**Chapter 3**

# INTRODUCTION OF TWO-DIMENSIONAL IMPRECISE NUMBERS




The work presented in this chapter is published in "International Journal of Information Engineering and Electronic Business", Vol. 7(5) (2015), 27-38, ISSN: 2074-9023 (Print), ISSN: 2074-9031 (Online).

## 3.1. Introduction

Dimension study plays a vital role in the field of mathematics. Since the problem of the object is not only in the single real line part. To get a complete solution of any physical problem we need the solution of the whole dimension of the object. For a surface study of any experiment, the two-dimensional imprecise number takes a part to identify properly, how much membership of the object is effecting on the surface of the object. So to study how much percentage is occupied as a member, called membership value along the two axes, this chapter is coming out.

Here, the concept of imprecise number over the real line defined by Baruah (2011) [6], [7] is extended into a particular form of two-dimensional imprecise number so that we can study the effect of fuzziness occupied over the two axes of the body. Identification of the effect of fuzziness characters in the specific dimension will help to solve many difficult practical problems.

Like the one-dimensional imprecise number intersection and the union of two imprecise numbers definition is also introduced in their own notation. It is expressed with the help of maximum and minimum operators. All the properties occurred under these operations on one-dimensional imprecise number are proposed in the two dimensional imprecise numbers and their proofs are done with the help of examples. Application of two-dimensional imprecise numbers in the field of economics is discussed with the examples.

## 3.2. Two-dimensional Imprecise Numbers

In the definition of Baruah (2011) [6], imprecise number is defined for only x-axis. This case is studied if the effect of fuzziness over the physical significance is along the x-axis, when all other remaining axes are already fully membership. In practice such a standard problem is limited. So, to study practical problems in more effectiveness it may be introduced two-dimensional imprecise numbers. Two-dimensional numbers are expressible in XY-plane. Here, imprecise number is defined in the two-dimensional form such that the full membership along the x-axis and the y-axis is considered one when other axes are already fully membership. For example, at any instant distance travelled by a wave is the x-axis and the height of the tide is y-axis.

***3.2.1. Definition:*** A two dimensional imprecise number

$$N_{XY} = [(\alpha_x, \alpha_y); (\beta_x, \beta_y); (\gamma_x, \gamma_y)] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.1)$$

is divided into sub intervals with a partial element is presence in both the intervals. Where all the points in this interval are the elements of Cartesian product of two sets $X \times Y$ and both the sets X and Y are the imprecise numbers.

***3.2.2. Definition:*** An element of partial presence in the two-dimensional imprecise number $N_{XY} = [(\alpha_x, \alpha_y); (\beta_x, \beta_y); (\gamma_x, \gamma_y)]$ is described by the present level indicator function $p(x, y)$ which is counted from the reference function $r(x, y)$ such that present level indicator for any $(x, y)$, $(\alpha_x, \alpha_y) \leq (x, y) \leq (\gamma_x, \gamma_y)$ is $(p(x, y) - r(x, y))$, where $(0,0) \leq r(x, y) \leq p(x, y) \leq (1,1)$.

***3.2.3. Definition:*** Indicator function of the two-dimensional imprecise number $N_{XY} = [(\alpha_x, \alpha_y); (\beta_x, \beta_y); (\gamma_x, \gamma_y)]$ is represented and defined by,

$$\rho_{N_{XY}}(x, y) = \begin{cases} \rho_{XY1}(x, y), & (\alpha_x, \alpha_y) \leq (x, y) \leq (\beta_x, \beta_y) \\ \rho_{XY2}(x, y), & (\beta_x, \beta_y) \leq (x, y) \leq (\gamma_x, \gamma_y) \\ 0, & otherwise \end{cases} \quad \ldots\ldots\ldots\ldots(3.2)$$

Such that $\rho_{XY1}(\alpha_x, \alpha_y) = \rho_{XY2}(\gamma_x, \gamma_y) = 0$ and $\rho_{XY1}(\beta_x, \beta_y) = \rho_{XY2}(\beta_x, \beta_y)$.
where $\rho_{XY1}(x, y)$ is increasing function over the interval $[(\alpha_x, \alpha_y), (\beta_x, \beta_y)]$ and $\rho_{XY2}(x, y)$ is decreasing over the interval $[(\beta_x, \beta_y), (\gamma_x, \gamma_y)]$. Then,

Case I: Two-dimensional normal imprecise number if,

$$\rho_{XY1}(\alpha_x, \alpha_y) = \rho_{XY2}(\gamma_x, \gamma_y) = 0$$

$$\text{and} \quad \rho_{XY1}(\beta_x, \beta_y) = \rho_{XY2}(\beta_x, \beta_y) = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.3)$$

Case II: Two-dimensional subnormal imprecise number if,

$$\rho_{XY1}(\alpha_x, \alpha_y) = \rho_{XY2}(\gamma_x, \gamma_y) = 0$$

$$\text{and} \quad \rho_{XY1}(\beta_x, \beta_y) = \rho_{XY2}(\beta_x, \beta_y) \neq 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.4)$$

$$And \qquad (\rho_{XY1}(x, y) - \rho_{XY2}(x, y)) = (\alpha_x - \beta_x) \times (\alpha_y - \beta_y) \quad \ldots\ldots\ldots\ldots\ldots(3.5)$$

is called membership value of the indicator function $\rho_{NXY}(x, y)$.

