Chapter 6

COVERING ENERGY OF A SEMIGRAPH

6.1 Introduction

Chemical graph energy is the concept stems from chemistry to approximate the total π-electron energy of a molecule. In chemistry the conjugate hydrocarbon can be represented by graph called molecular graph, in which the atoms of a molecule are represented by the vertices and the chemical bonds are represented by the edges. Ivan Gutman [22] first introduced the energy for chemical graphs in the year 1978 and defined as the sum of the absolute value of eigenvalues of the adjacency matrix of a graph. Further, many authors conceived on different types of graph energy like color energy [6, 34, 35], the minimum covering energy [5], distance energy [19] etc. of a graph.

In the year 2012 Adiga *et.al*. [5] introduced a matrix, called minimum covering matrix of a graph and its energy, and defined as follows:

Suppose $G(V, X)$ be a graph of order *n* and size *m*, with vertex set *V* and edges set *X*. Let *C* subset of *V* be the minimum covering set of a graph *G*. The minimum covering matrix of *G* is the square matrix $A_{mc}(G) = (a_{ij})$ of order *n*, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 1 & \text{if } i = j \text{ and } v_i \in C \\ 0 & \text{otherwise.} \end{cases}$$

And the minimum covering energy of the graph *G* is defined as $E_{mc}(G) = \sum_{i=1}^{n} |\lambda_i|$ where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of the minimum covering matrix $A_{mc}(G)$.

Adiga *et.al.* [6], have introduced the concept of color matrix and energy of a graph and investigated many properties and results. Further in the year 2015, M. R. Rajesh Kanna [30] *et.al*. investigated minimum covering color energy of a graph and their definitions are given below:

Let G be a vertex-colored graph of order *n*. Let *C* subset of *V* be the minimum covering set of a graph *G*. Then the minimum covering color matrix of G is the matrix $A_{mc}^{c}(G) = (a_{ij})_{n\times n}$ of which,

$$a_{ij}(v_i, v_j) = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent or if } i = j \text{ and } v_i \in C \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i) = c(v_j), \\ 0 & \text{otherwise.} \end{cases}$$

where $c(v_i)$ is the color of the vertex v_i in G. Recall that, the vertices of the graph G are colored so that two adjacent vertices always have different colors. The minimum covering color energy E_{mc}^{c} of a graph G with respect to a given coloring is the sum of the absolute value of eigenvalues of the minimum covering color matrix $A_{mc}^{c}(G)$.

Motivated by these, we have extended the minimum covering energy of semigraphs in section 6.2 and minimum covering color energy of a semigraph in section 6.3.

6.2 On minimum covering matrix and energy of semigraphs

In this section a new type of matrix, called minimum covering matrix of a semigraph was introduced and obtained its energy. The minimum covering matrix of a semigraph is defined as follows:

The minimum covering matrix of a semigraph:

If $G(V, X)$ be a semigraph of order n size m. Let *C* be the minimum covering set, then the minimum covering matrix of *G* is the square matrix $A_{mc}(G) = (a_{ij})$ of order n, where

i. For every edge e_i of *X* of cardinality, say *k*, let $e_i = (i_1, i_2, i_3, \ldots, i_k)$ such that $i_1, i_2, i_3, \ldots, i_k$ are distinct vertices in V , for all $i_r \in e_i$; r $= 1,2, \ldots, k$

   (a) $a_{i_1 i_r} = r - 1,$

   (b) $a_{i_k i_r} = k - r$

ii. $a_{ij} = 1$ if $i = j$ and $v_i \in C$.

iii. All the remaining entries of *A* are zero.

The minimum covering energy of semigraphs:

Nikiforov [60] defined the energy of a general matrix (of any size) as the summation of the singular values of that matrix.

Thus, if $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the singular values of minimum covering matrix $A_{mc}(G)$ of the semigraph G, then the minimum covering energy of a semigraph denoted by $E_{mc}(G)$, is defined as the summation of its singular values. i.e.

$$E_{mc}(G) = \sum_{i=1}^{n} \sigma_i$$

We observe that, $A_{mc}(G)A'_{mc}(G)$ is a positive semidefinite matrix. So, its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are non-negative and therefore the singular values of $A_{mc}(G)$ are non-negative real numbers. Thus $E_{mc}(G) \geq 0$, equality holds if and only if the number of edges in G is zero. Minimum covering energy of a semigraph is well defined, as if G' be a semigraph obtained by relabeling of the vertices of G, then $A_{mc}(G')A'_{mc}(G')$ is obtained by interchanging the rows and the corresponding columns of $A_{mc}(G)A'_{mc}(G)$. Hence the eigenvalues of $A_{mc}(G)A'_{mc}(G)$ and $A_{mc}(G')A'_{mc}(G')$ are same, and so the singular values of G and $G'$are also same.

