Chapter 1

GENERAL INTRODUCTION

1.1. History and Development

An imprecise number is a fuzzy number obtained with the help of extension definition of the fuzzy number. In this definition the elements belong to the imprecise number is due to the degree of real number membership value lies between 0 and 1. And the difference between membership function and reference function is the membership value. This term has arrived in the field of study after the fuzzy numbers are not satisfied the universal laws property of the classical set.

Earlier existence of the elements of sets was studied in the definition of the classical set. Here, the elements belong to and not belongs to the set is due to either 100% or zero% similarity in the behavior. It means that an element can be a member of the set when it has a behavior hundred percent similar to the family of the group. If the behavior is a slight difference then it is not considered as a member of the classical set. But there are many real-life problems which are composed a set in the certain time-bound even not hundred percent similar in their behavior. This type of set is the collection of all the behaviors lies between 0% and 100%. From this point, we can observe that all the real-life problems are not possible to express in the classical set.

Later the elements belong to a set is studied in the crisp set. Here, if an element is a member of the set, then it is declared by, 1(one) and non-member is declared by 0(zero). But there are so many real numbers lies in between 0 and 1, which can construct a set and are not studied in the crisp set. So the crisp set is also not possible to construct the set for all the behavior of the objects whose degree of membership is lies between 0 and 1. Later mathematics theory has been upgraded by different personalities to give the existence of an element in the set is a probability. Membership of an element of the probability definition is not a real number. It is the study of membership percentage. Thus to study the element's degree of the membership of set into real number concept, the probability is transformed into possibility in their own definition and notation. Electrical Engineer L.A. Zadeh [96] was first identified this concept in the year (1965). The theory is a more generalized form of a crisp set and is named as Fuzzy Set.

Here the element of the fuzzy set is due to the degree of membership lies between 0 and 1. According to the definition of real line there are infinite number of intervals can be created in between the two real numbers. Thus, L.A. Zadeh (1975) has extended the concept of fuzzy set into fuzzy number to deal with imprecise numerical quantities in a numerical way in the article [97] and develop the concept of fuzzy set from the theory of possibility in the year (1978) in the article [98].

The fuzzy concept has been developed by different authors from the different sources to study and define the uncertain condition into certainty in the different field of study. The possibility of a fuzzy set comes from probability theory. Fuzzy numbers come from un-sharp boundary studies at the different intervals. To study those theories for the different problems of the un-sharp boundary, Nahmias S (1978) has contributed an article [57] of fuzzy Variables. Dubois D and Prade H (1978) have contributed articles [20] regarding the operations of fuzzy numbers. To study the properties of fuzzy numbers, Mizumoto M and Tanaka K (1979) have contributed the article [52]. An article of the interactive of fuzzy numbers [22] is contributed by Dubois D and Prade H (1981) to study the fuzzy number in the field of fuzzy study. Mares M (1989) has discussed regarding the disjunction and conjunction of the approach of the addition of fuzzy quantities in the article [53]. Giachetti R E and Young R E (1997) have discussed regarding the representation and the operators used in the different operations of fuzzy numbers in the article [33]. Cheng C H (1998) has explained regarding the ranking of fuzzy in the field of fuzzy number study in the article [14]. Irion A (1998) has contributed article [46] to study the fuzzy logic and fuzzy numbers by using the Fuzzy rules and fuzzy functions. Delgado M, Vila M A and Voxman W (1998) have discussed the representation of fuzzy numbers in the article [25]. Cheng C H (1999) has contributed article [15] regarding the discussion of triangular fuzzy numbers product for the study of fuzzy product quantity. Ma M, Friedman M and Kandel A (1999) have contributed article [54] regarding the study of new fuzzy Arithmetic in the field of fuzzy number study. Hong D H (2003) has discussed regarding the decomposition of fuzzy numbers in the article [41]. Discussion about the operations of fuzzy number Guerra M L and Stefanini L (2005) contributed an article of fuzzy arithmetic operations [35] in the field of study. Nasseri H (2008) has discussed the fuzzy number with the help of Zadeh's definition in the article [59] and found that it is a positive and non-negative number. Roohollah A S and Majid D (2013) ,[73] have introduced the new parametric method for ranking of fuzzy numbers in the field of fuzzy study. A fuzzy set is helpful to study complex relationships among the different members and is discussed by the authors Bentkowska U and Anna A (2016) in the article [12].