Where $\rho_{XY1}(x, y) = (\alpha_x, \alpha_y)$ and $\rho_{XY2}(x, y) = (\beta_x, \beta_y)$

***3.2.4. Definition:*** Complement of the normal imprecise number

$N_{XY} = \{ \rho_{N_{XY}}(x, y), (0,0): (x, y) \in X \times Y\}$ is defined by

$$N_{XY}{}^{c} = \{(1,1), \rho_{NXY}(x, y): (x, y) \in R \times R\} \text{ ......................(3.6)}$$

with membership function equal to (1,1) and the reference function $\rho_{N_{XY}}(x, y) < 1$ for $-\infty < (x, y) < \infty$.

Two-dimensional imprecise number is characterized by,

$$\{\rho_{XY1}(x, y), \rho_{XY2}(x, y): (x, y) \in X \times Y \}$$

Where $\rho_1(x, y)$ and $\rho_2(x, y)$ are called membership function and the reference function of the indicator function $\rho_{N_{XY}}(x, y)$ defined in equation (3.2) and

$$\left(\rho_{XY1}(x, y) - \rho_{XY2}(x, y)\right) = (x_1 - x_2) \times (y_1 - y_2) \text{........................(3.7)}$$

is called the membership value of the indicator function.

Where $\rho_{XY1}(x, y) = (x_1, y_1)\ and\ \rho_{XY2}(x, y) = (x_2, y_2)$ respectively.

If the membership value is equal to 1, then the two-dimensional imprecise number is called two-dimensional normal imprecise number otherwise subnormal.

***3.2.5. Definition:*** If $A(\rho_{XY}(x, y)) = \{(\rho_{XY1}(x, y), \rho_{XY2}(x, y)): (x, y) \in X \times Y\}$ and

$$B(\rho_{XY}(x, y)) = \{\left(\rho_{XY3}(x, y), \rho_{XY4}(x, y)\right): (x, y) \in X \times Y\},$$

Then intersection and the union of imprecise numbers is defined by,

$$A(\rho_{XY}(x, y)) \cap B(\rho_{XY}(x, y)) = \begin{Bmatrix} min\left(\rho_{XY1}(x, y), \rho_{XY3}(x, y)\right), \\ max\left(\rho_{XY2}(x, y), \rho_{XY4}(x, y)\right): (x, y) \in X \times Y \end{Bmatrix} \text{...(3.8)}$$

$$A(\rho_{XY}(x, y)) \cup B(\rho_{XY}(x, y)) = \begin{Bmatrix} max\left(\rho_{XY1}(x, y), \rho_{XY3}(x, y)\right), \\ min\left(\rho_{XY2}(x, y), \rho_{XY4}(x, y)\right): (x, y) \in X \times Y \end{Bmatrix} \text{...(3.9)}$$

## 3.3. Properties of Two-dimensional Imprecise Numbers

Classical set theory properties under the operations of maximum and minimum is proposed in two-dimensional imprecise numbers.

***3.3.1. Property (Universal Laws)***

(i) $A(\rho_{XY}(x, y)) \cap A^C\left(\rho_{XY}(x, y)\right) = \emptyset(\rho_{XY}(x, y))$

(ii) $A(\rho_{XY}(x, y)) \cup A^C\left(\rho_{XY}(x, y)\right) = \Omega(\rho_{XY}(x, y))$

Where, $A^C\left(\rho_{XY}(x, y)\right)$, $\emptyset(\rho_{XY}(x, y))$ and $\Omega(\rho_{XY}(x, y))$ are complement of set, empty set and the universal set respectively.

Since the one-dimensional imprecise number is always satisfy all properties of classical set. First it can be claimed that two-dimensional imprecise numbers also satisfies the above Universal Laws. For this purpose, let a dram 'A" filled with a portion of mercury and is shown in the figure below.

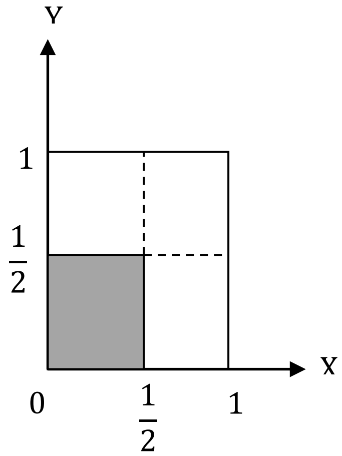


Fig.3.1. Two-dimensional imprecise number

Imprecise number is $A((\rho_{XY}(x,y)) = \left\{\left(\frac{1}{2},\frac{1}{2}\right),(0,0)\right\}$ with the membership value $\left(\frac{1}{2}-0\right)\times\left(\frac{1}{2}-0\right)=\left(\frac{1}{2}\times\frac{1}{2}\right)=\frac{1}{4}$ and the portions of complement of imprecise number are $A^C\left(\rho_{XY}(x,y)\right)$ are $\left\{(1,1),\left(0,\frac{1}{2}\right)\right\}$ and $\left\{\left(1,\frac{1}{2}\right),\left(\frac{1}{2},0\right)\right\}$.