Example 6.1 $G(V, X)$ be a connected semigraph as shown in **Figure 6.1** having vertex set $V = \{1,2,3,4,5,6,7,8\}$ and let $C = \{3, 4, 7\}$ be the minimum covering set. And $X = \{(1,2,3), (3,4), (4,5,6), (6,7,3), (7,8)\}$ be the edge set of G. Then,

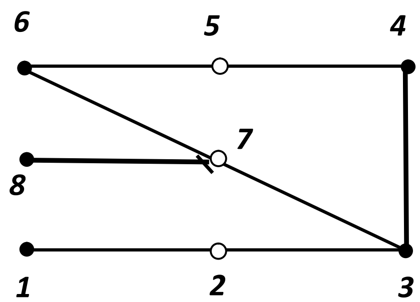

Figure 6.1

Then, Minimum covering matrix $A_{mc}(G)$ of the semigraph $G(V,X)$ is

$$A_{mc}(G) = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

6.2.1 Properties of minimum covering energy of semigraphs

Lemma 6.1 Let $A_{mc}(G)$ is the minimum covering matrix of a semigraph G, and C is its minimum covering set. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $A_{mc}(G)A'_{mc}(G)$. Then

$$\sum_{i=1}^{n} \lambda_i = 2\sum_{e\in X}\left(1^2 + 2^2 + \ldots + {k_e}^2\right) + |C|$$

where the cardinality of an edge $e \in X$ of the semigraph is $k_e + 1$and $k_e \geq 1$.

Proof: In the minimum covering matrix $A_{mc}(G)$, corresponding to every edge $e \in X$ of cardinality $k_e + 1$, there is a sequence $\{1,2,\ldots,k_e\}$ in the rows corresponding to the end vertices of that edge. And there are $|C|$ nos. of 1's in the diagonal of $A_{mc}(G)$. Thus every edge contributes $2\sum_e\left(1^2 + 2^2 + \ldots + {k_e}^2\right)$ and the diagonal elements contribute $|C| \times 1^2$ in the trace of $A_{mc}A'_{mc}$.

$$Therefore \qquad Trac(A_{mc}A'_{mc}) = 2\sum_{e\in X}\left(1^2 + 2^2 + \ldots + {k_e}^2\right) + |C| \times 1^2$$

$$Hence \qquad \sum_{i=1}^{n} \lambda_i = 2\sum_{e\in X}\left(1^2 + 2^2 + \ldots + {k_e}^2\right) + |C|$$

Theorem 6.1 The minimum covering energy $E_{mc}(G)$ of a semigraph G, is a square root of an even or odd integer according as $|C|$ is even or odd.

Proof: If $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the singular values of minimum covering matrix $A_{mc}(G)$ of the semigraph G, then

$$(\sigma_1 + \sigma_2 + \ldots\ldots + \sigma_n)^2 = \sum_{i=1}^{n} \sigma_i^2 + 2\sum_{i<j} \sigma_i \sigma_j$$

Thus

$$[E_{mc}(G)]^2 = \sum_{i=1}^{n} \lambda_i + 2\sum_{i<j} \sigma_i \sigma_j$$

$$= 2\sum_{e\in X}\left(1^2 + 2^2 + \ldots + {k_e}^2\right) + |C| + 2\sum_{i<j} \sigma_i \sigma_j$$

$$= 2\left[\sum_{e\in X}\left(1^2 + 2^2 + \ldots + {k_e}^2\right) + \sum_{i<j} \sigma_i \sigma_j\right] + |C|$$

$$E_{mc}(G) = \sqrt{2\left[\sum_{e\in X}\left(1^2 + 2^2 + \ldots + {k_e}^2\right) + \sum_{i<j} \sigma_i \sigma_j\right] + |C|}$$

Thus the minimum covering energy $E_{mc}(G)$ of a semigraph *G*, is a square root of an even or odd integer according as $|C|$ is even or odd.

Theorem 6.2 The minimum covering energy $E_{mc}(G)$ of a semigraph *G*, then

$$[E_{mc}(G)]^2 = |C| (mod\, 2)$$

Proof: By **Theorem 6.1,** the minimum covering energy $E_{mc}(G)$ of a semigraph *G*, is a square root of an even or odd integer according as $|C|$ is even or odd.

$$E_{mc}(G) = \sqrt{2t + |C|}$$

i.e. $\quad [E_{mc}(G)]^2 = 2t + |C|$

Thus, $\quad [E_{mc}(G)]^2 = |C| (mod\, 2).$

6.2.2 Some bounds on minimum covering energy of semigraphs

Theorem 6.3 If *G* be a semigraph having *n* vertices and *m* edges,

$$\sqrt{2\sum_{e\in X}\left(1^2 + 2^2 + \ldots + {k_e}^2\right) + |C|} \le E_{mc}(G) \le \sqrt{n\left[2\sum_{e\in X}(1^2 + 2^2 + \ldots + k_e^2) + |C|\right]}$$