In this way, the concept of fuzzy number is developed from different types of fuzzy study in the field of mathematics. It is a member of the fuzzy set which can construct a shape of triangular or trapezoidal or bell-shaped within some time bound. But all the elements of the fuzzy set may not have this character. Thus, we found in the study that every fuzzy number can be an element of fuzzy sets, but the converse is not always true.

The researchers adopted fuzzy concept to construct and study the slight difference behavior of the objects having unsharp boundary within the single set. Dubois D and Prade H (1992) have developed the theory of upper probabilities into possibilities with the help of fuzzy definition to construct a fuzzy set in the field of study in the article [24]. Verkuilen J (2005) has contributed article [83] to study how membership is assigned in the Fuzzy Set Analysis. Dubois D and Hullermeier E, (2007), [27] have discussed the comparison of probability measures using possibility theory in the field of study.

Using the fuzzy definition different authors has developed the properties of the different type of fuzzy set in the field of mathematics. Rodabaugh S E (1982) [69] has contributed an article of expression of the fuzzy addition over the real line in the field of fuzzy study. In related to fuzzy number properties, Sinha D (1990) has contributed article [74] of the general theory of fuzzy arithmetic in the field of study. Still, some of the properties of possibilities theories of fuzzy numbers are not found as similar to the classical set in their studies and we are considered all of them to review in our study with the help of the definition of imprecise numbers.

Classical matrices properties have been developed into the form of fuzzy matrix by the different authors in the field of study to represent complex links among the variables. Thomson M G (1977), [82] has introduced fuzzy matrix in the field of fuzzy study. Kim J B (1978) has contributed an article [47] to study some of the formulae of determinant using the definition of the fuzzy matrix. Hasimato H (1983) has discussed in the article [39] regarding the Convergence of powers of the fuzzy transitive Matrices. Here, he has used the fuzzy concept to study the properties of the fuzzy matrix. In related to the fuzzy matrix, Kolodziejezyk W (1988) has also contributed an article [48] of Convergence of the s-transitive fuzzy matrices. Another author, Xin L J (1992) (1994) has discussed in the articles [92] and [93] about the fuzzy matrices with the help of fuzzy theory. Ragab M Z and Emam E G (1994) have contributed an article [70] to define the formula of the determinant of the square fuzzy matrix for the study of fuzzy matrices. The same authors (1995) have expressed the different operations of fuzzy matrices with the help of min-max operators and used own notation to discuss the different properties of the fuzzy matrices in the article [71]. Shymal A K and Pal M (2004) have also contributed an article [78] in related to the operators on fuzzy matrices for which fuzzy matrix operation is definable.

Later the concept of fuzzy has been developed to construct the fuzzy function and relation. Regarding the structure of fuzzy relation, Ovehinnikov S V(1981) has contributed article [62]. Pedrycz W, Hirota K and Sessa S (2001) have discussed in their article [65] regarding the decomposition of fuzzy relations. Shimoda M (2002),(2003) has contributed article [77] and [75] regarding the fuzzy relations and fuzzy mapping in the field of fuzzy study. Bica A M (2002), (2007) discussed regarding the structures of fuzzy numbers in the article [4] and [5]. Zhu Y and Ji X (2006) discussed the uses of fuzzy concept to identify the suitable variable in the fuzzy functions in the article [100]. Hu D, Li H and Yu X (2009) contributed article [42] to study the rules for which fuzzy relationship can be constructed among the members. Qiu D and Zhang W Q (2013) has discussed regarding the symmetric of the fuzzy number and the equivalence of fuzzy number in the article [68].

