Chapter 3

SIGNED SEMIGRAPH AND ITS ADJACENCY MATRIX

3.1 Introduction

The concept of a signed graph was introduced by the author F. Heider's [17]**.** A signed semigraph is the generalized form of a signed graph. An *e*-signed, *v*-signed and *ve*-signed semigraphs were introduced by P. R. Hampiholi, H. S. Ramane, Shailaja S. Shirkol, Meenal M. Kaliwal, and Saroja R. Hebbar in the year 2017 [43] and finding their balancing conditions.

In this chapter, the adjacency matrix associated with *e*-signed, *v*-signed and *ve*-signed semigraphs has been studied and obtained some necessary and sufficient conditions for a matrix to be the adjacency matrix of an *e*-signed, *v*-signed, or *ve*-signed semigraph.

3.2 Adjacency matrix of an *e*-signed semigraph [50]

Definition 3.2.1 **Adjacency matrix associated with an *e*-signed semigraph**:

Suppose $S(V, E)$ be a e-signed semigraph with vertex set $V = \{v_1, v_2, \ldots, v_m\}$ and edge set $E = E^+ \cup E^-$, where $E^+ = \{P_1, P_2, P_3 \ldots, P_r\}$ is the set of all positive signed edges and $E^- = \{N_1, N_2, N_3 \ldots, N_s\}$ is the set of all negative signed edges with $E^+ \cap E^- = \varphi$ also where $P_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,2\mu})$, $i = 1, 2, 3, \ldots, r$ and $N_j = (b_{j,1}, b_{j,2}, \ldots, b_{j,2\lambda-1})$, $j = 1,2,\ldots, s$ with all $a_{i,k}$ and $b_{j,t}$ are distinct elements of V.

Adjacency Matrix C of $S(V, E)$ is an $m \times m$ matrix whose entries are given by

$c_{i,j}$= cardinality of *fp*-edge$(v_i, v_j) - 1$, if v_i, v_j are adjacent in a positive signed edge.

= $-$[cardinality of *fp*-edge$(v_i, v_j) - 1$], if v_i, v_j are adjacent in a negative signed edge.

= 0, otherwise.

As every p-edge of cardinality greater than equal to 2 belongs to exactly one f-edge whether it may be either positive or negative signed edges of *S*, the above matrix is well defined.

We label the rows and columns of *C* as 1, 2, 3,…, *m*; the same as vertex set of *S*. A row/column corresponding to an isolated vertex is a zero row/column. In this paper, without loss of generality, we consider e-signed semigraphs without isolated vertices.

We discuss the definition with an example below and explain some important observations.

Example 3.1 $S = (V, E)$ be a e-signed with vertex set $V = \{1,2,3,4,5,6,7,8,9,10\}$ and edge set $E = E^+ \cup E^-$ where $E^+ = \{e_2(3,4),\ e_5(6,8,9,10)\}$ and $E^- = \{e_1(1,2,3),\ e_3(4,5,7), e_4(3,6,7)\}$ are respectively positive and negative signed edges of S. Then adjacency matrix *C* of the e-signed semigraph S is

$$C = \begin{bmatrix} 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 & 0 & -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 2 & 3 \\ 0 & 0 & -2 & -2 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 2 & 1 & 0 \end{bmatrix}$$

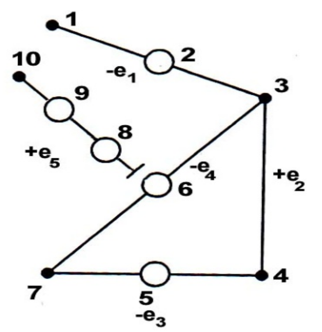

Figure 3.1

We make some observations as follows:

i. *C* is a real valued symmetric matrix, and $c_{i,i} = 0 \quad \forall i$.

ii. $c_{i,j} \in \{-(p-1), \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\ldots, p-1.\}\ \forall i, j$ Where p is the number of vertices.

iii. If there is an entry $p-1$ or $-(p-1)$ then the row and column containing that entry of C must have all the entries from 1 to $p-1$ or -1 to $-(p-1)$. In this case the semigraph will necessarily contain an edge of maximum length p.

iv. The sub matrices of C corresponding to positive edges e_2, e_5 are respectively given by

$$C_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } C_2 = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}.$$

And the sub matrices of C corresponding to negative edges e_1, e_3, e_4 are respectively given by

$$D_1 = D_2 = D_3 = \begin{bmatrix} 0 & -1 & -2 \\ -1 & 0 & -1 \\ -2 & -1 & 0 \end{bmatrix}$$

and all the remaining entries of C are 0.

v. If $A = [a_{i,j}]$ be a symmetric matrix of order m obtain by taking it $(i,j)^{th}$element $a_{i,j} = |c_{i,j}|$ where $c_{i,j}$ is the $(i,j)^{th}$ element of $C = [c_{i,j}]_{m\times m}$, whether $A = [a_{i,j}]$ is the adjacency matrix of a semigraph corresponding to the e-signed semigraph $S = (V, E)$.