Proof: Let $\sigma_i, i = 1,2,\ldots,n$ be the singular values of minimum covering matrix A_{mc}, and $\lambda_i, i = 1,2,\ldots,n$ be the eigenvalues of $A_{mc}A'_{mc}$. By Cauchy- Schwarz's inequality on two vector $(\sigma_1, \sigma_2, \ldots\ldots, \sigma_n)$ and $(1,1,\ldots\ldots,1)$, we have

$$(\sigma_1 + \sigma_2 + \ldots\ldots + \sigma_n)^2 \leq n \sum_{i=1}^{n} \sigma_i^2 = n \sum_{i=1}^{n} \lambda_i$$

$$Thus, \qquad [E_{mc}(G)]^2 \leq n \left[2 \sum_{e \in X} \left(1^2 + 2^2 + \ldots + {k_e}^2 \right) + |C| \right]$$

Again, we have

$$[E_{mc}(G)]^2 = \left(\sum_{i=1}^{n} \sigma_i \right)^2 \geq \sum_{i=1}^{n} \sigma_i^2 = \sum_{i=1}^{n} \lambda_i$$

$$i.e. \qquad [E_{mc}(G)]^2 \geq 2 \sum_{e \in X} \left(1^2 + 2^2 + \ldots + {k_e}^2 \right) + |C|$$

Hence

$$\sqrt{2 \sum_{e \in X} \left(1^2 + 2^2 + \ldots + {k_e}^2 \right) + |C|} \leq E_{mc}(G) \leq \sqrt{n \left[2 \sum_{e \in X} (1^2 + 2^2 + \ldots + k_e^2) + |C| \right]}$$

Theorem 6.4. If *G* be a semigraph having *n* vertices and *m* edges, then

$$[E_{mc}(G)]^2 \geq 2 \sum_{e \in X} \left(1^2 + 2^2 + \ldots + {k_e}^2 \right) + |C| + n(n-1)\Delta^{1/n},$$

Where $\Delta = det(A_{mc}A'_{mc})$.

Proof: Let $\sigma_i,\ i = 1,2,\ldots\ldots,n$ be the singular values of A_{mc}, then we have,

$$[E_{mc}(G)]^2 = \left(\sum_{i=1}^{n} \sigma_i \right)^2 = \sum_{i=1}^{n} \sigma_i^2 + 2 \sum_{i<j} \sigma_i \sigma_j = \sum_{i=1}^{n} \lambda_i + 2 \sum_{i \neq j} \sigma_i \sigma_j$$

As $\sigma_i,\ i = 1,2,\ldots\ldots,n$ are non-negative, so $n(n-1)$ nos. of $\sigma_i \sigma_j$ are also non-negative number.

Therefore, applying $AM \geq GM$ on $n(n-1)$ nos. of non-negative numbers $\sigma_i \sigma_j$. We have

$$\frac{1}{n(n-1)}\sum_{i\neq j}\sigma_i\sigma_j \geq \left(\prod_{i\neq j}\sigma_i\sigma_j\right)^{\frac{1}{n(n-1)}} = \left(\prod_{i=1}^{n}\sigma_i^{2(n-1)}\right)^{\frac{1}{n(n-1)}}$$

$$i.e. \qquad \sum_{i\neq j}\sigma_i\sigma_j \geq n(n-1)\left(\prod_{i=1}^{n}\lambda_i^{n-1}\right)^{\frac{1}{n(n-1)}} = n(n-1)\left(\prod_{i=1}^{n}\lambda_i\right)^{\frac{1}{n}}$$

$$Thus \qquad \sum_{i\neq j}\sigma_i\sigma_j \geq n(n-1)\Delta^{\frac{1}{n}}$$

$$Where \qquad \Delta = \prod_{i=1}^{n}\lambda_i = det(A_{mc}A'_{mc})$$

Therefore, we get

$$[E_{mc}(G)]^2 \geq \sum_{i=1}^{n}\lambda_i + n(n-1)\Delta^{\frac{1}{n}}$$

by **Lemma 6.1** we obtain

$$[E_{mc}(G)]^2 \geq 2\sum_{e\in X}\left(1^2+2^2+\ldots+{k_e}^2\right)+|C|+n(n-1)\Delta^{1/n}$$

Lemma 6.2 [60] If $A=[a_{ij}]$ is any non-constant matrix and its norm defined as

$$||A||_2 = \sqrt{\sum_{ij}a_{ij}^2}$$

Suppose $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$ are singular values of A, then $E(A) \geq \sigma_1 + \frac{||A||_2^2-\sigma_1^2}{\sigma_2}$.

