Chapter 1

GENERAL INTRODUCTION AND REVIEW OF LITERATURE

In the contemporary world, every part of our environment or society faces a lot of interconnected items. The concept of networking will come into the light from such types of interconnected items. In mathematical literature, we study the networks by using the concept of graph theory. The famous Konigberg's Bridge Problem (1736) is one of the fundamental pillars of development, in the field of networks. The modern theory of networks was very much guided and influenced by Frank Harary [17], who is recognized as the father of modern graph theory.

A network is a set of items, called vertices (or nodes) which represent the entities, with connections between them, called edges (or links) which represent a particular kind of interconnection between those entities. Examples of networks include the internet, the world wide web, social networks, information networks, food webs, reaction and metabolic networks, protein-protein interaction networks, etc.

1.1 Introduction to graph and its generalizations:

In the present civilization, no one can deny the importance of graph theory and its application to various aspects of theoretical and practical fields of a real-life situation.

A **graph** G consists of a finite nonempty set V of p points together with a prescribed set X of q unordered pairs of distinct points of V. The edges of a graph G can be interpreted in the following two ways:

G1: Each edge uv of G is a 2-element subset of the vertex set V of G.

G2: Edges of G are 2-tuple (u, v) of vertices of G satisfying the following:

Two 2-tuples (u, v) and (u', v') are equal if and only if either $u = u'$ and $v = v'$ or $u = v'$ and $v = u'$.

Though graph theory has wide application in our daily life it describes some binary relations only. Thus, in some situations e.g., Family Relationship [15], Optimal Routing [15], Projective Geometries [15], etc. where graph theory is not always sufficient for modelling problems or data which involve relations of order higher than binary. Looking at the applicability of graph theory in diverse fields of knowledge, attempts were made to generalize it so that its new version can be applied to those fields where the results of graph theory cannot be applied directly.

Graph theory is generalized by C. Berge [8, 9] into hypergraph by using the approach (G1). A hypergraph is defined as follows:

Let $X = \{v_1, v_2, v_3, \ldots., v_n\}$ be a finite set. A **hypergraph** on X is a family $H = \{E_1, E_2, E_3, \ldots, E_m\}$ of subsets of X such that
HG1: $E_i \neq \varphi,\ i = 1,2,3, \ldots., m$
HG2: $\bigcup_{i=1}^{m} E_i = X$

A hypergraph is called **simple** if it satisfies the additional condition
HG3: $E_i \subset E_j \Rightarrow i = j$

The elements of X are called vertices while, the sets $E_1, E_2, E_3, \ldots., E_m$ are called edges of the hypergraph H on X. The number of vertices on an edge of a hypergraph is called the cardinality of the edge. A graph is always a simple hypergraph each of whose edges has cardinality 2. A hypergraph H is **uniform** if each of its edges has the same cardinality.

A hypergraph $H = \{E_1, E_2, E_3, \ldots., E_m\}$ is **linear** if any two of its edges have at most one vertex in common i.e., $|E_i \cap E_j| \leq 1$ for $i \neq j$. Obviously, a **linear hypergraph** is a generalization of a graph.

In 1994, the second attempt was made by E. Sampathkumar [15] to propose a new kind of generalization of graph originally named graphoid was renamed later on as semigraph by B.D. Acharya [2] and was structurally designed to apply to problems as demanded by situations in real life that involve relations more than binary ones. Here he attempts a generalization of graphs using the approach (G2) in the following way:

A **semigraph** is a pair (V, X) where V is a non-empty set of elements called vertices and X is a set of n-tuples called edges of distinct vertices for various $n \geq 2$ satisfying the conditions:

SG1: Any two edges have at most one vertex in common.

SG2: Two edges $E_1 = (u_1, u_2, u_3, \ldots., u_n)$ and $E_2 = (v_1, v_2, v_3, \ldots., v_n)$ are considered to be equal if and only if (a) m = n and (b) either $u_i = v_i$ for $1 \leq i \leq n$ or $u_i = v_{n-i+1}$ for $1 \leq i \leq n$. In other words, the edge $(u_1, u_2, u_3, \ldots., u_n)$ is the same as the edge $(u_n, u_{n-1}, \ldots. u_2, u_1)$.