The theory of fuzzy concept has been adopted by different authors on their studies due to its effectiveness of the definition, which can represent many complex problems under their model. Due to this effectiveness of the model, the fuzzy concept has been applied in the many fields of our study. Hutton B (1975) discussed the application of fuzzy set theory to the study of topology in the article [38]. Dubois D and Prade H (1980) ,[21] have discussed the application of fuzzy theory in the field of study. Fuzzy number theory has been applied to represent the fuzzy event by the authors Lee E S and Li R J (1988) in the article [51]. Kuncheva L I (1992) has discussed regarding the helpfulness of fuzzy concept for the future selection in the article [50]. A comparative assessment for the measurement of similarity of fuzzy values was discussed by the authors, Pappis C P and Karacapilidis N I (1993) in the article [63]. William T A (1995) has contributed article [88] regarding the importance of fuzzy set theory in the study of the Movement of Social Sciences. Hielperm S (1997) has discussed regarding the representation and the application of fuzzy numbers in the article [40]. Adzic S and Sedlak O (1998) have discussed in the article [1] about the application of Macroeconomic Planning within the Process of Transition with help of fuzzy definition. Guijun W and Xiaoping L (1998) have discussed the importance of fuzzy number for the study of different fuzzy intervals in the article [34]. Pavlica V and Petrovacki D (1999) have contributed article [64] to study how a fuzzy problem can be controlled based on fuzzy relation equations. Demirci M (2003) [26] discussed regarding the importance of fuzzy function for the study of complex mapping among the different objects. Vasantha Kandasamy W B, Smarandache F and Iilanthenral K (2007) discussed regarding the uses of fuzzy matrix theory to study our Social system in the fuzzy model in the article [84]. Mas M, Monserrat M, Torrens J and Trillas E (2007) have discussed in the article [55] how a fuzzy concept can be used for the construction of fuzzy function. Wei S H and Chen S M (2009) have contributed article [89] to study Fuzzy risk analysis based on interval-valued fuzzy numbers. Shaw A K and Roy K (2012) discussed regarding the application of arithmetic operations fuzzy number on the reliability evaluation in the article [81]. Glad D (2013) has also contributed article [37] about the importance of fuzzy interval in the definition of the fuzzy function. Narayanamoorthy S, Saranya S and Maheswari S (2013) have used the fuzzy technique to solve the transportation problem in the article [61]. Chehlabi M (2014) also discussed regarding the importance of fuzzy concept for the interpolation of tabular functions in the article [16]. Wang Y (2014) has contributed article [91] to study the fuzzy arithmetic using the definition of fuzzy number. Harliana P and Rahim R (2017) have contributed article [45] to study about the importance of fuzzy function for the conclusion of decision-making.

The relationship among the different objects can be represented with the assistance of Cartesian product. So the fuzzy Cartesian products are discussed by different authors to represent the complex relationship among the different objects in the field of study. At the beginning of fuzzy age, the concept of fuzzy was used to study single line problem in the form of real line. Later different authors have included multiple problems in the fuzzy concept to have Cartesian product study of the fuzzy set. Dib K A and Youssef N L (1991) were extended the fuzzy concept into a fuzzy Cartesian product for the fuzzy relations and fuzzy functions in the article [23]. In this regards, Kim J B, Hee Kim Y and Kim C B (1991) have contributed article [49] regarding the fuzzy Cartesian product of fuzzy sets. Young J B and Chang H (1991) discussed the resolution of the Cartesian product of fuzzy sets to develop the Cartesian of fuzzy study in the article [95]. Muralikrishna P and Ouleeswaran M C (2011) discussed regarding the implementation of the Cartesian product of Intuitionistic L-Fuzzy in the article [56]. Varghese A and Kuriakose S (2012) have also contributed article [85] to discuss regarding the Cartesian product over Intuitionistic Fuzzy Sets. Vijayabalaji S and Sivaramakrishnan S (2012) have contributed article [86] to develop the Cartesian products in the study of Interval-Valued fuzzy linear space. Priya T and Ramachandran T (2014) have also discussed fuzzy PS-algebra in terms of Cartesian product in the article [67]. In this regards, another author, Xie J and Liu S (2017) have also contributed article [94] regarding the Cartesian product over the interval-valued intuitionistic fuzzy. In those articles, Cartesian products are studied with the help of fuzzy definition. Here, elements are defined with respect to membership function only. So, we have discussed the Cartesian product of fuzzy numbers in terms of multiple dimensional imprecise numbers with the help of membership and the reference function.