Thus, evaluate a lower bound for $E_{mc}(G)$ as follows:

Theorem 6.5 For a semigraph G on n vertices, if σ_1 and σ_2 are respectively largest and second largest singular values of its minimum covering matrix $A_{mc}(G)$. Then we have

$$E_{mc}(G) \geq \sigma_1 + \frac{1}{\sigma_2}[2\sum_{e\in X}\left(1^2+2^2+\ldots+{k_e}^2\right)+|C|-\sigma_1^2]$$

Proof: By **Lemma 6.2**, for the minimum covering matrix $A_{mc}(G)$ of G, we have

$$E_{mc}(G) \geq \sigma_1 + \frac{||A_{mc}||_2^2-\sigma_1^2}{\sigma_2}$$

Clearly, form definition of norm of a matrix we have

$$||A_{mc}(G)||_2^2 = trace(A_{mc}A'_{mc})$$

$$= 2\sum_{e\in X}\left(1^2 + 2^2 + \ldots + {k_e}^2\right) + |C|$$

Hence, we get

$$E_{mc}(G) \geq \sigma_1 + \frac{1}{\sigma_2}[2\sum_{e\in X}\left(1^2 + 2^2 + \ldots + {k_e}^2\right) + |C| - \sigma_1^2]$$

Which give another lower bound of $E_{mc}(G)$.

6.2.3 Relation between energy and minimum covering energy of a semigraph

Theorem 6.6 Let $G(V, X)$ be a semigraph of order *n*, size m then $E_{mc}(G) \geq \frac{E(G)}{\sqrt{n}}$, where $E(G)$ is the energy of the semigraph *G*.

Proof: If $G(V, X)$ be a semigraph of order *n*, size *m*, and if $E(G)$ be the energy of the semigraph. Then by **Theorem 2.4.1** we have

$$\sqrt{2\sum_{e\in X}\left(1^2 + 2^2 + \ldots + {k_e}^2\right)} \leq E(G) \leq \sqrt{2n\sum_{e\in X}(1^2 + 2^2 + \ldots + K_e^2)}$$

$$i.e. \qquad 2\sum_{e\in X}(1^2 + 2^2 + \ldots. + k_e^2) \leq [E(G)]^2 \leq 2n\sum_{e\in X}(1^2 + 2^2 + \ldots + k^2)$$

$$Thus \qquad [E(G)]^2 \leq 2n\sum_{e\in X}(1^2 + 2^2 + \ldots + k^2)$$

$$Therefore \qquad \frac{[E(G)]^2}{n} \leq 2\sum_{e\in X}(1^2 + 2^2 + \ldots + k^2)$$

If $E_{mc}(G)$ be the minimum covering energy of a semigraph $G(V, X)$,

By **Theorem 6.4,** we get

$$[E_{mc}(G)]^2 \geq 2\sum_{e\in X}\left(1^2 + 2^2 + \ldots + {k_e}^2\right) + |C| + n(n-1)\Delta^{1/n}$$

$$i.e. \qquad [E_{mc}(G)]^2 \geq 2\sum_{e\in X}\left(1^2 + 2^2 + \ldots + {k_e}^2\right)$$

$$Thus \qquad [E_{mc}(G)]^2 \geq \frac{[E(G)]^2}{n}$$

$$Hence \qquad E_{mc}(G) \geq \frac{E(G)}{\sqrt{n}}$$

Theorem 6.7 For a semigraph $G(V,X)$ of order *n*, size *m*, if σ_1 and σ_2 are respectively largest and second largest singular values of its minimum covering matrix $A_{mc}(G)$. Then we have

$$nE_{mc}(G) \geq \frac{[E(G)]^2 - n\sigma_1^2}{\sigma_2}$$

Where $E(G)$ is the energy of the semigraph.

Proof : If $G(V,X)$ be a semigraph of order *n*, size *m*, and if $E(G)$ be the energy of the semigraph. Then by **Theorem 2.4.1** we have

$$Then, \qquad \sqrt{2\sum_{e\in X}\left(1^2 + 2^2 + \ldots + {k_e}^2\right)} \leq E(G) \leq \sqrt{2n\sum_{e\in X}(1^2 + 2^2 + \ldots + K_e^2)}$$

$$Thus \qquad [E(G)]^2 \leq 2n\sum_{e\in X}(1^2 + 2^2 + \ldots. + k_e^2)$$

By **Theorem 6.5** we have,

$$E_{mc}(G) \geq \sigma_1 + \frac{2\sum_{e\in X}\left(1^2 + 2^2 + \ldots. + {k_e}^2\right) + |C| - \sigma_1^2}{\sigma_2}$$

$$Thus \qquad \sigma_2 E_{mc}(G) - \sigma_1\sigma_2 + \sigma_1^2 \geq 2\sum_{e\in X}(1^2 + 2^2 + \ldots. + k_e^2) + |C|$$