A semigraph with $n = 2$ is a graph. Hence it can be seen as a natural generalization of the graph. Also, a semigraph is a linear hypergraph *H* with an order given to each edge of *H*.

The particular order imposes upon vertices of an edge and two distinct edges intersect in at most one vertex makes the semigraph more closed to a graph than a hypergraph in spite of the fact that both hypergraph and semigraph generalize an edge to contain more than two vertices. Advantages of semigraph due to arrangements of vertices of an edge in a particular order over its counterpart viz., hypergraph as mentioned in [15]. These are:

i. Semigraphs look like graphs when drawn in a plane, because each edge can be written as a line with its vertices arranged in an order.
ii. The important concept of planarity of ordinary graph theory can be translated straightforwardly to semigraphs while the same cannot be enjoyed for hypergraphs [15, 37, 52].

iii. The concept of Eulerian and Hamiltonian paths of ordinary graphs can be easily introduced in semigraphs leading to the definition of Eulerian and Hamiltonian graphs which is not possible in hypergraphs.

iv. Since each edge in a semigraph looks like an edge in a graph, one can easily give a direction to each edge and obtain a structure called directed semigraph analogue of which is not possible in hypergraphs.

In the graph, many properties enjoyed by vertices are not true for edges. E. Sampathkumar [15] the originator of semigraph, points out some defects in graph theory these are also given as follows:

i. Any number of mutually nonadjacent vertices may be adjacent to the same vertex, this is not true for edges.

ii. Analogous to the concept of block graph *B(G)* of graph *G*, where every vertex represents a block of *G*, and two vertices in *B(G)* are adjacent if and only if, the corresponding blocks in *G* are adjacent, we do not have a concept of the graph where each edge represents a block of *G*.

iii. Analogous to the concept of a line graph of a graph *G*, we do not have a concept of a point graph where each edge represents a vertex of *G*.

While the justifications as enumerated in the preceding lines were sufficient for the new generalization of the graph called semigraph.

1.2 Aim and objective

As found in the above discussion, the concept of semigraph theory is more effective than a graph. We observed that, the concept of adjacency matrices of graphs and their energies are still not developed in semigraph theory.

So, this study focused on finding out the energies of different matrices, in the field of semigraph.

The objective of the presents study will be directed towards this end and to be specific, our primary focus will be on the following points:

To introduce the concept of:

i. Adjacency matrix of e-signed and v-signed semigraphs.
ii. Adjacency matrix of *ve*-signed semigraphs and their properties.
iii. Distance matrix and distance energy of a semigraph.
iv. Minimum covering distance energy of a semigraph.
v. Color matrix and its energy of a semigraph.
vi. Minimum covering energy of a semigraph.
vii. Minimum covering color energy of a semigraph.

We are trying to investigate on the above mentioned points, and develop some theory and structural properties on semigraphs through the matrix.

The nature of the investigation of the proposed work is theoretical and analytical unless demanded otherwise by unforeseen circumstantial factors.

1.3 Review of literature

Semigraph the generalization process of the graph due to E. Sampathkumar in his innovative project "Semigraphs and Their Application" [15] brings a new idea work and its impact on the existing terminology of graph theory. In his work, E. Sampathkumar introduced many new concepts like dendroids, isomorphisms, covering, connectivity, planarity, traversability, coloring, etc. on semigraphs. S. S. Kamath and R. S. Bhat [57] in 2003, introduced the concept of domination in semigraph and derived some of its important properties. In continuation of the concept of domination, S. S. Kamath and Saroja R. Hebbar [58, 59] obtained some more results viz strong and weak domination, full sets, domination balance in semigraph, and domination critical semigraphs. In the year 2004, B. Y. Bam [4] of Pune University carried out the research on semigraph to give a solution to the *e*-semigraphical problem raised by E. Sampathkumar [15]. Further N. S. Bhave *et al.* [33] consider the issue of line semigraph and presented a definition of line semigraph different from that given by E. Sampathkumar and obtained a characterization of line semigraph.