Now by the intersection and union definition of imprecise numbers we have,

$$A\left(\rho_{XY}(x,y)\right)\cap A^C\left(\rho_{XY}(x,y)\right)$$

$$=\left\{\left(\min\left(\frac{1}{2},1\right),min\left(\frac{1}{2},\frac{1}{2}\right)\right),\left(\max\left(0,\frac{1}{2}\right),\max(0,0)\right)\right\}$$

$$=\left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},0\right)\right\}$$

It's membership value: $\left(\frac{1}{2}-\frac{1}{2}\right)\times\left(\frac{1}{2}-0\right)=0.$

And $\left\{\left(\min\left(\frac{1}{2},1\right),min\left(\frac{1}{2},\frac{1}{2}\right)\right),\left(\max(0,0),\max\left(0,\frac{1}{2}\right)\right)\right\}=\left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(0,\frac{1}{2}\right)\right\}$

Its membership value: $\left(\frac{1}{2}-0\right)\times\left(\frac{1}{2}-\frac{1}{2}\right)=0$ .

Since, the union of two empty sets is again empty set. So, intersection of the membership and its complement is forming a two-dimensional null or empty imprecise number to satisfy the property 3.3.1(i).

$$\left(A\rho_{XY}(x,y)\right)\cup A^C\left(\rho_{XY}(x,y)\right)=\left\{\left(\frac{1}{2},\frac{1}{2}\right),(0,0)\right\}\cup\left(\left\{(1,1),\left(0,\frac{1}{2}\right)\right\}\cup\left\{\left(1,\frac{1}{2}\right),\left(\frac{1}{2},0\right)\right\}\right)$$

$$= \left\{ \begin{matrix} \left(\max\left(\frac{1}{2},1\right), \max\left(\frac{1}{2},1\right)\right), \\ \left(\min(0,0), \min\left(0,\frac{1}{2}\right)\right) \end{matrix} \right\} \cup \left\{ \begin{matrix} \left(\max\left(\frac{1}{2},1\right), max(\frac{1}{2},\frac{1}{2})\right), \\ \left(\min\left(0,\frac{1}{2}\right), \min(0,0)\right) \end{matrix} \right\}$$

$$= \{(1,1),(0,0)\} \cup \left\{\left(1,\tfrac{1}{2}\right),(0,0)\right\}$$

$$= \left\{\left(\max(1,1), \max\left(1,\tfrac{1}{2}\right)\right), (\min(0,0), \min(0,0))\right\}$$

$$= \{(1,1),(0,0)\}$$

It has the membership value: $(1-0)\times(1-0)=1$.

Thus the union of the membership and the complement of two-dimensional imprecise numbers is the universal two-dimensional imprecise number to satisfy the property 3.3.1.(ii).

### *3.3.2. Property (Commutative laws)*

Let $A(\rho_{XY}(x,y)) = \{(\rho_{XY1}(x,y)), (\rho_{XY2}(x,y)) : (x,y) \in X \times Y\}$ and $B(\rho_{XY}(x,y)) = \{\rho_{XY3}(x,y), \rho_{XY4}(x,y) : (x,y) \in X \times Y\}$ be two-dimensional imprecise numbers. Then,

(i) $A(\rho_{XY1}(x,y)) \cup B(\rho_{XY2}(x,y)) = B(\rho_{XY2}(x,y)) \cup A(\rho_{XY1}(x,y))$

(ii) $A(\rho_{XY1}(x,y)) \cap B(\rho_{XY2}(x,y)) = B(\rho_{XY2}(x,y)) \cap A(\rho_{XY1}(x,y))$

**Proof:** If $A(\rho_{XY1}(x,y)) = \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{4},\frac{1}{4}\right)\right\}$ and $B(\rho_{XY2}(x,y)) = \left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{6},\frac{1}{6}\right)\right\}$ be two dimensional imprecise numbers, then

(i) $$A\big(\rho_{XY1}(x,y)\big) \cup B\big(\rho_{XY2}(x,y)\big) = \left\{ \begin{matrix} \left(\max\left(\frac{1}{2},\frac{1}{3}\right), \max\left(\frac{1}{2},\frac{1}{3}\right)\right), \\ \left(min\left(\frac{1}{4},\frac{1}{6}\right), min\left(\frac{1}{4},\frac{1}{6}\right)\right) \end{matrix} \right\}$$

$$= \left\{\left(\tfrac{1}{2},\tfrac{1}{2}\right),\left(\tfrac{1}{6},\tfrac{1}{6}\right)\right\}$$

And $$B(\rho_{XY2}(x,y)) \cup A(\rho_{XY1}(x,y)) = \left\{ \begin{pmatrix} \max\left(\frac{1}{3},\frac{1}{2}\right), \\ \min\left(\frac{1}{3},\frac{1}{2}\right) \end{pmatrix}, \begin{pmatrix} max\left(\frac{1}{6},\frac{1}{4}\right), \\ min\left(\frac{1}{6},\frac{1}{4}\right) \end{pmatrix} \right\}$$

$$= \left\{\left(\tfrac{1}{2},\tfrac{1}{3}\right),\left(\tfrac{1}{4},\tfrac{1}{6}\right)\right\}$$

Hence proved

(ii) $$A\big(\rho_{XY1}(x,y)\big) \cap B\big(\rho_{XY2}(x,y)\big) = \left\{ \begin{matrix} \left(\min\left(\frac{1}{2},\frac{1}{3}\right), \min\left(\frac{1}{2},\frac{1}{3}\right)\right), \\ \left(max\left(\frac{1}{4},\frac{1}{6}\right), max\left(\frac{1}{4},\frac{1}{6}\right)\right) \end{matrix} \right\}$$

$$= \left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{4},\frac{1}{4}\right)\right\}$$