$$i.e. \qquad \sigma_2 E_{mc}(G) - \sigma_1\sigma_2 + \sigma_1^2 \geq 2\sum_{e\in X}(1^2 + 2^2 + \ldots. + k_e^2)$$

$$i.e. \qquad n(\sigma_2 E_{mc}(G) - \sigma_1\sigma_2 + \sigma_1^2) \geq 2n\sum_{e\in X}(1^2 + 2^2 + \ldots. + k_e^2)$$

$$i.e. \qquad n(\sigma_2 E_{mc}(G) - \sigma_1\sigma_2 + \sigma_1^2) \geq [E(G)]^2$$

$$i.e. \qquad nE_{mc}(G) \geq \frac{[E(G)]^2}{\sigma_2} - n\frac{\sigma_1^2}{\sigma_2} + n\sigma_1$$

$$i.e. \qquad nE_{mc}(G) \geq \frac{[E(G)]^2}{\sigma_2} - n\frac{\sigma_1^2}{\sigma_2}$$

$$Hence \qquad nE_{mc}(G) \geq \frac{[E(G)]^2 - n\sigma_1^2}{\sigma_2}$$

6.3 Minimum covering color matrix and color energy of semigraphs

In this section another type of matrix called minimum covering color matrix of a semigraph was introduced and obtained energy of the matrix, and established some bonds to realizing the mathematical aspects of the minimum covering color energy of a semigraph. The minimum covering color matrix of a semigraph is defined as follows:

Minimum covering color matrix and energy of semigraph: Suppose $G(V, X)$ be a vertex-colored semigraph of order *n* and size *m,* and if $c(v_i)$ denote the color of the vertex v_i. Let $C \subseteq V$ be a minimum covering set, then the minimum covering color matrix of G is defined by the square matrix $A^c_{mc}(G) = (a_{ij})_{n\times n}$, and of which

$$a_{ij}(v_i, v_j) = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent or if } i = j \text{ and } v_i \in C, \\ -1 & \text{if } v_i \text{ and } v_j \text{ are non-adjacent with } c(v_i) = c(v_j), \\ 0, & \text{otherwise.} \end{cases}$$

The minimum covering color matrix $A^c_{mc}(G)$ of a semigraph G is symmetric and hence its eigenvalues $\xi_1, \xi_2, \xi_3, \dots, \xi_n$ are all real, called minimum covering color eigenvalues of G. The minimum covering color energy of a semigraph G is denoted by $E^c_{mc}(G)$ and defined as $E^c_{mc}(G) = \sum_{i=1}^{n} |\xi_i|$.

Example 6.2 $G(V, X)$ be a connected semigraph as shown in **Figure 6.2** having vertex set $V = \{1,2,3,4,5,6,7,8\}$ with the minimum colors C1, C1, C2, C1, C2, C2, C1 and C2 respectively and edge set $X = \{(1,2,3), (3,4), (4,5,6), (6,7,3), (7,8)\}$. Let $C = \{3, 4, 7\}$ be the minimum covering set. Then,

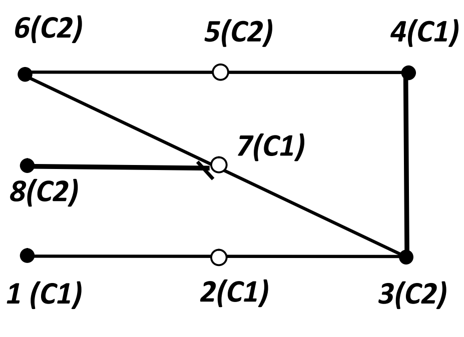

Figure 6.2

Then, the minimum covering color matrix $A^c_{mc}(G)$ of the semigraph $G(V,X)$ is

$$A^c_{mc}(G) = \begin{bmatrix} 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & -1 \\ -1 & -1 & 1 & -1 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 & 1 & 0 \end{bmatrix}$$

6.3.1 Properties of minimum covering color energy of semigraphs

Suppose $G(V,X)$ be a vertex-colored semigraph order *n* and size *m,* and if $c(v_i)$ denote the color of the vertex v_i and let *C* be the minimum covering set. Suppose $A^c_{mc}(G) = (a_{ij})_{n\times n}$ be the minimum covering color matrix of G. Suppose characteristic polynomial of $A^c_{mc}(G)$ be

$$P^c_{mc}(G,\xi) = det\big(\xi I - A^c_{mc}(G)\big) = a_0\xi^n + a_1\xi^{n-1} + a_2\xi^{n-2} + a_3\xi^{n-3} + \ldots + a_n$$

Theorem 6.8 Using the notations given above, we have

(a) $a_0 = 1$

(b) $a_1 = -|C|$

(c) $a_2 = \binom{|C|}{2} - \sum_{i=1}^{m} \binom{|e_i|}{2} - m'_c$

Where m'_c = number of pairs of non-adjacent vertices receiving the same color in G.