In 2007, E. Sampathkumar in collaboration with L. Pushpalata [14] presented the concept of representation of semigraph by matrices and subsequently defined different types of matrices viz adjacency matrix, incidence matrix, consecutive adjacency matrix, and the 3-matrix of a semigraph G. Note that the adjacency matrix determines the adjacency graph G_a of the semigraph G uniquely, but it does not determine the semigraph G uniquely. The incidence matrix, together with the consecutive adjacency matrix determines a semigraph uniquely. Again the 3-matrix of a semigraph G determines the semigraph uniquely.

Inspired by E. Sampathkumar's work many authors generalized most of the concepts and important results of graph theory into semigraph. Further development was brought to the theory of semigraphs through a series of four collaborated papers by K. Kayathri, Mary Sunithi Vijayan, and S. Pethanachi Selvam, the first three of which were published in 2007 and the last in 2010. The first paper authored by K. Kayathri and Mary Sunithi Vijayan [25] dealt with the problem of the coloring of complete semigraph and obtained chromatic numbers for some special types of semigraphs. The second and third papers in the series authored by K. Kayathri and S. Pethanachi Selvam [26, 27] dealt with edge completeness of (p, 2) and (p, 3) semigraphs and the fourth paper authored by K. Kayathri and S. Pethanachi Selvam [28] in 2010 introduced the concept of enumeration of edge complete (p,3) semigraph in which they found 20 categories of edge complete (p,3) semigraphs and no two of these 20 categories of semigraphs were isomorphic.

The study of B. Y. Bam [4] on (m, e)-degree of vertices, etc. was revived again in 2009 by S. Gomathi, R. Sundareswaran, and V. Swaminathan [46] and they extended it to the concept of (m, e)-strong dominating set, (m, e)-dominating set, (m, e)-domination number along with a characterization theorem of (m, e)-strong dominating set. Based on various types of domination concepts as presented in the work of S. Gomathi, R. Sundareswaran, and V. Swaminathan [46], in the year 2012.

Y. B. Venkatakrishnan and V. Swaminathan [62, 63] published two papers. In the first paper, the authors introduced some bipartite graphs corresponding to a semigraph and in the second paper, they worked out results on the concept of bipartite semigraphs and successfully introduced hyper domination of vertex, hyper domination of a set, hyper domination number of a set, hyper independent set and hyper irredundant set.

Encouraged by new developments in various directions of semigraph structure, new faces are attracted into its fold day by day. In the year 2011, P. Das and Surajit Kr. Nath [36, 37] introduced two papers of which one dealt with factorization in semigraphs and the other with the genus of semigraphs. Surajit Kr. Nath and P. Das [52] also introduced some topological entities like the thickness, coarseness, and crossing number of semigraphs in the year 2014. Further, they worked on matching in semigraphs in the year 2013 [53]. Shailaja S. Shirkol, Prabhakar. R. Hampiholi and Meenal M. Kaliwal [55] of Karnataka, India published a paper on signed adjacent domination function and its properties for a class of semigraphs and present their singed adjacent domination number in the year 2016. Also in the year 2017, Professor Prabhakar. R. Hampiholi and Meenal M. Kaliwal [42] introduced a paper titled 'Operations on semigraphs' which explored the structural equivalence of union, intersection ring sum and decomposition of semigraphs and various types of isomorphisms.

Recently, in the year 2018, N. Murugesan and D. Narmatha [32] studied on e-domination number of the cartesian product of some simple path semigraphs. Group study by the authors P. R. Hampiholi, H. S. Ramane, Shailaja S. Shirkol, Meenal M. Kaliwal, and Saroja R. Hebbar [43] of India, introduced a generalized concept of signed graphs to semigraphs and defined some terms like e-signed semigraph, v-signed semigraph, and ve-signed semigraph and discuss their balanced conditions.

The strength of semigraph as a mathematical model and its application is more powerful than the graph model, and this was realized when the concept of semigraph was used for DNA splicing. In this context, we refer to the works of S. Jeyabharathi, J. Padmashree, S. S. Selvi, and K. Thiagarajan [48] in the year 2011. Again, in December 2012, S. Jeyabharathi, M. Angayarkanni, S. Sinthanai. Selvi and R. Anusha [49], observed that at the time of splicing DNA molecule holds the characterization of semigraph, which is more powerful than graph model. In this paper, they also discussed the properties of the odd and even cut bipartite semigraph structures of double-stranded DNA molecules and characterized the number of independent sets and languages produced by them. Also, S. J. Bharathi and P. R. Indhu [47] in 2015 introduced the finite automaton for n-cut splicing semigraph system and show that any k^{th} product of automaton of splicing system leads to the automaton of splicing system. Recently in April 2018, S. Saravanan, R. Poovazhaki N. R. Shanker [56], published a paper titled 'Cluster Topology in WSN with SCPS for QoS' and proposed a semigraph contiguous prevalent set (SCPS) algorithm for semigraph structure to reduce NP-hard and size of the virtual backbone.