Definition 3.2.2 *e*-Signed semigraphical matrix:

A matrix M of order $p \times p$ is said to be *e*-signed semigraphical if there exists an *e*-signed semigraph S on p vertices having adjacency matrix as M.

Theorem 3.1 (Necessary and sufficient conditions for a matrix to be e-signed semigraph):

A $m \times m$ matrix $C = (c_{i,j})$ is the adjacency matrix of an e-signed semigraph iff it satisfies the following conditions:

i. $C = (c_{i,j})$ is a real valued symmetric matrix of order *m*, with its diagonal elements 0.

ii. $c_{i,j} \in \{-(m-1), \dots, -3, -2, -1, 0, 1, 2, 3, \dots, m-1.\}\ \forall i, j$

iii. $V = \{v_1, v_2, v_3, \dots, v_m\}$ can be expressed as a union of subsets $P_1, P_2, \dots, P_k$ with even cardinality such that $|P_i| \geq 2, |P_i \cap P_j| \leq 1\ \forall\ i \neq j$ and N_1, $N_2, \dots, N_t$ with odd cardinality such that $|N_i| \geq 3, |N_i \cap N_j| \leq 1\ \forall\ i \neq j$.

iv. If P_i is a $(2r)$-subset containing $\{x_1, x_2, x_3, \dots x_{2r}\} \subseteq V$, then P_i is a positive signed edge and C_i, the square sub matrix of C associated with P_i of order $2r$, obtained by considering $x_1, x_2, x_3, \dots {x_{2r}}^{th}$ row and column entries of C is given by

$$C_i = \begin{bmatrix} 0 & 1 & 2 & \dots & 2r-1 \\ 1 & 0 & 1 & \dots & 2r-2 \\ 2 & 1 & 0 & \dots & 2r-3 \\ \dots & \dots & \dots & \dots & \dots \\ 2r-1 & 2r-2 & 2r-3 & \dots & 0 \end{bmatrix}$$

v. And if N_i is a $(2r+1)$-subset containing $\{y_1, y_2, y_3, \dots y_{2r+1}\} \subseteq V$, then N_i is a negative signed edge and D_i, the square sub matrix of C associated with N_i of order $2r+1$, obtained by considering $y_1, y_2, y_3, \dots {y_{2r+1}}^{th}$ row and column entries of C is given by

$$D_i = \begin{bmatrix} 0 & -1 & -2 & \dots & -2r \\ -1 & 0 & -1 & \dots & -(2r-1) \\ -2 & -1 & 0 & \dots & -(2r-2) \\ \dots & \dots & \dots & \dots & \dots \\ -2r & -(2r-1) & -(2r-2) & \dots & 0 \end{bmatrix}$$

and all the remaining entries, if any, of C are 0.

Proof: Suppose S is an e-signed semigraph on *m* vertices $\{v_1, v_2, v_3, \dots, v_m\}$ and with positive edges $P_1, P_2, \dots, P_k$ and negative edges $N_1, N_2, \dots, N_t$ and having adjacency matrix $C = (c_{i,j})$. By definition (i), (ii) and (iii) are satisfied. If $\{x_1, x_2, x_3, \dots x_{2r}\}$ is a positive edge of S then $x_1, x_2, x_3, \dots {x_{2r}}^{th}$ row and column entries, by definition of adjacency matrix C, are as given in C_i . Again if $\{y_1, y_2, y_3, \dots y_{2r+1}\}$ is a negative edge of S then $y_1, y_2, y_3, \dots {y_{2r+1}}^{th}$ row and column entries, by definition of C are also given in D_i. Hence (iv) and (v) are satisfied.

Conversely if (i), (ii), (iii), (iv) and (v) are satisfied then we uniquely define an e-signed semigraph S with vertex set and with positive and negative edges as $P_1, P_2, \ldots, P_k$ and $N_1, N_2, \ldots, N_t$ respectively.