Since the fuzzy set of Zadeh is not similar to the classical set. Many of the properties of set study in the field of mathematics are not possible to discuss in this definition. Different authors in their articles have discussed remarks of these problems. Nola A D, Sessa S and Pedrycz W (1985) discussed in their article [58] regarding the decomposition problem of fuzzy relations with the help of existing definition of a fuzzy set. Zimmermann K (1991) also contributed article [99] about the Fuzzy Set Covering Problem in the field of fuzzy study. So to represent a complex problem in the theory of fuzzy, Pieget A (2005) contributed an article [64], where he has discussed the new definition of fuzzy sets in the field of fuzzy mathematics. Gao Q S, Yu X and Hu Y (2009) have also introduced a new definition of fuzzy set theory can be possible to have similar to the classical sets in the article [36].

Fuzzy definition of Zadeh is not similar to classical set and the definition has been developed by different authors to write various articles and their application in the field of mathematics. Some of the new authors have identified that assimilation of a fuzzy concept of Zadeh can be constructed with respect to the membership function and the reference function for the study of interval created by the objects of the unsharp boundary. These numbers can satisfy all the classical set theory properties under the complexities situation. So the unsatisfied properties of the fuzzy numbers are undertaken to study in the imprecise numbers.

New complement definition of the fuzzy set with their own notation is introduced by Baruah H K (2011) [6], [7], [8], [9], [10] and (2012), [11] in their article. According to this author fuzzy set of Zadeh defined for both set and the complement of the set is by using membership function. Due to this reason, it doesn’t have a character similar to the classical set. To discuss this matter in the various situations, Baruah H K has introduced new complement definition of the fuzzy set with respect to the membership function and the reference function. Later found that it satisfies all the properties of the classical set. But the boundary between fuzzy set and the complement of the fuzzy set is not so clear. Still the definition of Baruah H K can satisfy all the properties of the classical set. So the new term of fuzzy set introduced by him is named as an imprecise set. This definition is used by Neog T J and Sut D K (2012) to study the soft set in the article [60]. In this way, the imprecise definition has arrived in the field of study.

Fuzzy Number is one of the special types of fuzzy set. It has particular importance in the field of fuzzy theory. If we draw the fuzzy number, then the graph is found as a triangular fuzzy number, trapezoidal fuzzy number or the bell shape fuzzy number. As for example riding a car at a speed of 60 Km. per hours in the certain time bound is constructing a trapezoidal fuzzy number. Here the numbers are represented only in the form of the membership function. So that complement of the fuzzy number and the complement of fuzzy number will have commonality in the graph. The idea of recovery from this definition is started in the mind of Baruah by defining a fuzzy number in both the membership and the reference function. The new term is known as the imprecise number. In this definition fuzzy number and its complement does not have a common value. So, we have used the definition of imprecise number in the various sections of our study.

In the imprecise number all the properties of fuzzy numbers, which do not hold good in the earlier definition are review with examples. Finally found that all the properties of the imprecise number are similar to the classical sets. This imprecise number is playing a key role in our study. It has more properties than the fuzzy number. So, we are undertaking to apply the definition of imprecise number in the field of transportation, economics. Imprecise matrix obtained with help of the definition of imprecise number is finding an application in the field of transportation.

The concept is extended up to the n^{th} dimensional imprecise number to study problems involving with the n^{th} number of variables. In particular, one, two and three-dimensional imprecise numbers are modeled to solve the economic and transportation problems in our study.