And $B(\rho_{XY2}(x,y)) \cap A(\rho_{XY1}(x,y)) = \left\{\begin{pmatrix}\min\left(\frac{1}{3},\frac{1}{2}\right),\\ \min\left(\frac{1}{3},\frac{1}{2}\right)\end{pmatrix},\begin{pmatrix}max\left(\frac{1}{6},\frac{1}{4}\right),\\ max\left(\frac{1}{6},\frac{1}{4}\right)\end{pmatrix}\right\}$

$$= \left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{4},\frac{1}{4}\right)\right\}$$

Hence proved

***3.3.3. Property (Distributive Laws)***

If $A(\rho_{XY1}(x,y))$, $B(\rho_{XY2}(x,y))$ and $C(\rho_{XY3}(x,y))$ be two-dimensional imprecise number, then

(i) $A(\rho_{XY1}(x,y)) \cap \left(B(\rho_{XY2}(x,y)) \cup C(\rho_{XY3}(x,y))\right)$

$= \left(A\left(\rho_{XY1}(x,y)\right) \cap B\left(\rho_{XY2}(x,y)\right)\right) \cup \left(A\left(\rho_{XY1}(x,y)\right) \cap C\left(\rho_{XY3}(x,y)\right)\right)$

(ii) $A(\rho_{XY1}(x,y)) \cup \left(B(\rho_{XY2}(x,y)) \cap C(\rho_{XY3}(x,y))\right)$

$= \left(A\left(\rho_{XY1}(x,y)\right) \cup B\left(\rho_{XY2}(x,y)\right)\right) \cap \left(A\left(\rho_{XY1}(x,y)\right) \cup C\left(\rho_{XY3}(x,y)\right)\right)$

Let us prove the property 3.3.3.(i) and 3.3.3.(ii) with example.

If $A(\rho_{XY1}(x,y)) = \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{4},\frac{1}{4}\right)\right\}$, $B(\rho_{XY2}(x,y)) = \left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{5},\frac{1}{5}\right)\right\}$

and $C(\rho_{XY3}(x,y)) = \left\{\left(\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{7},\frac{1}{7}\right)\right\}$. Then,

**Proof:**

(i) $A(\rho_{XY1}(x,y)) \cap \left(B(\rho_{XY2}(x,y)) \cup C(\rho_{XY3}(x,y))\right)$

$$= \left(\left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{4},\frac{1}{4}\right)\right\} \cap \left\{\begin{matrix}\left(\max\left(\frac{1}{3},\frac{1}{6}\right),\max\left(\frac{1}{3},\frac{1}{6}\right)\right),\\ \left(min\left(\frac{1}{5},\frac{1}{7}\right),min\left(\frac{1}{5},\frac{1}{7}\right)\right)\end{matrix}\right\}\right)$$

$$= \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{4},\frac{1}{4}\right)\right\} \cap \left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{7},\frac{1}{7}\right)\right\}$$

$$= \left\{\begin{matrix}\left(\min\left(\frac{1}{2},\frac{1}{3}\right),\min\left(\frac{1}{2},\frac{1}{3}\right)\right),\\ \left(max\left(\frac{1}{4},\frac{1}{7}\right),max\left(\frac{1}{4},\frac{1}{7}\right)\right)\end{matrix}\right\}$$

$$= \left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{4},\frac{1}{4}\right)\right\}$$

and $\left(A(\rho_{XY1}(x,y)) \cap B(\rho_{XY2}(x,y))\right) \cup \left(A(\rho_{XY1}(x,y)) \cap C(\rho_{XY3}(x,y))\right)$

$$= \left\{ \begin{matrix} \left(\min\left(\frac{1}{2},\frac{1}{3}\right),\min\left(\frac{1}{2},\frac{1}{3}\right)\right), \\ \left(max\left(\frac{1}{4},\frac{1}{5}\right),max\left(\frac{1}{4},\frac{1}{5}\right)\right) \end{matrix} \right\} \cup \left\{ \begin{matrix} \left(\min\left(\frac{1}{2},\frac{1}{6}\right),\min\left(\frac{1}{2},\frac{1}{6}\right)\right), \\ \left(max\left(\frac{1}{4},\frac{1}{7}\right),max\left(\frac{1}{4},\frac{1}{7}\right)\right) \end{matrix} \right\}$$

$$= \left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{4},\frac{1}{4}\right)\right\} \cup \left\{\left(\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{4},\frac{1}{4}\right)\right\}$$

$$= \left\{ \begin{matrix} \left(\max\left(\frac{1}{3},\frac{1}{6}\right),\max\left(\frac{1}{3},\frac{1}{6}\right)\right), \\ \left(min\left(\frac{1}{4},\frac{1}{4}\right),min\left(\frac{1}{4},\frac{1}{4}\right)\right) \end{matrix} \right\}$$

$$= \left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{4},\frac{1}{4}\right)\right\}$$

Hence proved

$$A(\rho_{XY1}(x,y)) \cup \left(B(\rho_{XY2}(x,y)) \cap C(\rho_{XY3}(x,y))\right)$$

$$= \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{4},\frac{1}{4}\right)\right\} \cup \left\{ \begin{matrix} \left(\min\left(\frac{1}{3},\frac{1}{6}\right),\min\left(\frac{1}{3},\frac{1}{6}\right)\right), \\ \left(max\left(\frac{1}{5},\frac{1}{7}\right),max\left(\frac{1}{5},\frac{1}{7}\right)\right) \end{matrix} \right\}$$

$$= \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{4},\frac{1}{4}\right)\right\} \cup \left\{\left(\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{5},\frac{1}{5}\right)\right\}$$

$$= \left\{ \begin{matrix} \left(\max\left(\frac{1}{2},\frac{1}{6}\right),\max\left(\frac{1}{2},\frac{1}{6}\right)\right), \\ \left(min\left(\frac{1}{4},\frac{1}{5}\right),min\left(\frac{1}{4},\frac{1}{5}\right)\right) \end{matrix} \right\}$$

$$= \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{5},\frac{1}{5}\right)\right\}$$