3.3 Adjacency matrix of a *v*-signed semigraph [50]

Definition 3.3.1 Adjacency matrix associated with a *v*-signed semigraph:

A Semigraph $S(V, E)$ is called *v*-signed semigraph if every end-vertex of S is assigned either positive or negative sign. Suppose $S(V, E)$ be a *v*-signed semigraph with vertex set $V = \{v_1, v_2, v_3, \ldots, v_m\}$ and edge set $E = \{E_1, E_2, E_3, \ldots, E_n\}$, then every end-vertices with even consecutive adjacent degrees is assigned positive sign, and end-vertices with odd consecutive adjacent degrees is assigned negative sign. Suppose $V^+ = \{p_1, p_2, p_3 \ldots, p_r\}$ is the set of all positive signed end vertex and $V^- = \{n_1, n_2, n_3 \ldots, n_s\}$ is the set of all negative signed end vertices then $V^+ \cap V^- = \varphi$ and $V^+ \cup V^- \subseteq V$.

Adjacency matrix D of $S(V, E)$ is a $m \times m$ matrix whose $(i, j)^{th}$entries are given by

$d_{i,j}$= i [cardinality of *fp*-edge (v_i, v_j) -1], if v_i, v_j are adjacent with same sign end vertices.

= -i [cardinality of *fp*-edge (v_i, v_j) -1], if v_i, v_j are adjacent with opposite sign end vertices.

= cardinality of *fp*-edge (v_i, v_j) -1, if v_i, v_j are adjacent with at least one is a *m*-vertex.

= 0, otherwise.

Where $i = \sqrt{-1}$

As every *p*-edge of cardinality greater than equal to 2 belongs to exactly one *f*-edge of S, the above matrix is well defined.

Here also we labelled the rows and columns of D as 1, 2, 3, …, m; the same as vertex set of S. A row/column corresponding to an isolated vertex is a zero row/column. In this paper, without loss of generality, we consider v-signed semigraphs without isolated vertices.

We discuss the definition with an example below and introduce some important observations.

Example 3.2 $S = (V, E)$ be a v-signed with vertex set $V = \{1,2,3,4,5,6,7,8,9,10\}$ and edge set $E = \{e_1(1,2,3), e_2(3,4), e_3(4,5,7), e_4(3,6,7), e_5(6,8,9,10)\}$ where $V_p = \{4,7\}$ and $V_N = \{1,3,6,10\}$ are positive and negative signed end vertices. Then adjacency matrix D of the v-signed semigraph S is

$$D = \begin{bmatrix} 0 & 1 & 2i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2i & 1 & 0 & -i & 0 & i & -2i & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 1 & 0 & 2i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & -i & 1 & 2 & 3i \\ 0 & 0 & -2i & 2i & 1 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3i & 0 & 2 & 1 & 0 \end{bmatrix}$$

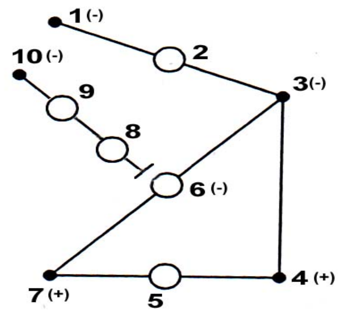

Figure 3.2

And we make the following important observations:

i. $D = (d_{i,j})_{p\times p}$ is a complex valued symmetric matrix, and $d_{ii} = 0 \;\; \forall i$.

ii. $d_{i,j} = 0, \;\; if \;\; i = j$

$d_{i,j} = c_{i,j}, \;\; or \;\; \pm ic_{i,j}, \;\; i \neq j$

Where $c_{i,j} \in \{0,1,2,3,\ldots\ldots.,(p-2),(p-1).\}$ and p is the number of vertices.

iii. The sub matrices corresponding to edges e_1, e_2, e_3, e_4, e_5 are respectively given by

$$A_1 = \begin{bmatrix} 0 & 1 & 2i \\ 1 & 0 & 1 \\ 2i & 1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & 1 & 2i \\ 1 & 0 & 1 \\ 2i & 1 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & i & -2i \\ i & 0 & -i \\ -2i & -i & 0 \end{bmatrix}, \qquad \text{and} \qquad A_5 = \begin{bmatrix} 0 & 1 & 2 & 3i \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3i & 2 & 1 & 0 \end{bmatrix}.$$

and all the remaining entries of D are 0.

iv. Rows (or columns) free from imaginary numbers are the middle vertex which are not the end vertex of any other edges.

v. If $A = [a_{i,j}]$ be a symmetric matrix of order m obtain by taking it $(i,j)^{th}$element $a_{i,j} = |d_{i,j}|$ where $d_{i,j}$ is the $(i,j)^{th}$ element of $D = [d_{i,j}]_{m\times m}$, whether $A = [a_{i,j}]$ is the adjacency matrix of a semigraph corresponding to the v-signed semigraph $S = (V, E)$.

Definition 3.3.2 *v*-Signed semigraphical matrix:

A matrix M of order $p \times p$ is said to be v-signed semigraphical if there exists a v-signed semigraph $S(V, E)$ on p vertices having adjacency matrix as M.