As the application is basically done in the field of transportation and economics. Many authors have discussed the problems of those areas with the help of classical definition and solved them with the different theory of the classical method. Dwyer P S and Galler B A (1957), (1966), (1967) have studied and solved many of transportation problems with the help of classical matrix definition in the article [17], [18], [19]. Chawla M (1992) discussed the use of the Gaussian elimination method to solve linear equations in the article [13]. Foster L V (1997) used the Gauss-elimination method to study the growth factor in the article [31]. This method of Gauss-elimination is later used to solve a chain of linear equations in our study. Adlakha V, Kowalski K and Lev B (2006), [2] solved the transportation problems with mixed constraints in the classical theory. Adlakha A, Kowalski K and Wang S (2014), [3] have contributed an article on the approximation of the fixed charge transportation problem. But problems with probability in nature are not all solvable in the classical definition. So, we represent those types of problems in the imprecise number definition and finally solved them with the help of the imprecise number model.

The definition of imprecise number helps us to obtain a new function called the imprecise function. Which is very much helpful to represent the behavior of any object in the form of function and control it accordingly. Thus the behavior of the imprecise function in various situations is also discussed in our study.

1.2. Preliminaries

Many authors in the name of fuzzy number with new complement definition have already studied the imprecise number. This notion of fuzzy helped us to study its application in the various field of science and technology. Baruah H K extended the definition of fuzzy number in terms of membership function and the reference function. This new term is an imprecise number. In his research articles, some of the definitions of imprecise form are discussed in their own form. The notions of imprecise help to construct the type of set which is composed due to not distinct in nature into the form of classical set. These types of set can satisfy all the classical set theory properties. The following are the basic definitions used by the earlier authors in the field of imprecise numbers and are frequently used by us to get extension study of imprecise numbers in the various chapters of this study.

1.2.1. Imprecise Number: Imprecise number $N = [\alpha; \beta; \gamma]$ is divided into closed sub-intervals with the partial presence of element β in both the intervals.

1.2.2. Partial presence: Partial presence of an element in an imprecise real number $[\alpha; \beta; \gamma]$ is described by the present level indicator function $p(x)$ which is counted from the reference function $r(x)$ such that present level indicator for any x, $\alpha \leq x \leq \gamma$ is $(p(x) - r(x))$, where $0 \leq r(x) \leq p(x) \leq 1$.

1.2.3. Membership value: Indicator function of an imprecise number $N = [\alpha; \beta; \gamma]$ is defined by,

$$\mu_N(x) = \begin{cases} \mu_1(x), & when\ \alpha \leq x \leq \beta \\ \mu_2(x), & when\ \beta \leq x \leq \gamma \\ 0, & otherwise \end{cases} \tag{1.1}$$

Where, 0 is the constant reference function over the real line.

Here $\mu_1(x)$ is continuous and non-decreasing in the interval $[\alpha, \beta]$ and $\mu_2(x)$ is a continuous and non-increasing in the interval $[\beta, \gamma]$.

And $(\mu_1(x) - \mu_2(x))$ is known as the membership value of an indicator function, $\mu_N(x)$ for all $0 \leq \mu_1(x) \leq \mu_2(x) \leq 1$.

Case I: Imprecise function having indicator function (1.1) is called a normal imprecise number if satisfy the following properties.

$$\mu_1(\alpha) = \mu_2(\gamma) = 0$$
$$\mu_1(\beta) = \mu_2(\beta) = 1 \tag{1.2}$$

Case II: Imprecise function having indicator function (1.1) is called a subnormal imprecise number if satisfy the following properties.

$$\mu_1(\alpha) = \mu_2(\gamma) = 0$$
$$\mu_1(\beta) = \mu_2(\beta) \neq 1 \qquad (1.3)$$

1.2.4. Representation of imprecise number: For a real line $0 \leq \mu_1(x) \leq \mu_2(x) \leq 1$, the imprecise numbers is characterized in the form $\{(\mu_1(x), \mu_2(x)): x \in R\}$.

Where $\mu_1(x)$ is called the membership function measured from the reference function $\mu_2(x)$ and $(\mu_1(x) - \mu_2(x))$ is called the membership value of the indicator function equation (1.3).

Here, the imprecise number is normal when membership value of indicator function $\mu_N(x)$ is equal to 1 otherwise subnormal.