And $\left(A(\rho_{XY1}(x,y)) \cup B(\rho_{XY2}(x,y))\right) \cap \left(A(\rho_{XY1}(x,y)) \cup C(\rho_{XY3}(x,y))\right)$

$$= \left\{ \begin{matrix} \left(\max\left(\frac{1}{2},\frac{1}{3}\right),\max\left(\frac{1}{2},\frac{1}{3}\right)\right), \\ \left(msin\left(\frac{1}{4},\frac{1}{5}\right),min\left(\frac{1}{4},\frac{1}{5}\right)\right) \end{matrix} \right\} \cap \left\{ \begin{matrix} \left(\max\left(\frac{1}{2},\frac{1}{6}\right),\max\left(\frac{1}{2},\frac{1}{6}\right)\right), \\ \left(min\left(\frac{1}{4},\frac{1}{7}\right),min\left(\frac{1}{4},\frac{1}{7}\right)\right) \end{matrix} \right\}$$

$$= \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{5},\frac{1}{5}\right)\right\} \cap \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{7},\frac{1}{7}\right)\right\}$$

$$= \left\{ \begin{matrix} \left(\min\left(\frac{1}{2},\frac{1}{2}\right),\min\left(\frac{1}{2},\frac{1}{2}\right)\right), \\ \left(max\left(\frac{1}{5},\frac{1}{7}\right),max\left(\frac{1}{5},\frac{1}{7}\right)\right) \end{matrix} \right\}$$

$$= \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{5},\frac{1}{5}\right)\right\}$$

Hence proved

### *3.3.4. Property(Idempotence Laws)*

*(i)* $A(\rho_{XY}(x,y)) \cap A\left(\rho_{XY}(x,y)\right) = A(\rho_{XY}(x,y))$

*(ii)* $A(\rho_{XY}(x,y)) \cup A(\rho_{XY}(x,y)) = A(\rho_{XY}(x,y))$

The properties are obviously true.

***3.3.5. Property (Identity Laws)***

(i) $A(\rho_{XY}(x,y)) \cap \emptyset(\rho_{XY}(x,y)) = \emptyset(\rho_{XY}(x,y))$

(ii) $A(\rho_{XY}(x,y)) \cup \emptyset(\rho_{XY}(x,y)) = A(\rho_{XY}(x,y))$

(iii) $A(\rho_{XY}(x,y)) \cap X(\rho_{XY}(x,y)) = A(\rho_{XY}(x,y))$

(iv) $A(\rho_{XY}(x,y)) \cup X(\rho_{XY}(x,y)) = X(\rho_{XY}(x,y))$

Where $X(\rho_{XY}(x,y))$ is universal set and $\emptyset(\rho_{XY}(x,y))$ is null set.

To prove the property 3.3.5. (i) and 3.3.5. (ii) let us consider,

$A(\rho_{XY1}(x,y)) = \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{3},\frac{1}{3}\right)\right\}$ and $\emptyset(\rho_{XY}(x,y)) = \left\{(0,0),\left(\frac{1}{3},\frac{1}{3}\right)\right\}$, be such that membership function of the imprecise number of $A(\rho_{XY1}(x,y))$ is $\left(\frac{1}{2},\frac{1}{2}\right)$ and is measured from the reference function $\left(\frac{1}{3},\frac{1}{3}\right)$, where $\left(\frac{1}{2},\frac{1}{2}\right)$ and $\left(\frac{1}{3},\frac{1}{3}\right)$ are the half portion, one third portion of the two dimensional object respectively.

$\emptyset(\rho_{XY}(x,y)) = \left\{(0,0),\left(\frac{1}{3},\frac{1}{3}\right)\right\}$ is a null imprecise number measured from the one third potion of the two dimensional objects. Here membership function is zero due to null .Then,

**Proof:**

(i) $$A(\rho_{XY1}(x,y)) \cap \emptyset(\rho_{XY}(x,y)) = \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{3},\frac{1}{3}\right)\right\} \cap \left\{(0,0),\left(\frac{1}{3},\frac{1}{3}\right)\right\}$$

$$= \left\{\begin{matrix}\left(\min\left(\frac{1}{2},0\right),\min\left(\frac{1}{2},0\right)\right), \\ \left(max\left(\frac{1}{3},\frac{1}{3}\right),max\left(\frac{1}{3},\frac{1}{3}\right)\right)\end{matrix}\right\}$$

$$= \left\{(0,0),\left(\frac{1}{3},\frac{1}{3}\right)\right\}$$

$$= \emptyset(\rho_{XY1}(x,y))$$

Hence proved

(ii) $$A(\rho_{XY1}(x,y)) \cup \emptyset(\rho_{XY}(x,y)) = \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{3},\frac{1}{3}\right)\right\} \cup \{(0,0),(1,1)\}$$

$$= \left\{\begin{matrix}\left(\max\left(\frac{1}{2},0\right),\max\left(\frac{1}{2},0\right)\right), \\ \left(min\left(\frac{1}{3},\frac{1}{3}\right),min\left(\frac{1}{3},\frac{1}{3}\right)\right)\end{matrix}\right\}$$