Theorem 3.2 (Necessary and sufficient conditions for a matrix to be *v*-signed semigraph):

A $m \times m$ matrix $D = [d_{ij}]$ is the adjacency matrix of a v-signed semigraph $S(V, E)$ iff it satisfies the following conditions:

i. D is a complex valued symmetric matrix of order m, with its diagonal elements 0.

ii. $d_{i,j} \in \{x/x \in Z, -(m-1) \le x \le (m-1)\} \cup \{ix/x \in Z, -(m-1) \le x \le (m-1), i = \sqrt{-1}\}$ $\forall i, j$

iii. The vertex set $V = \{v_1, v_2, v_3, \ldots\ldots, v_m\}$ can be expressed as a union of subsets $E_1, E_2, \ldots., E_n$ such that $|E_i| \geq 2, |E_i \cap E_j| \leq 1 \ \forall i \neq j$ and if E_i is a $(r+1)$-subset containing $\{x_0, x_1, x_2, \ldots.. x_r\} \subseteq V$ then, the square sub matrix of D of order $(r+1)$, obtained by considering $x_0, x_1, x_2, \ldots x_r{}^{th}$row and column entries of D is given by C_i and represented as

Case 1. If both the end-vertices x_0 and x_r of the edges E_i are in the same signed and middle vertices which are not end vertices of any other edges:

$$C_i = \begin{bmatrix} 0 & 1 & 2 & \ldots & r-1 & ir \\ 1 & 0 & 1 & \ldots & r-2 & r-1 \\ 2 & 1 & 0 & \ldots & r-3 & r-2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ r-1 & r-2 & r-3 & \ldots & 0 & 1 \\ ir & r-1 & r-2 & \ldots & 1 & 0 \end{bmatrix}$$

Case 2. If both the end-vertices x_0 and x_r of the edges E_i are in opposite sign and middle vertices which are not end vertices of any other edges:

$$C_i = \begin{bmatrix} 0 & 1 & 2 & \ldots & r-1 & -ir \\ 1 & 0 & 1 & \ldots & r-2 & r-1 \\ 2 & 1 & 0 & \ldots & r-3 & r-2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ r-1 & r-2 & r-3 & \ldots & 0 & 1 \\ -ir & r-1 & r-2 & \ldots & 1 & 0 \end{bmatrix}$$

Case 3. If some middle vertices of an edge E_i are end-vertex of any other edge E_j:

$C_i = [c_{i,j}]_{(r+1)X(r+1)}$

where $c_{i,j}$ is defined as

$c_{i,j} = ib_{i,j}$ or $-ib_{i,j}$, if the starting end-vertex of the edge and the middle vertex which is an end vertex of another edge have same or opposite signed;

$c_{i,j} = ib_{i,j}$ or $-ib_{i,j}$, if the starting end-vertex of the edge and another end vertex of the edge have same or opposite signed;

$c_{i,j} = b_{i,j}$ Otherwise

where $b_{i,j} \in B_i, \forall i \neq j$ and B_i is a symmetric matrix of order $(r+1)$ and defined as

$$B_i = \begin{bmatrix} 0 & 1 & 2 & \dots & r-1 & r \\ 1 & 0 & 1 & \dots & r-2 & r-1 \\ 2 & 1 & 0 & \dots & r-3 & r-2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ r-1 & r-2 & r-3 & \dots & 0 & 1 \\ r & r-1 & r-2 & \dots & 1 & 0 \end{bmatrix}$$

and all the remaining entries, if any, of D are 0.

Proof: Suppose $S(V,E)$ is a v-signed semigraph on m vertex set $V = \{v_1, v_2, \dots, v_m\}$ and edge set $E = \{E_1, E_2, \dots, E_n\}$ where $V^+ = \{p_1, p_2, \dots, p_s\}$ and $V^- = \{n_1, n_2, \dots, n_t\}$ are the set of all positive and negative signed end-vertex respectively, having adjacency matrix $D = [d_{i,j}]$.

By definition of the matrix (i) and (ii) are satisfied.

If $E_i = \{x_0, x_1, x_2, \dots\dots x_r\}$ is an edge of order $(r+1)$ of $S(V,E)$ where both the end-vertices of the edges are in same signed and middle vertices which are not end vertices of any other edges then, $x_0, x_1, x_2, \dots\dots x_r{}^{th}$ row and column entries of D as given in C_i Case.1 of (iii).

If $E_i = \{x_0, x_1, x_2, \dots\dots x_r\}$ is an edge of order $(r+1)$ of $S(V,E)$ where both the end-vertices of the edge are opposite signed and middle vertices which are not end vertices of any other edges then, $x_0, x_1, x_2, \dots\dots x_r{}^{th}$ row and column entries of D as given in C_i Case.2 of (iii).