1.2.5. Complement: For an imprecise number N= $\{(\mu_N(x), 0) : x \in R\}$ as defined above, the complement $N^c = \{(1, \mu_N(x)): x \in R\}$ will have indicator function having membership value equal to 1 and the reference function $\mu_N(x) < 1$ for $-\infty < x < \infty$

Graphically complement of this definition is very difficult to show for all the problems. Still the problems solvable with help of this complement definition. So, there is a partial presence element hidden as a boundary of the fuzzy number and its complement. The new term obtained due to exclusion of partial element is named as the imprecise number in the article of Baruah H K. Gradually the concept is used by many other authors to prove that the properties of fuzzy numbers which are not possible to discuss in their definition are possible for the definition of imprecise number.

1.3. Outlines of the Thesis

The whole chapter of the thesis is divided into eight chapters and each chapter is divided into subsections for the study of imprecise numbers.

In chapter 1 includes a brief history regarding the theory and the origin of the imprecise number. For this reason, a literature survey is done in brief along with the importance and motivation of the study of imprecise numbers. Different types of imprecise numbers, Complement of the imprecise number used by earlier authors in their study are brought in the preliminary section. With the help of those definitions, extension study of imprecise number in various forms is discussed in the remaining chapters. This chapter is also included a summary of research works, which are carried out in the thesis.

Chapter 2 includes the definition of one-dimensional imprecise number, properties of one-dimensional imprecise number. In order to prove the properties of one-dimensional imprecise number, various examples are considered for the discussion. For the construction of imprecise matrix, the definition of imprecise number is used. Later in the definition of the imprecise number we have discussed various types of imprecise matrices with their own notation. To enrich the application of one-dimensional imprecise number, properties of the imprecise matrix is discussed with the help of the network diagram.

Chapter 3 includes the discussion of two-dimensional imprecise numbers. To obtain this study, One-dimensional imprecise number is extended into more dimensional form in the chapter. Along with this, properties of the classical theory under the operations of intersection and unions are discussed in the two-dimensional imprecise numbers. Those properties are proved with counter examples. A two-dimensional model for the study of the applications of two variables involving economics problems is also included at the end of this chapter.

Chapter 4 includes the discussion regarding the definition of three-dimensional imprecise numbers and their notations. Similar to the two-dimensional imprecise number all the properties of the classical set theory under the operation of the intersection and the union are proposed to discuss in the three-dimensional imprecise number. Prove of the properties of three-dimensional imprecise number is discussed with examples and found that those imprecise numbers are similar to the classical set. Model of three-dimensional imprecise numbers is framed for the problems involving three variables of fuzzy type.

Chapter 5 includes the discussion of imprecise number concept into N^{th} dimensional imprecise numbers. Different N^{th} dimensional imprecise numbers definition is discussed with own notation. With the help of those definitions, properties of classical set theory under the operation of the intersection and the union are discussed in the N^{th} dimensional imprecise numbers. Prove of those properties are done with help of the examples of N^{th} dimensional imprecise numbers.

Chapter 6 includes the uses of imprecise number definition in the construction of imprecise function from the ordinary function. After the study of the definition of imprecise function, basic properties of imprecise functions are identified and are discussed in the chapter. The points, from which a general function is transformed into the imprecise function is discussed for the different type of functions. Different methods to obtain imprecise function from the ordinary function are also introduced. The condition of convergence rate of an imprecise function is discussed. This value helps us to know that whether a function is usable in a particular place or not. Discussion of the different types of the imprecise function is done in the light.

Chapter 7 includes details about the general method of construction of the imprecise function using sine function. Example of a particular data collection of points is considered to define sine imprecise function for the study. Conversion points of the various sine imprecise functions are discussed with the help of examples. Area formula of sine imprecise function is obtained in the form of summation with the help of integration and is discussed for the different algebraic polynomial functions. Different problems of sine imprecise functions are representing different curves as an example of the most economical sine imprecise curve of a particular region.

Finally, chapter 8 includes details about the general method for the construction of cosine imprecise function. Examples of the construction of cosine imprecise function are discussed with the assistance of particular data collection of points. The technique of defining conversion point of the different cosine imprecise functions is discussed for the different examples. In defining area formula and the reason why imprecise numbers are used as the limit of the integration is discussed. Different examples of cosine imprecise function are found as the example of the most economical curves of a particular region.