Chapter-9

Polytropic gas has Thermodynamic manners in Cosmology

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9.1 Introduction

Cosmologist's belief that our Universe expands under an accelerated expansion A new energy with negative pressure, called dark energy (DE), is responsible for this expansion (Kalita, 2015). The action of the DE is still uncertain. The cosmological constant is the earliest, easiest and most conventional candidate for dark energy. But it has some problems like fine-tuning and cosmic coincidence puzzles (Copland et al., 2006; Weinberg, 1989). Besides the cosmological constant, the other dark energy models are Quintessence (Wetterich, 1988). Phantom (Caldwell, 2002)), Tachyon (Setare et al., 2009), Holographic dark energy (Setare, 2007 & 2009), K-essence (Afshordi et al., 2007), Chaplygin gas and Polytropic gas models with various equation of state (Mukhopadhyay et al., 2008; Das & Singh, 2020a). In this chapter, we have investigated the thermodynamic manners of the Polytropic gas. K. Karami et al. investigated the interaction between the Polytropic gas and cold dark matter and found that the Polytropic gas behaves as the Phantom dark energy (Karami et al., 2009). K. Karami and S. Ghaffari showed that the generalized second law of thermodynamics is always satisfied by a Universe filled with a Polytropic gas and a cold dark matter (Karami & Ghaffari, 2010). K. Kleidis and N. K. Spyron used the first law of thermodynamics in the Polytropic gas model and they show that the Polytropic gas behaves as dark energy and this model leads to a suitable fitting with the observational data about the current expanding era (Kleidis & Spyron, 2015). H. Moradpour et al. investigated the thermo dynamical behavior and stability of the Polytropic gas (Moradpour et al., 2016).

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9.2 Formalism

In this work, we consider the following equation of state which is well known as Polytropic gas equation of state (Karami et al., 2009; Das & Singh, 2020b).

$$P_\Lambda = K{\rho_\Lambda}^{1+\frac{1}{n}} \tag{9.1}$$

Here, the Polytropic constant and the Polytropic index are K (>0) and n (<0). In addition, the fluid energy density(ρ) is such that

$$\rho_\Lambda = \frac{U}{V} \tag{9.2}$$

The internal energy and the volume occupied by the fluid are denoted by $U$and V respectively (Das & Singh, 2020b).

First of all, we will try to find the internal energy U and energy density $\rho_\Lambda$of the polytropic gas as a function of its volume V and entropy S.

From the general thermodynamics, we have

$$\left(\frac{\partial U}{\partial V}\right)_S = -P_\Lambda \tag{9.3}$$

From the equations (9.1), (9.2) and (9.3), we get

$$\left(\frac{\partial U}{\partial V}\right)_S = -K\left(\frac{U}{V}\right)^{1+\frac{1}{n}} \tag{9.4}$$

Integrating the equation (9.4), we get

$$U = (-1)^{-n}\left(KV^{-\frac{1}{n}} + \xi\right)^{-n} \tag{9.5}$$

Where the parameter ξ is the constant of integration which may be a universal constant or a function of entropy S only

The equation (9.5) also can rewrite in the following form

$$U = (-1)^{-n}K^{-n}V\left(1 + \left(\frac{\mathrm{V}}{\varepsilon}\right)^{\frac{1}{\mathrm{n}}}\right)^{-n} \tag{9.6}$$

$$\text{Where } \varepsilon = \left(\frac{K}{\xi}\right)^n \tag{9.7}$$

And it has a dimension of volume.

Therefore, the energy density ρ_Λ of the Polytropic gas is

$$\rho_\Lambda = \frac{U}{V} = (-1)^{-n} K^{-n} \left(1 + \left(\frac{V}{\epsilon}\right)^{\frac{1}{n}}\right)^{-n} \tag{9.8}$$

When $n < 0$ then equation (9.8) gives

$$\rho_\Lambda \sim (-1)^{-n} K^{-n} \frac{\varepsilon}{V} \tag{9.9}$$

Now we will use these equations to discuss different physical parameters.