$$= \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{3},\frac{1}{3}\right)\right\}$$

$$= A(\rho_{XY1}(x,y))$$

Hence proved

To prove the property 3.3.5. (iii) and 3.3.5. (iv), let us consider $A((\rho_{XY1}(x,y)) = \left\{\left(\frac{1}{4},\frac{1}{4}\right),\left(\frac{1}{5},\frac{1}{5}\right)\right\}$ and $X((\rho_{XY}(x,y)) = \left\{\left(\frac{3}{4},\frac{3}{4}\right),\left(\frac{1}{5},\frac{1}{5}\right)\right\}$, be such that membership function of the imprecise number of $A((\rho_{XY1}(x,y))$ is $\left(\frac{1}{4},\frac{1}{4}\right)$ and measured from reference function, $\left(\frac{1}{5},\frac{1}{5}\right)$, where $\left(\frac{1}{4},\frac{1}{4}\right)$ and $\left(\frac{1}{5},\frac{1}{5}\right)$ are the one fourth portion, one fifth portion of the two-dimensional object respectively.

$X((\rho_{XY}(x,y)) = \left\{\left(\frac{3}{4},\frac{3}{4}\right),\left(\frac{1}{5},\frac{1}{5}\right)\right\}$ is the universal imprecise number measured from the one fifth potion of the two-dimensional object. Here membership function is three fourth portion of two-dimensional object and is greater than the membership value of $A((\rho_{XY1}(x,y))$. Then

(iii) $$A(\rho_{XY1}(x,y)) \cap \emptyset(\rho_{XY}(x,y)) = \left\{\left(\frac{1}{4},\frac{1}{4}\right),\left(\frac{1}{5},\frac{1}{5}\right)\right\} \cap \left\{\left(\frac{3}{4},\frac{3}{4}\right),\left(\frac{1}{5},\frac{1}{5}\right)\right\}$$
$$= \left\{\begin{matrix}\left(\min\left(\frac{1}{4},\frac{3}{4}\right),\min\left(\frac{1}{4},\frac{3}{4}\right)\right),\\ \left(max\left(\frac{1}{5},\frac{1}{5}\right),max\left(\frac{1}{5},\frac{1}{5}\right)\right)\end{matrix}\right\}$$
$$= \left\{\left(\frac{1}{4},\frac{1}{4}\right),\left(\frac{1}{5},\frac{1}{5}\right)\right\}$$
$$= \emptyset(\rho_{XY1}(x,y))$$

Hence proved

(iv) $$A(\rho_{XY1}(x,y)) \cup \emptyset(\rho_{XY}(x,y)) = \left\{\left(\frac{1}{4},\frac{1}{4}\right),\left(\frac{1}{5},\frac{1}{5}\right)\right\} \cup \left\{\left(\frac{3}{4},\frac{3}{4}\right),\left(\frac{1}{5},\frac{1}{5}\right)\right\}$$
$$= \left\{\begin{matrix}\left(\max\left(\frac{1}{4},\frac{3}{4}\right),\max\left(\frac{1}{4},\frac{3}{4}\right)\right),\\ \left(min\left(\frac{1}{5},\frac{1}{5}\right),min\left(\frac{1}{5},\frac{1}{5}\right)\right)\end{matrix}\right\}$$
$$= \left\{\left(\frac{3}{4},\frac{3}{4}\right),\left(\frac{1}{5},\frac{1}{5}\right)\right\}$$
$$= X((\rho_{XY1}(x,y))$$

Hence proved

### 3.3.6. *Property (Associatively Laws)*

If $A(\rho_{XY}(x,y)) = \{\rho_{XY1}(x,y),\rho_{XY2}(x,y):(x,y) \in X \times Y\}$,

$B(\rho_{XY}(x,y)) = \{\rho_{XY3}(x,y),\rho_{xy4}(x,y))(x,y) \in X \times Y\}$

and $C(\rho_{XY}(x,y)) = \{\rho_{XY5}(x,y),\rho_{XY6}(x,y):(x,y) \in X \times Y\}$ be two-dimensional imprecise numbers, then

(i) $A\big(\rho_{XY}(x,y)\big) \cup \Big(B\big(\rho_{XY}(x,y)\big) \cup C\big(\rho_{XY}(x,y)\big)\Big) = \Big(A\big(\rho_{XY}(x,y)\big) \cup B\big(\rho_{XY}(x,y)\big)\Big) \cup C\big(\rho_{XY}(x,y)\big)$

(ii) $A\big(\rho_{XY}(x,y)\big) \cap \Big(B\big(\rho_{XY}(x,y)\big) \cap C\big(\rho_{XY}(x,y)\big)\Big) = \Big(A\big(\rho_{XY}(x,y)\big) \cap B\big(\rho_{XY}(x,y)\big)\Big) \cap C(\rho_{XY}(x,y))$

To prove this property, let us consider $A(\rho_{XY}(x,y)) = \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}$, $B(\rho_{XY}(x,y)) = \left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}$ and $C(\rho_{XY}(x,y)) = \left\{\left(\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}$. Then,