If $E_i = \{x_0, x_1, x_2, \ldots\ldots x_r\}$ is an edge of order $(r+1)$ of $S(V,E)$ where both the end-vertices of an edge is either in the same or opposite signed and some middle vertices are end-vertex of any other edges then, $x_0, x_1, x_2, \ldots\ldots {x_r}^{th}$ row and column entries of D as given in C_i Case.3 of (iii).
Hence (iii) is satisfied.

Conversely if (i), (ii) and (iii) are satisfied then we define a semigraph S with vertex set $V = \{v_1, v_2, v_3, \ldots\ldots, v_m\}$ and edges $E_1, E_2, E_3, \ldots\ldots, E_n$ obtain from adjacent matrix $A = [a_{i,j}]_{m\times m}$ where $a_{i,j} = |d_{i,j}|$, $d_{i,j} \in D \ \ \forall i, j$.

If C_i be a square sub-matrix of D of order$(r+1)$, obtained by considering $x_0, x_1, x_2, \ldots\ldots {x_r}^{th}$row and column entries of D as in Case 1 or Case 2 of (iii), then we get an edge $E_i = \{x_0, x_1, x_2, \ldots\ldots x_r\}$ with end-vertices are of same sign or opposite sign and whose middle vertices are not an end vertex of any other edges.

Also if we have a square sub-matrix C_i of D of order $(r+1)$, obtained by considering $x_0, x_1, x_2, \ldots\ldots {x_r}^{th}$row and column entries of D as in Case 3 of (iii), then we obtained an edge $E_i = \{x_0, x_1, x_2, \ldots\ldots x_r\}$ with end-vertices are of same sign or opposite sign and whose middle vertices are an end vertex of any other edges.

The sign of each end-vertex of an edge is found positive or negative according as even or odd numbers of square sub matrix C_i of D as in the case 1, 2 and 3 of (iii) associate with that vertex.

Thus, we get two subsets of V denoted by $V^+ = \{p_1, p_2, \ldots, p_s\}$ and $V^- = \{n_1, n_2, \ldots, n_t\}$ of positive and negative signed end-vertex of S respectively. Hence $S(V,E)$ is a *v*-signed semigraph.

3.4 Adjacency matrix of an *ve*-signed semigraph

Definition 3.4.1 Adjacency matrix associated with an *ve*-signed semigraph:

Suppose $S(V,E)$ be an ve-signed semigraph having m vertices and n edges, where V is the vertex set of S and where every end-vertices with even consecutive adjacent degrees is assigned positive sign, and end-vertices with odd consecutive adjacent degrees is assigned negative sign. Suppose $V^+ = \{p_1, p_2, p_3 \ldots, p_r\}$ is the set of all positive signed end vertex and $V^- = \{n_1, n_2, n_3 \ldots, n_s\}$ is the set of all negative signed end vertices then $V^+ \cap V^- = \varphi$ and $V^+ \cup V^- \subseteq V$. And the edge set $E = E^+ \cup E^-$, where $E^+ = \{P_1, P_2, P_3 \ldots, P_\gamma\}$ is the set of all positive signed edges and $E^- = \{N_1, N_2, N_3 \ldots, N_\eta\}$ is the set of all negative signed edges with $E^+ \cap E^- = \varphi$ also where $P_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,2\mu})$, $i = 1,2,\ldots,\gamma$ and $N_j = (b_{j,1}, b_{j,2}, \ldots, b_{j,2\lambda-1})$, $j = 1,2,..,\eta$, with all $a_{i,k}$ and $b_{j,t}$ are distinct elements of *V*.

Adjacency matrix X of $S(V,E)$ is of order *m* square matrix whose $(i,j)^{th}$entries are given by

$x_{i,j} = \mu(1+i)$, if v_i and v_j are adjacent with same sign end vertices of a positive signed edge.

$= \mu(-1+i)$, if v_i and v_j are adjacent with same sign end vertices of a negative signed edge.

$= \mu(1-i)$, if v_i and v_j are adjacent with opposite sign end vertices of a positive signed edge.

$= \mu(-1-i)$, if v_i and v_j are adjacent with opposite sign end vertices of a negative signed edge.

$= \mu$, if v_i and v_j are adjacent with at least one is a m-vertex of a positive signed edge.

$= -\mu$, if v_i and v_j are adjacent with at least one is a m-vertex of a negative signed edge.

$= 0$, otherwise.

Where, μ = [cardinality of *fp*-edge(v_i, v_j) -1], and $i = \sqrt{-1}$

As every *fp*-edge(v_i, v_j) of cardinality greater than equal to 2 belongs to exactly one *f*-edge(v_i, v_j) whether it may be either positive or negative signed edges of *S*, the above matrix is well defined.