(a) Pressure:

From the equations (9.1) & (9.8), the pressure of the Polytropic gas is

$$P_\Lambda = (-1)^{n+1} K^{-n} \left(1 + \left(\frac{V}{\epsilon}\right)^{\frac{1}{n}}\right)^{-(n+1)} \tag{9.10}$$

We can rewrite the equation (9.10) in the following form

$$P_\Lambda = -\frac{\rho_\Lambda}{1+\left(\frac{V}{\epsilon}\right)^{\frac{1}{n}}} \tag{9.11}$$

When $n < 0$ and ε does not diverge then for small volume i.e. at early stage of Universe $V \ll \varepsilon$ ie $\frac{V}{\varepsilon} \ll 1$ we get

$P_\Lambda \sim 0$, which represents a dust dominated Universe.

When $n < 0$ and ε does not diverge then for large volume i.e. at late stage of Universe $V \gg \varepsilon$ie $\frac{V}{\varepsilon} \gg 1$ we get

$P_\Lambda \sim -\rho_\Lambda$, which indicates an accelerated expansion of the Universe.

(b) Caloric equation of state:

Now from the equations (9.8) and (9.10) we get the caloric equation of state parameter as

$$\omega_\Lambda = \frac{\rho_\Lambda}{\rho_\Lambda} = -\frac{1}{1+\left(\frac{V}{\epsilon}\right)^{\frac{1}{n}}} \tag{9.12}$$

When$n<0$ and ε does not diverge then for small volume $V \ll \varepsilon$ ie $\frac{V}{\varepsilon} \ll 1$ we get

$$\omega_{\Lambda} \simeq 0$$

When$n<0$ and ε does not diverge then for large volume $V \gg \varepsilon$ ie $\frac{V}{\varepsilon} \gg 1$ we get

$$\omega_{\Lambda} \simeq -1 (\text{Cosmological constant})$$

Thus the equation of state parameter (ω_{Λ}) of the Polytropic gas with $n<0$ is decreased from $\omega_{\Lambda} \simeq 0$(for small volume) to $\omega_{\Lambda} \simeq -1$ (for large volume). It indicates that the Universe expands from the dust dominated era to the present expanding era.

(c) Deceleration parameter:

We get the deceleration parameter of the Polytropic gas with the help of equation (9.12)

$$q = \frac{1}{2} + \frac{3}{2}\frac{P_{\Lambda}}{\rho_{\Lambda}} = \frac{1}{2} - \frac{3}{2}\frac{1}{1+\left(\frac{V}{\epsilon}\right)^{\frac{1}{n}}} \tag{9.13}$$

When$n<0$ and ε does not diverge then for small volume $V \ll \varepsilon$ ie $\frac{V}{\varepsilon} \ll 1$ we get

$q > 0$, Thiscorrespond to the deceleration Universe

When$n<0$ and ε does not diverge then for large volume $V \gg \varepsilon$ ie $\frac{V}{\varepsilon} \gg 1$ we get $q < 0$, this correspond to the accelerated Universe.

(d) Square velocity of sound:

From the equation (9.11) we get the velocity of sound (V_s) as

$$V_s^{\,2} = \left(\frac{\partial P_{\Lambda}}{\partial \rho_{\Lambda}}\right)_S = -\frac{1}{1+\left(\frac{V}{\epsilon}\right)^{\frac{1}{n}}} \tag{9.14}$$

When$n<0$ and ε does not diverge then for small volume $V \ll \varepsilon$ ie $\frac{V}{\varepsilon} \ll 1$ we get $V_s^{\,2} \simeq 0$ since velocity of sound is zero in vacuum. Therefore the Polytropic gas behaves like a pressure less fluid at the early stage of the Universe.

When$n<0$ and ε does not diverge then for large volume $V \gg \varepsilon$ ie $\frac{V}{\varepsilon} \gg 1$ we get $V_s^{\,2} \simeq -1$, this gives an imaginary speed of sound leading to a perturbation cosmology.