Proof: (a) From the definition of the characteristic polynomial $P^c_{mc}(G,\xi) = det\big(\xi I - A^c_{mc}(G)\big)$ of $A^c_{mc}(G)$, it is clear that $a_0 = 1$.

(b) $(-1)^1 a_1$ = Sum of all first order principal minors of $A^c_{mc}(G)$ = Trace of $A^c_{mc}(G) = |C|$

Thus $a_1 = -|C|$

(c) $(-1)^2 a_2$ = Sum of all the 2×2 principal minors of $A^c_{mc}(G)$

$$= \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} = \sum_{1\le i<j\le n} (a_{ii}a_{jj} - a_{ij}a_{ji}) = \binom{|C|}{2} - \sum_{1\le i<j\le n} a^2{}_{ij}$$

$$Thus, \quad a_2 = \binom{|C|}{2} - \sum_{i=1}^{m} \binom{|e_i|}{2} - m'_c$$

Where, m'_c = number of pairs of non-adjacent vertices receiving the same color in G.

Theorem 6.9 If $\xi_1, \xi_2, \xi_3, \ldots, \xi_n$ are the eigenvalues of the minimum covering color matrix $A^c_{mc}(G)$ of a semigraph $G(V,E)$ of order *n*, having *m* edges and if *C* be a minimum covering set of G, then

$$i. \quad \sum_{i=1}^{n} \xi_i = |C|$$

$$ii. \quad \sum_{i=1}^{n} \xi_i^2 = 2\left[\sum_{i=1}^{m} \binom{|e_i|}{2} + m'_c\right] + |C|$$

Where m'_c is the number of pairs of non-adjacent vertices receiving the same color and $|e_i|$ is the number of vertices in the edge $e_i \in E$.

Proof: i. Since, the sum of the eigenvalues of $A^c_{mc}(G)$ is equal to the trace of $A^c_{mc}(G)$

$$Hence \qquad \sum_{i=1}^{n} \xi_i = \sum_{i=1}^{n} a_{ii} = |C|$$

ii. Consider

$$\sum_{i=1}^{n} \xi_i^2 = \sum_{i=1}^{n} ((A^c_{mc})^2)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\, a_{ji}$$

As $A^c_{mc}(G)$ is a symmetric matrix,

$$\sum_{i=1}^{n} \xi_i^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}{}^2$$

$$= 2\sum_{i<j} (a_{ij})^2 + \sum_{i=1}^{n} (a_{ii})^2$$

$$= 2\left[\sum_{i=1}^{m} \binom{|e_i|}{2} + m'_c\right] + |C| \qquad Since, \ \sum_{i=1}^{n} (a_{ii})^2 = |C|$$

Where, m'_c is the number of pairs of non-adjacent vertices receiving the same color.

6.3.2 Some bounds for minimum covering color energy of Semigraphs

Theorem 6.10 Let $G(V,E)$ be the minimum covering colored semigraph having *n* vertices and *m* edges with a minimum covering set *C*. Then

$$E_{mc}^{c}(G) \le \sqrt{2n\left[\sum_{i=1}^{m}\binom{|e_i|}{2} + m_c'\right] + n|C|}$$

Where, m_c' is the number of pairs of non-adjacent vertices in G receiving the same color.

Proof: The minimum covering color matrix of a semigraph, $A_{mc}^{c}(G)$ is symmetric and hence its eigenvalues are real and can be ordered as $\xi_1 \ge \xi_2 \ge \xi_3 \ge \ldots \ge \xi_n$.

Appling the Cauchy-Schwarz inequality, we have

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 \le \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right)$$

Substituting $u_i = 1$, $v_i = |\xi_i|$ in the above inequality and by **Theorem 6.9** we have

$$[E_{mc}^{c}(G)]^2 = \left(\sum_{i=1}^{n}|\xi_i|\right)^2 \le n\left(\sum_{i=1}^{n}|\xi_i|^2\right)$$

$$= n\sum_{i=1}^{n}\xi_i^2$$

$$= n\left[2\left\{\sum_{i=1}^{m}\binom{|e_i|}{2} + m_c'\right\} + |C|\right]$$

$$Hence, \qquad E_{mc}^{c}(G) \le \sqrt{2n\left[\sum_{i=1}^{m}\binom{|e_i|}{2} + m_c'\right] + n|C|}$$

Theorem 6.11 Let $G(V,E)$ be a minimum covering colored semigraph having n vertices and m edges with a minimum covering set C. Let m'_c be the number of pairs of non-adjacent vertices receiving the same color in G. If $\Delta = |det\, A^c_{mc}(G)|$ then

$$\boldsymbol{E^c_{mc}(G)} \geq \sqrt{2\left(\sum_{i=1}^{m}\binom{|e_i|}{2} + m'_c\right) + |C| + n(n-1)\Delta^{2/n}}$$