Here also we labelled the rows and columns of *X* as 1, 2, 3,…, m; the same as vertex set of *S*. A row/column corresponding to an isolated vertex is a zero row/column. In this paper, without loss of generality, we consider *ve*-signed semigraphs without isolated vertices.

We discuss the definition with an example given below and obtain some important observations.

Example 3.3 If $S = (V, E)$ be a ve-signed with vertex set $V = V^+ \cup V^-$ where $V^+ = \{4, 7\}$ and $V^- = \{1, 3, 6, 10\}$ are the sets with positive and negative signed end vertices, and edge set $E = E^+ \cup E^-$where $E^+ = \{e_2(3,4), e_5(6,8,9,10)\}$ and $E^- = \{e_1(1,2,3), e_3(4,5,7), e_4(3,6,7)\}$ are sets of positive and negative signed edges. Then Adjacency matrix *X* of the *ve*-signed semigraph *S* is

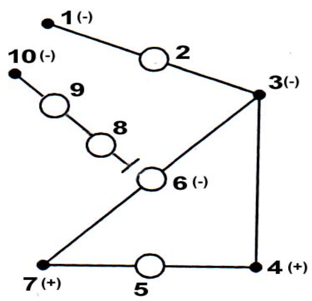

Figure 3.3

$$X = \begin{bmatrix} 0 & -1 & -2+2i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2+2i & -1 & 0 & 1-i & 0 & -1+i & -2-2i & 0 & 0 & 0 \\ 0 & 0 & 1-i & 0 & -1 & 0 & -2+2i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1+i & 0 & 0 & 0 & -1-i & 1 & 2 & 3+3i \\ 0 & 0 & -2-2i & -2+2i & -1 & -1-i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3+3i & 0 & 2 & 1 & 0 \end{bmatrix}$$

And we make the following important observations:

i. $X = \left(x_{i,j}\right)_{p\times p}$ is a complex valued symmetric matrix, and $x_{i,i} = 0 \quad \forall\, i.$

ii. $x_{i,j} = 0,\ \ if\ i = j$

$x_{i,j} = \pm c_{i,j}\ or \pm (1+i)c_{i,j}, i \neq j$ where $c_{i,j} \in \{0,1,2,\dots,(p-2),$ $(p-1)\}$.

iii. The sub matrices corresponding to edges e_1, e_2, e_3, e_4, e_5 are respectively given by

$$A_1 = \begin{bmatrix} 0 & -1 & -2+2i \\ -1 & 0 & 1 \\ -2+2i & 1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1-i \\ 1-i & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & -1 & -2+2i \\ -1 & 0 & -1 \\ -2+2i & -1 & 0 \end{bmatrix}, \ A_4 = \begin{bmatrix} 0 & -1+i & -2-2i \\ -1+i & 0 & -1-i \\ -2-2i & -1-i & 0 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0 & 1 & 2 & 3+3i \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3+3i & 2 & 1 & 0 \end{bmatrix}$$

and all the remaining entries of X are 0.

iv. Rows (or columns) with real number, entries are the middle vertex which are not the end vertex of any other edges.

v. If $A = [a_{i,j}]$ be a symmetric matrix of order m obtain by taking it $(i,j)^{th}$ element $a_{i,j} = Re(x_{i,j})$ where $x_{i,j}$ is the $(i,j)^{th}$ element of $X = [x_{i,j}]_{m\times m}$

then, A is the adjacency matrix of the semigraph $S(V, E)$ due to sign of their edges.

vi. If $B = [b_{i,j}]$ be a symmetric matrix of order m obtain by taking it $(i,j)^{th}$ element $b_{i,j} = x_{i,j}\ or\ Im(x_{i,j})$ where $x_{i,j}$ is the $(i,j)^{th}$ element of $X = [x_{i,j}]_{m\times m}$ according as one of them is not an end vertex of any other edge or both of them are end vertex of any other edges of $S(V, E)$. Then B is the adjacency matrix of the semigraph $S(V, E)$ due to sign of their end vertices.

Definition 3.4.2 *ve*-Signed semigraphical matrix

A matrix M of order $m \times m$ is said to be *ve*-signed semigraphical if there exists a *ve*-signed semigraph $S(V, E)$ on m vertices having adjacency matrix as M.