**Proof:**

(i) $A\big(\rho_{XY}(x,y)\big) \cup \Big(B\big(\rho_{XY}(x,y)\big) \cup C\big(\rho_{XY}(x,y)\big)\Big)$

$$= \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\} \cup \left(\left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\} \cup \left\{\left(\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}\right)$$

$$= \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\} \cup \left\{\begin{matrix}\left(\max\left(\frac{1}{3},\frac{1}{6}\right),\max\left(\frac{1}{3},\frac{1}{6}\right)\right), \\ \left(min\left(\frac{1}{9},\frac{1}{9}\right),min\left(\frac{1}{9},\frac{1}{9}\right)\right)\end{matrix}\right\}$$

$$= \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\} \cup \left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}$$

$$= \left\{\begin{matrix}\left(\max\left(\frac{1}{2},\frac{1}{3}\right),\max\left(\frac{1}{2},\frac{1}{3}\right)\right), \\ \left(min\left(\frac{1}{9},\frac{1}{9}\right),min\left(\frac{1}{9},\frac{1}{9}\right)\right)\end{matrix}\right\}$$

$$= \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}$$

$\Big(A\big(\rho_{XY}(x,y)\big) \cup B\big(\rho_{XY}(x,y)\big)\Big) \cup C\big(\rho_{XY}(x,y)\big)$

$$= \left(\left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\} \cup \left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}\right) \cup \left\{\left(\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}$$

$$= \left\{\begin{matrix}\left(\max\left(\frac{1}{2},\frac{1}{3}\right),\max\left(\frac{1}{2},\frac{1}{3}\right)\right), \\ \left(min\left(\frac{1}{9},\frac{1}{9}\right),min\left(\frac{1}{9},\frac{1}{9}\right)\right)\end{matrix}\right\} \cup \left\{\left(\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}$$

$$= \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\} \cup \left\{\left(\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}$$

$$= \left\{\begin{matrix}\left(\max\left(\frac{1}{2},\frac{1}{6}\right),\max\left(\frac{1}{2},\frac{1}{6}\right)\right), \\ \left(min\left(\frac{1}{9},\frac{1}{9}\right),min\left(\frac{1}{9},\frac{1}{9}\right)\right)\end{matrix}\right\}$$

$$= \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}$$

Hence proved

(ii) $A(\rho_{XY}(x,y)) \cap \big(B(\rho_{XY}(x,y)) \cap C(\rho_{XY}(x,y))\big)$

$$= \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\} \cap \left(\left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\} \cap \left\{\left(\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}\right)$$

$$= \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\} \cap \left\{\begin{matrix}\left(\min\left(\frac{1}{3},\frac{1}{6}\right),\min\left(\frac{1}{3},\frac{1}{6}\right)\right),\\ \left(max\left(\frac{1}{9},\frac{1}{9}\right),max\left(\frac{1}{9},\frac{1}{9}\right)\right)\end{matrix}\right\}$$

$$= \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\} \cap \left\{\left(\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}$$

$$= \left\{\begin{matrix}\left(\min\left(\frac{1}{2},\frac{1}{6}\right),\min\left(\frac{1}{2},\frac{1}{6}\right)\right),\\ \left(max\left(\frac{1}{9},\frac{1}{9}\right),max\left(\frac{1}{9},\frac{1}{9}\right)\right)\end{matrix}\right\}$$

$$= \left\{\left(\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}$$

$\big(A(\rho_{XY}(x,y)) \cap B(\rho_{XY}(x,y))\big) \cap C(\rho_{XY}(x,y))$

$$= \left(\left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\} \cap \left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}\right) \cap \left\{\left(\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}$$

$$= \left\{\begin{matrix}\left(\min\left(\frac{1}{2},\frac{1}{3}\right),\min\left(\frac{1}{2},\frac{1}{3}\right)\right),\\ \left(max\left(\frac{1}{9},\frac{1}{9}\right),max\left(\frac{1}{9},\frac{1}{9}\right)\right)\end{matrix}\right\} \cap \left\{\left(\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}$$

$$= \left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\} \cap \left\{\left(\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}$$

$$= \left\{\begin{matrix}\left(\min\left(\frac{1}{3},\frac{1}{6}\right),\min\left(\frac{1}{3},\frac{1}{6}\right)\right),\\ \left(max\left(\frac{1}{9},\frac{1}{9}\right),max\left(\frac{1}{9},\frac{1}{9}\right)\right)\end{matrix}\right\}$$

$$= \left\{\left(\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}$$

Hence proved

### *3.3.7. Property (De-Morgan's Laws)*

If $A(\rho_{XY}(x,y)) = \{\rho_{XY1}(x,y), \rho_{XY2}(x,y): (x,y) \in X \times Y\}$,
$B(\rho_{XY}(x,y)) = \{\rho_{XY3}(x,y), \rho_{XY4}(x,y): (x,y) \in X \times Y\}$ be two-dimensional imprecise numbers. Then,

(i) $\big(A(\rho_{XY}(x,y)) \cup B(\rho_{XY}(x,y))\big)^C = A^C(\rho_{XY}(x,y)) \cap B^C(\rho_{XY}(x,y))$

(ii) $\big(A(\rho_{XY}(x,y)) \cap B(\rho_{XY}(x,y))\big)^C = A^C(\rho_{XY}(x,y)) \cup B^C(\rho_{XY}(x,y))$

To prove this property let us take the above two dimensional imprecise numbers $A(\rho_{XY}(x,y)) = \left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}$ and $B(\rho_{XY}(x,y)) = \left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}$. Then,