Proof: We have,

$$[\boldsymbol{E^c_{mc}(G)}]^2 = \left(\sum_{i=1}^{n}|\xi_i|\right)^2$$

$$= \sum_{i=1}^{n}\xi_i^2 + \sum_{i\neq j}|\xi_i|\,|\xi_j|$$

By applying $\boldsymbol{AM} \geq \boldsymbol{GM}$, we have

$$\frac{1}{n(n-1)}\sum_{i\neq j}|\xi_i|\,|\xi_j| \geq \left(\prod_{i\neq j}|\xi_i||\xi_j|\right)^{1/n(n-1)}$$

$$= \left(\prod_{i\neq j}|\xi_i|^{2(n-1)}\right)^{1/n(n-1)}$$

$$= \left|\prod_{i\neq j}\xi_i\right|^{2/n}$$

$$= \Delta^{2/n}$$

$i.e.$
$$\sum_{i\neq j}|\xi_i|\,|\xi_j| \geq n(n-1)\Delta^{2/n}$$

$Thus$
$$[\boldsymbol{E^c_{mc}(G)}]^2 \geq \sum_{i=1}^{n}\xi_i^2 + n(n-1)\Delta^{2/n}$$

By **Theorem 6.9** we get

$$[\boldsymbol{E^c_{mc}(G)}]^2 \geq 2\left(\sum_{i=1}^{m}\binom{|e_i|}{2} + m'_c\right) + |C| + n(n-1)\Delta^{2/n}$$

Hence the result.

Theorem 6.12 Let $G(V, E)$ be a minimum covering colored semigraph of order *n*, size *m* and having *C* be a minimum covering set. Then $\alpha \leq E_{mc}^{c}(G) \leq \beta$,

$$Where,\ \alpha = \sqrt{2\left[\sum_{i=1}^{m}\binom{|e_i|}{2} + m_c' + \left|\binom{|C|}{2} - \sum_{i=1}^{m}\binom{|e_i|}{2} - m_c'\right|\right] + |C|}$$

$$\textbf{and} \qquad \beta = \mathbf{2}\left[\sum_{i=1}^{m}\binom{|e_i|}{2} + m_c'\right] + |C|$$

Where, m_c' be the number of pairs of non-adjacent vertices in G receiving the same color.

Proof : Consider

$$[E_{mc}^{c}(G)]^2 = \left(\sum_{i=1}^{n}|\xi_i|\right)^2$$

$$= \sum_{i=1}^{n}|\xi_i|^2 + \sum_{i\neq j}|\xi_i|\,|\xi_j|$$

$$= \sum_{i=1}^{n}|\xi_i|^2 + 2\sum_{i<j}|\xi_i|\,|\xi_j| \qquad (6.1)$$

We have,

$$\sum_{1\leq i<j\leq n}\xi_i\xi_j = \sum_{1\leq i<j\leq n}\begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj}\end{vmatrix}$$

$$= \sum_{1\leq i<j\leq n}\left(a_{ii}a_{jj} - a_{ij}a_{ji}\right)$$

For minimum covering color matrix $A_{mc}^{c}(G)$ is symmetric, $a_{ij} = a_{ji}$ Thus,

$$\sum_{1\leq i<j\leq n}\xi_i\xi_j = \sum_{1\leq i<j\leq n}a_{ii}a_{jj} - \sum_{1\leq i<j\leq n}a_{ij}a_{ji}$$

$$= \sum_{1\leq i<j\leq n}a_{ii}a_{jj} - \sum_{1\leq i<j\leq n}\left(a_{ij}\right)^2$$

$$= \binom{|C|}{2} - \sum_{i=1}^{m}\binom{|e_i|}{2} - m_c'$$

We know that,

$$\sum_{i<j} |\xi_i|\,|\xi_j| \geq |\sum_{i<j} \xi_i \xi_j$$

Thus $$\sum_{i<j} |\boldsymbol{\xi_i}|\,|\boldsymbol{\xi_j}| \geq \left| \binom{|C|}{2} - \sum_{i=1}^{m} \binom{|e_i|}{2} - m'_c \right| \qquad (6.2)$$

Using inequation (6.1) and (6.2) and **Theorem 6.9**, we get

$$[\boldsymbol{E^c_{mc}(G)}]^{\boldsymbol{2}} \geq 2\left[\sum_{i=1}^{m} \binom{|e_i|}{2} + m'_c + \left| \binom{|C|}{2} - \sum_{i=1}^{m} \binom{|e_i|}{2} - m'_c \right| \right] + |C|$$