In the year 2012 paper due to C. M. Deshpande and Y. S. Gaidhani [10] gives a new definition for the adjacency matrix of a semigraph and states necessary and sufficient conditions for a matrix to be semigraphical. They also defined the spectrum of a semigraph and studied its spectral properties. C. M. Deshpande, Y. S. Gaidhani and B.P. Athawale [11] also in the year 2015 published a paper on an incident matrix that represents a semigraph uniquely and obtained some properties and structure of it for some special classes of semigraphs. Recently in the year 2017, Y. S. Gaidhani, C. M. Deshpande, and B. P. Athawale [65] again introduced an adjacency matrix that represents a semigraph uniquely and develop an algorithm to construct the semigraph from a given square, if semigraphical is given. P. R. Hampiholi and J. P. Kitturkar [40, 41] introduced two papers, first in the year 2014 on partial edge incidence matrix of semigraph over $GF(2^2)$ and obtain its rank, second in the year 2015 on strong circuit matrix and strong path matrix of a semigraph. From adjacency matrix and incidence matrix defined by C. M. Deshpande, Y. S. Gaidhani and B. P. Athawale [10, 11] for a semigraph, Ambika K. Biradar and D. Y. Patil [1] in the year 2018 obtained the Laplacian matrix of a semigraph.

In 2010, R. B. Bapat [44] in his book titled "Graphs and Matrices" discussed the results in graph theory in which linear algebra and matrix theory play an important role. The results discussed here are usually treated under algebraic graph theory, as outlined in the classic book by N. Biggs [31].

Graph energy is the concept that comes from chemistry to approximate the total π-electron energy [21] of a molecule. In chemistry, the conjugate hydrocarbon can be represented by a graph called a molecular graph. The study of graph energy for all arbitrary graphs was initiated by Ivan Gutman [22] in the year 1978 and defined as the sum of the absolute values of its eigenvalues. An extensive study has been done on graph energy and various energies like distance energy [19], color energy [6, 13, 34, 35], minimum covering energy [5, 29, 30], etc. have been studied over the past few years.

Gopalapillai Indulal *et al.* [19] study on distance energy of graphs of diameter 2 and obtain bounds for distance spectral radius and their distance energy. A vertex labeled graph G can be uniquely represented by a matrix called L-matrix, which was introduced by E. Sampathkumar *et al.* [13] in the year 2013 and obtained its characteristics. They also show that the L-matrix of a colored graph with usual coloring is the adjacency matrix of a signed graph on the same vertex set of G.

In the year 2010 C. Adiga *et al.* [6] introduced the concept of color energy of a graph and compute the color energy of a few families of graphs with the minimum number of colors on its vertices. They also established an upper bound and lower bound for color energy and compute energies of the complement of colored graphs of a few families of graphs. Further, P. B. Joshi *et al.*, [34, 35] obtained some new bounds for the color energy of graphs in terms of Zagreb index, Laplacian energy and signless Laplacian energy. Also established the relationship between color energy and the energy of a graph.

C. Adiga *et al*., [5] introduce a new kind of graph energy called minimum covering energy in the year 2012. They computed the energies of some well-known graphs and found that it depends both on the structure of graph G and on its particular minimum cover C. Further, minimum covering distance energy and minimum covering color energy of graphs were studied by M. R. R. Kanna *et al.*, [29, 30] and computed the energies for star graph, complete graph, crown graph, bipartite graph and cocktail graphs. Upper and lower bounds for the energies are also established.