**Proof:**

(i) $\left(A(\rho_{XY}(x,y)) \cup B(\rho_{XY}(x,y))\right)^C$

$$= \left(\left\{\begin{matrix}\left(\max\left(\frac{1}{2},\frac{1}{3}\right),\max\left(\frac{1}{2},\frac{1}{3}\right)\right),\\ \left(min\left(\frac{1}{9},\frac{1}{9}\right),min\left(\frac{1}{9},\frac{1}{9}\right)\right)\end{matrix}\right\}\right)^C$$

$$= \left(\left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}\right)^C$$

$$= \left\{(1,1),\left(\frac{1}{2},\frac{1}{2}\right)\right\}$$

$$A^C(\rho_{XY}(x,y)) \cap B^C(\rho_{XY}(x,y)) = \left(\left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}\right)^C \cap \left(\left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}\right)^C$$

$$= \left\{(1,1),\left(\frac{1}{2},\frac{1}{2}\right)\right\} \cap \left\{(1,1),\left(\frac{1}{3},\frac{1}{3}\right)\right\}$$

$$= \left\{\begin{matrix}(\min(1,1),\min(1,1)),\\ \left(max\left(\frac{1}{2},\frac{1}{3}\right),max\left(\frac{1}{2},\frac{1}{3}\right)\right)\end{matrix}\right\}$$

$$= \left\{(1,1),\left(\frac{1}{2},\frac{1}{2}\right)\right\}$$

Hence proved

(ii) $$\left(A(\rho_{XY}(x,y)) \cap B(\rho_{XY}(x,y))\right)^C = \left(\left\{\begin{matrix}\left(\min\left(\frac{1}{2},\frac{1}{3}\right),\min\left(\frac{1}{2},\frac{1}{3}\right)\right),\\ \left(max\left(\frac{1}{9},\frac{1}{9}\right),max\left(\frac{1}{9},\frac{1}{9}\right)\right)\end{matrix}\right\}\right)^C$$

$$= \left(\left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}\right)^C$$

$$= \left\{(1,1),\left(\frac{1}{3},\frac{1}{3}\right)\right\}$$

$$A({\rho_{XY}}^C(x,y)) \cup B({\rho_{XY}}^C(x,y)) = \left(\left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}\right)^C \cup \left(\left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{9},\frac{1}{9}\right)\right\}\right)^C$$

$$= \left\{(1,1),\left(\frac{1}{2},\frac{1}{2}\right)\right\} \cup \left\{(1,1),\left(\frac{1}{3},\frac{1}{3}\right)\right\}$$

$$= \left\{\begin{matrix}(\max(1,1),\max(1,1)),\\ \left(min\left(\frac{1}{2},\frac{1}{3}\right),min\left(\frac{1}{2},\frac{1}{3}\right)\right)\end{matrix}\right\}$$

$$= \left\{(1,1),\left(\frac{1}{3},\frac{1}{3}\right)\right\}$$

Hence proved

Here, membership function of $A(\rho_{XY}(x,y))$ and $B(\rho_{XY}(x,y))$ are $\left(\frac{1}{2},\frac{1}{2}\right)$ and $\left(\frac{1}{3},\frac{1}{3}\right)$ and are measured from the reference function, $\left(\frac{1}{9},\frac{1}{9}\right)$. So the complement of $A(\rho_{XY}(x,y))$ and $B(\rho_{XY}(x,y))$ are measured from reference function $\left(\frac{1}{2},\frac{1}{2}\right)$ and $\left(\frac{1}{3},\frac{1}{3}\right)$ till to the highest membership function of the two dimensional imprecise function $(1,1)$.

## 3.4. Application of Two-dimensional Imprecise Numbers

Two-dimensional imprecise numbers can be obtained as an application in the field of economics. Assume that 60% of the effort of production of different crops of our country is done every year to fulfill the 75% needs or demand of our people. So, the production of crops should be increased to 85% so that demand can be fulfill hundred percent. In this case demand and the production situation can be expressed in two-dimensional imprecise number. Thus, imprecise number is obtained in the following form,

$$A\big(\rho_{XY}(x,y)\big) = \{(100\%, 85\%), (75\%\ ,60\%)\}$$
$$= \left\{\left(1,\frac{17}{20}\right),\left(\frac{3}{4},\frac{3}{5}\right)\right\}.$$

Here, membership value can be model in the following form,

$$A\{\rho_{XY1}(x,y), \rho_{XY2}(x,y): (x,y) \in X \times Y\} = |x_1 - x_2|, |y_1 - y_2|;$$
$$\rho_{XY1}(x,y) = (x_1, y_1), \rho_{XY2}(x,y) = (x_2, y_2)$$

Where $|x_1 - x_2|$ and $|y_1 - y_2|$ are distinct behaviors. In case the behaviors of variables are similar, then their product will be the membership function of the above imprecise number. Otherwise membership function will be counted separately.

Thus, membership function of above demand and production problem is, $\left|1-\frac{3}{4}\right|, \left|\frac{17}{20}-\frac{3}{5}\right| = \left|\frac{1}{4}\right|, \left|\frac{5}{20}\right| = \left|\frac{1}{4}\right|, \left|\frac{1}{4}\right|$. Which shows that 25% more effort of production and demand has to increase to fulfill the people need of our country.

## 3.5. Conclusions

Solution of practical problem is the main objective of this study. So the possible effect of fuzziness of any particle is suggested to study for all the axes. If the problem is solved in one part of the axis may not overcome for whole of the body because of not study of another part of the body. All types of imprecise numbers are assumed as classical set to have the same characters and properties. So, the characters of classical sets are helped to identify the characters of two-dimensional imprecise numbers. Operations of intersection and union of sets are the tools to obtain the properties of two-dimensional imprecise numbers.