Taking positive square-root, we get

$$\boldsymbol{E^c_{mc}(G)} \geq \sqrt{2\left[\sum_{i=1}^{m} \binom{|e_i|}{2} + m'_c + \left| \binom{|C|}{2} - \sum_{i=1}^{m} \binom{|e_i|}{2} - m'_c \right| \right] + |C|}$$

Again, we obtain

$$\boldsymbol{n} \leq \boldsymbol{2}\sum_{i=1}^{m} \binom{|e_i|}{2} \leq \boldsymbol{2}\left[\sum_{i=1}^{m} \binom{|e_i|}{2} + m'_c\right] + |C|$$

Thus $$\boldsymbol{n}\left[2\left[\sum_{i=1}^{m} \binom{|e_i|}{2} + m'_c\right] + |C|\right] \leq \left[\boldsymbol{2}\left[\sum_{i=1}^{m} \binom{|e_i|}{2} + m'_c\right] + |C|\right]^{\boldsymbol{2}}$$

Taking positive square-root, we get

$$\sqrt{2n\left[\sum_{i=1}^{m} \binom{|e_i|}{2} + m'_c\right] + n|C|} \leq \boldsymbol{2}\left[\sum_{i=1}^{m} \binom{|e_i|}{2} + m'_c\right] + |C|$$

Thus, by using **Theorem 6.10**

$$\boldsymbol{E^c_{mc}(G)} \leq \boldsymbol{2}\left[\sum_{i=1}^{m} \binom{|e_i|}{2} + m'_c\right] + |C|$$

Hence the result.

Theorem 6.13 Let $G(V, E)$ be a minimum covering colored semigraph of order *n* and size *m*, with minimum covering set *C*. Let minimum covering color eigenvalues of $A^c_{mc}(G)$ be $\xi_1 \geq \xi_2 \geq \xi_3 \geq \ldots \geq \xi_n$. Then

$$E^c_{mc}(G) \leq |\xi_1| + \sqrt{(n-1)\left[2\left[\sum_{i=1}^{m}\binom{|e_i|}{2} + m'_c\right] + |C| - \xi_1^2\right]}$$

Where, m'_c is the number of pairs of non-adjacent vertices in G receiving the same color.

Proof: Let $\xi_1 \geq \xi_2 \geq \xi_3 \geq \ldots\ldots\ldots \geq \xi_n$ be the minimum covering color eigenvalues of $A^c_{mc}(G)$. Appling the Cauchy-Schwarz inequality on to vectors $(|\xi_2|, |\xi_3|, \ldots, |\xi_n|)$ and $(1,1,\ldots,1)$ with $n-1$ entries,

$$\left(\sum_{i=2}^{n}|\xi_i|\right)^2 \leq (n-1)\left(\sum_{i=2}^{n}|\xi_i|^2\right)$$

$i.e.$
$$\left(\sum_{i=2}^{n}|\xi_i|\right) \leq \sqrt{(n-1)\left(\sum_{i=2}^{n}|\xi_i|^2\right)}$$

$i.e.$
$$\sum_{i=1}^{n}|\xi_i| - |\xi_1| \leq \sqrt{(n-1)\left(\sum_{i=1}^{n}{\xi_i}^2 - \xi_1^2\right)}$$

By using **Theorem 6.9,** we have

$$E^c_{mc}(G) \leq |\xi_1| + \sqrt{(n-1)\left[2\left[\sum_{i=1}^{m}\binom{|e_i|}{2} + m'_c\right] + |C| - \xi_1^2\right]}$$

Theorem 6.14 Let $G(V,E)$ be a minimum covering colored semigraph of order *n* and size *m* with minimum covering set *C*. Let ξ_{max} be the largest absolute value of minimum covering color eigenvalue. Then

$$E^c_{mc}(G) \geq \frac{2\left[\sum_{i=1}^{m}\binom{|e_i|}{2} + m'_c\right] + |C|}{\xi_{max}}$$

Where, m'_c is the number of pairs of non-adjacent vertices in G receiving the same color.

Proof: Let ξ_{max} be the largest absolute value of the minimum covering color eigenvalue of $A^c_{mc}(G)$. Then

$$\xi_{max}|\xi_i| \geq \xi_i^2$$

Thus
$$\sum_{i=1}^{n} \xi_{max}\,|\xi_i| \geq \sum_{i=1}^{n} \xi_i^2$$

By **Theorem 6.9,** we have

$$\xi_{max}\sum_{i=1}^{n}|\xi_i| \geq \mathbf{2}\left[\sum_{i=1}^{m}\binom{|e_i|}{2} + m'_c\right] + |C|$$

Hence
$$E^c_{mc}(G) \geq \frac{2\left[\sum_{i=1}^{m}\binom{|e_i|}{2} + m'_c\right] + |C|}{\xi_{max}}\ .$$

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