Theorem 3.3 (Necessary and sufficient conditions for a matrix to be *ve*-signed semigraph):

A square matrix $X = [x_{i,j}]$ of order m is the adjacency matrix of an *ve*-signed semigraph $S(V, E)$ iff it satisfies the following conditions:

i. X is a complex valued symmetric matrix of order m, with its diagonal elements 0.

ii. $x_{i,j} \in \{x_1 + ix_2/x_1, x_2 \in Z; -(m-1) \le x_k \le (m-1), k = 1,2\ and\ i = \sqrt{-1}\}\ \forall\, i, j$

iii. The vertex set $V = \{v_1, v_2, v_3, \dots, v_m\}$ can be expressed as a union of subsets $E_1, E_2, \dots, E_n$ such that $|E_i| \ge 2, |E_i \cap E_j| \le 1\ \forall i \ne j$ and if E_i is a $(r+1)$-subset containing $\{x_0, x_1, x_2, \dots x_r\} \subseteq V$ then, the square submatrix of X of order $(r+1)$, obtained by considering $x_0, x_1, x_2, \dots x_r{}^{th}$ row and column entries of X is given by C_i and represented as

Case 1. If both the end-vertices x_0 and x_r of the positive signed edges E_i are in the same signed and middle vertices which are not end vertices of any other edges:

$$C_i = \begin{bmatrix} 0 & 1 & 2 & \dots & r-1 & r(1+i) \\ 1 & 0 & 1 & \dots & r-2 & r-1 \\ 2 & 1 & 0 & \dots & r-3 & r-2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ r-1 & r-2 & r-3 & \dots & 0 & 1 \\ r(1+i) & r-1 & r-2 & \dots & 1 & 0 \end{bmatrix}$$

Case 2. If both the end-vertices x_0 and x_r of the negative signed edges E_i are in the same signed and middle vertices which are not end vertices of any other edges:

$$C_i = \begin{bmatrix} 0 & -1 & -2 & \dots & -(r-1) & -r(1-i) \\ -1 & 0 & -1 & \dots & -(r-2) & -(r-1) \\ -2 & -1 & 0 & \dots & -(r-3) & -(r-2) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -(r-1) & -(r-2) & -(r-3) & \dots & 0 & -1 \\ -r(1-i) & -(r-1) & -(r-2) & \dots & -1 & 0 \end{bmatrix}$$

Case 3. If both the end-vertices x_0 and x_r of the positive signed edges E_i are in opposite sign and middle vertices which are not end vertices of any other edges:

$$C_i = \begin{bmatrix} 0 & 1 & 2 & \dots & r-1 & r(1-i) \\ 1 & 0 & 1 & \dots & r-2 & r-1 \\ 2 & 1 & 0 & \dots & r-3 & r-2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ r-1 & r-2 & r-3 & \dots & 0 & 1 \\ r(1-i) & r-1 & r-2 & \dots & 1 & 0 \end{bmatrix}$$

Case 4. If both the end-vertices x_0 and $x_r$of the negative signed edges E_i are in opposite sign and middle vertices which are not end vertices of any other edges:

$$C_i = \begin{bmatrix} 0 & -1 & -2 & \dots & -(r-1) & -r(1+i) \\ -1 & 0 & -1 & \dots & -(r-2) & -(r-1) \\ -2 & -1 & 0 & \dots & -(r-3) & -(r-2) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -(r-1) & -(r-2) & -(r-3) & \dots & 0 & -1 \\ -r(1+i) & -(r-1) & -(r-2) & \dots & -1 & 0 \end{bmatrix}$$

Case 5. If some middle vertices of a positive signed edge E_i, are end-vertex of any other edge E_j:

$C_i = [c_{i,j}]_{(r+1)X(r+1)}$

where $c_{i,j}$ is defined as

$c_{i,j} = (1+i)b_{i,j}$ or $(1-i)b_{i,j}$, if the starting end-vertex (x_0) of the edge and the middle vertex $(x_k), k < r$ which is an end vertex of any another edge, have same or opposite signed.

$c_{i,j} = (1+i)b_{i,j}$ or $(1-i)b_{i,j}$, if the starting end-vertex (x_0) of the edge and another end vertex (x_r) of the edge have same or opposite signed.

$c_{i,j} = b_{i,j}$ Otherwise

Case 6. If some middle vertices of a negative signed edge E_i, are end-vertex of any other edge E_j:

$C_i = [c_{i,j}]_{(r+1)X(r+1)}$

where $c_{i,j}$ is defined as

$c_{i,j} = (-1+i)b_{i,j}$ or $(-1-i)b_{i,j}$, if the starting end-vertex (x_0) of the edge and the middle vertex $(x_k), k < r$ which is an end vertex of another edge, have same or opposite signed.

$c_{i,j} = (-1+i)b_{i,j}$ or $(-1-i)b_{i,j}$, if the starting end-vertex (x_0) of the edge and another end vertex (x_r) of the edge have same or opposite signed.

$c_{i,j} = -b_{i,j}$ Otherwise.

