]$is

$$\int_0^{\frac{n\pi}{2}} \left((c_0 + c_1x + c_2x^2)\cos(2x)\right)dx$$

$$= \frac{c_0}{2}(2n) + \frac{c_1}{2^2}\pi\left\{\sum_{k=1}^{\frac{n}{2}}(2k-1)\right\} + \frac{c_2}{2^3}\left[\frac{\pi^2}{2}\left\{\sum_{k=1}^{\frac{n}{2}}(2k-1)^2\right\} + 2n(\pi-2)\right];$$

……………………………(8.25)

Thus the result is true for *l*=1,2.

So, it is true for positive integers.

Proceeding in the same way we get the result is true for negative integers.

Thus the result is true $\forall\ l \in Z$

8.5. Conclusions

Defining a general form of Cosine imprecise function and the identification of the conversion point of the imprecise function in different situations is one of the objectives of this chapter. So the cosine imprecise function is framed with the assistance of the multiplication factor and the finite numbers of data collection of points. For the different angle of the cosine function, we have obtained a different conversion point. Later to obtain the area of different cosine imprecise functions, we have defined different summation formulae with the assistance of integration. Maximum and minimum Points of imprecise numbers are used as the limits of the integration for the cosine imprecise function.