Color Laplacian energy of graphs was introduced by V. S. Shigehalli *et al.* [61] in the year 2015. In their paper, they extend the concept of color energy, and chromatic energy to color Laplacian energy, and chromatic Laplacian energy of graph and established the relationship between color eigenvalues and color Laplacian eigenvalues and some properties of chromatic Laplacian eigenvalues. Some bounds for the chromatic Laplacian energy of graphs are obtained. P. G. Bhat *et al.* [38] introduced color signless Laplacian energy of graphs in the year 2017. In their work they studied the new concept of color signless Laplacian energy of a graph G. It depends on the graphs and colors of the vertices of a graph. The color signless Laplacian energy for the complement of some colored graphs is also obtained.

Recently, Y. S. Gaidhani *et al*. [64] introduced the energy of semigraph in 2019. In their paper, they studied the energy of semigraphs in two ways, one, the matrix energy, as a summation of singular values of the adjacency matrix of a semigraph, and the other, polynomial energy, as the energy of the characteristic polynomial of the adjacency matrix and obtained some bounds for matrix energy and show that matrix energy is never a square root of an odd integer and polynomial energy cannot be an odd integer. Also investigate the matrix energy of a partial semigraph and change in the matrix energy due to edge deletion.

So, we observed that the foregoing paragraph provides sufficient grounds for the identification of the subject "Adjacency matrix of different types of semigraph and their energy” as an area of active research because of their importance in the field of chemistry, physics, computer science, biotechnology, and social science, etc.

However, despite a considerable number of useful results, semigraphs and their energy are still in their nascent stage and many areas of general graph theory, topological graph theory, and algebraic graph theory are yet to find analogues in semigraphs and ramifications if successfully formulated.

1.4 Organization of the thesis:

This thesis makes effort to introduce and analysed some important graph theoretic concepts like adjacency matrix, distance matrix, and energy, color matrix and energy, minimum covering matrix and energy, minimum covering color energy, and minimum covering distance energy along with their implications in the context of semigraph and obtained some new results. The thesis consists of six chapters. The organization of the thesis is as follows.

In the **first chapter,** a general introduction regarding graphs and their generalizations is given. A literature survey on recent development through works of almost all well-known researchers working on graph and semigraph after E. Sampathkumar up to date has been included in this chapter which justified the relevance and importance of the problem under consideration.

The **second chapter** introduces the reader to a collection of notations and ideas in the form of definitions, axioms, and results which can be used as a ready reference for easy understanding of the subsequent topics.

The **third chapter** deals with signed semigraph. A signed semigraph is a generalization of a signed graph. In this chapter, we introduce the adjacency matrix for *e*-signed, *v*-signed, and *ve*-signed semigraph. And discuss them with suitable examples and derived some properties of adjacency matrix associated with the respectively signed semigraphs. A necessary and sufficient condition for a matrix to be *e*-signed, *v*-signed, or *ve*-signed semigraph is successfully established. A part of the work has been published in [50].

The **fourth chapter** in the present thesis discusses on distance matrix and the energy of a semigraph. The eigenvalues of the distance matrix are called *D*-eigenvalues. The energy of the distance matrix of a semigraph i.e. *D*-energy is defined as the sum of the absolute values of *D*-eigenvalues. we investigate some properties and bounds for the distance spectral radius and *D*-energy of semigraph of diameter 2. Further, the minimum covering distance energy of semigraphs and their properties are also discussed. A part of the work has been published in [51].

A coloring of a semigraph G is an assignment of colors to its vertices such that not all vertices in the same edge are colored the same. In the **fifth chapter,** we introduced the concept of color matrix and energy of semigraphs. We defined the color energy of a semigraph as the sum of the absolute values of the eigenvalues of the color matrix. Obtained some properties and bounds on color energy of semigraphs.

In a semigraph G a vertex v and an edge E are incidents to each other if $v \in E$ and in that case, v and E are said to cover each other. A set S of vertices that cover all the edges of a semigraph G is a vertex cover for G. In the **sixth chapter** we introduce the concept of minimum covering matrix and minimum covering energy $E_{mc}(G)$ of a semigraph G. The minimum covering energy is the summation of singular values of the minimum covering matrix which is not symmetric. Upper and lower bounds for $E_{mc}(G)$ are established, and also derive some relationship between minimum covering energy and energy of semigraph $E(G)$. Further, minimum covering color energy of semigraphs and their properties are also discussed.

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