Where, $b_{i,j} \in B_i$, $\forall i \neq j$ and B_i is a symmetric matrix of order $(r+1)$ and defined as

$$B_i = \begin{bmatrix} 0 & 1 & 2 & \dots & r-1 & r \\ 1 & 0 & 1 & \dots & r-2 & r-1 \\ 2 & 1 & 0 & \dots & r-3 & r-2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ r-1 & r-2 & r-3 & \dots & 0 & 1 \\ r & r-1 & r-2 & \dots & 1 & 0 \end{bmatrix}$$

iv. The remaining entries, if any, of X are all 0.

Proof: Suppose $X = [x_{i,j}]$ be the adjacency matrix of *ve*-signed semigraph $S(V,E)$ with vertex set $V = \{v_1, v_2, v_3, \dots, v_m\}$ and edge set $E = \{E_1, E_2, E_3, \dots, E_n\}$ where $V^+ = \{p_1, p_2, \dots, p_r\}$ and $V^- = \{n_1, n_2, \dots, n_s\}$ are the set of all positive and negative signed end-vertex set, and having positive signed edges $P_1, P_2, \dots, P_k$ and negative signed edges $N_1, N_2, \dots, N_t$. Then, by definition of the matrix, (i) and (ii) are satisfied.

If $E_i = \{x_0, x_1, x_2, \dots x_r\}$ is an edge of order $(r+1)$ of $S(V,E)$ where both the end-vertices of the edges are in same signed and middle vertices which are not end vertices of any other edges then, $x_0, x_1, x_2, \dots x_r{}^{th}$ row and column entries of X as given in C_i of Case 1 or Case 2 in (iii) according as E_i is positive or negative signed edges.

If $E_i = \{x_0, x_1, x_2, \dots x_r\}$ is an edge of order $(r+1)$ of $S(V,E)$ where both the end-vertices of the edge are opposite signed and middle vertices which are not end vertices of any other edges then, $x_0, x_1, x_2, \dots x_r{}^{th}$ row and column entries of X as given in C_i of Case 3 or Case 4 in (iii) according as E_i is positive or negative signed edges.

If $E_i = \{x_0, x_1, x_2, \dots x_r\}$ is an edge of order $(r+1)$ of $S(V,E)$ where both the end-vertices of an edge is either in the same or opposite signed and some middle vertices are end-vertex of any other edges then, $x_0, x_1, x_2, \dots x_r{}^{th}$ row and column entries of X as given in C_i of Case 5 or Case 6 in (iii) according as E_i is positive or negative signed edges.

Hence (iii) is satisfied. And by definition (iv) is also satisfied.

Conversely, if (i), (ii) (iii) and (iv) are satisfied then we uniquely defined an *e*-signed semigraph S with vertex set $V = \{v_1, v_2, v_3, \dots, v_m\}$ obtain from adjacent matrix $A = [a_{i,j}]_{m\times m}$, where $a_{i,j} = \left|Re(x_{i,j})\right|$, $x_{i,j} \in X \ \ \forall i, j$ with n positive or negative signed edges $E_1, E_2, \dots, E_n$ according as $|E_i|$ is even or odd.

If C_i is $(r+1)$-order square sub-matrix of X, obtained by considering $x_0, x_1, x_2, \dots x_r{}^{th}$ row and column entries of X as in Case 1 or Case 2 of (iii), then we get an edge $E_i = \{x_0, x_1, x_2, \dots x_r\}$ with end-vertices are in same sign and whose middle vertices are not an end vertex of any other edges.

If C_i be a square sub-matrix of X of order $(r+1)$, obtained by considering $x_0, x_1, x_2, \dots x_r{}^{th}$ row and column entries of X as in Case 3 or Case 4 of (iii), then we get an edge $E_i = \{x_0, x_1, x_2, \dots x_r\}$ with end-vertices are in opposite sign and whose middle vertices are not an end vertex of any other edges.

Also if we have a square sub-matrix C_i of X of order $(r+1)$, obtained by considering $x_0, x_1, x_2, \dots x_r{}^{th}$row and column entries of X as in Case 5 or Case 6 of (iii), then we obtained an edge $E_i = \{x_0, x_1, x_2, \dots x_r\}$ with end-vertices are of same sign or opposite sign and whose middle vertices are an end vertex of any other edges.

The sign of each end-vertex of an edge is found positive or negative according as even or odd numbers of square sub matrix C_i of X as in the Case 1, 2, 3, 4, 5 and 6 of (iii) associate with that vertex.

Thus, we get two subsets of V denoted by $V^+ = \{p_1, p_2, \dots, p_s\}$ and $V^- = \{n_1, n_2, \dots, n_t\}$ of positive and negative signed end-vertex of S respectively.

Hence $S(V, E)$ is *ve*-signed semigraph.

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