(e) Thermo dynamical stability:

The conditions of the thermo dynamical stability of a fluid are

$$\left(\frac{\partial P_\Lambda}{\partial V}\right)_S < 0 \tag{9.15}$$

$$\text{And } C_V > 0 \tag{9.16}$$

Here C_V is the thermal capacity at constant volume.

From the equation (9.10) we have

$$\left(\frac{\partial P_\Lambda}{\partial V}\right)_S = -\left(1+\frac{1}{n}\right)\frac{P_\Lambda}{V}\frac{1}{1+\left(\frac{V}{\epsilon}\right)^{-\frac{1}{n}}} \tag{9.17}$$

If $-1 < n < 0$ and $\varepsilon < 0$ then from (9.17), we have

$$\left(\frac{\partial P_\Lambda}{\partial V}\right)_S < 0$$

Thus the stability condition (9.15) of thermodynamics is satisfied.

Now we have to verify the positivity of the thermal capacity at constant volume C_V where $C_V = T\left(\frac{\partial S}{\partial T}\right)_V$ (9.18)

Now we determine the temperature T of the Polytropic gas as a function of its entropy S and its volume V. The temperature T of the Polytropic gas is determined from the relation

$$T = \left(\frac{\partial U}{\partial S}\right)_V \tag{9.19}$$

Using (9.6) in (9.19) we get

$$T = (-1)^{n+1}V^{1+\frac{1}{n}}\left(K+\xi V^{\frac{1}{n}}\right)^{-(n+1)}\frac{d\xi}{dS} \tag{9.20}$$

This gives the temperature of the Polytropic gas.

We can rewrite the equation (9.20) in the following form

$$T = -n\frac{\rho_\Lambda V^{1+\frac{1}{n}}}{1+\left(\frac{V}{\epsilon}\right)^{\frac{1}{n}}}\frac{d\xi}{dS} \tag{9.21}$$

From (9.5) we have

$$[\xi]^{-n} = [U] \tag{9.22}$$

$$\text{Since } [U] = [TS] \tag{9.23}$$

Therefore from the equations (9.22) & (9.23) we get

$$\xi = [U]^{-\frac{1}{n}} = [T_* S]^{-\frac{1}{n}} \tag{9.24}$$

Here T_* is a universal constant with temperature dimension

Differentiating (9.24) with respect to 'S' we get

$$\frac{d\xi}{dS} = -\frac{1}{n} T_*^{-\frac{1}{n}} S^{-\frac{1}{n}-1} \tag{9.25}$$

Using (9.8) & (9.24) in (9.25) we get

$$T = (-1)^n V^{1+\frac{1}{n}} \left(T_*^{-\frac{1}{n}} S^{-\frac{1}{n}-1} \right) \left[K + T_*^{-\frac{1}{n}} S^{-\frac{1}{n}} V^{\frac{1}{n}} \right]^{-(n+1)} \tag{9.26}$$

This leads to the entropy of the Polytropic gas as

$$S = \left[(-1)^{\frac{n}{n+1}} \left(\frac{T_*}{T} \right)^{\frac{1}{n+1}} - 1 \right]^n \frac{V}{K^n T_*} \tag{9.27}$$

We know that entropy (S) of a thermo dynamical system should be positive ie $S > 0$ (Callen, 1985)

Here $S > 0$ if $K^n T_* > 0$

Now the thermal capacity at constant volume is

$$C_V = T \left(\frac{\partial S}{\partial T} \right)_V$$

$$= (-1)^{\frac{2n+1}{n+1}} \left(\frac{n}{n+1} \right) \frac{S}{\left[(-1)^{\frac{n}{n+1}} \left(\frac{T_*}{T} \right)^{\frac{n}{n+1}} - 1 \right]} \left(\frac{T_*}{T} \right)^{\frac{1}{n+1}} \tag{9.28}$$

Therefore, the condition $C_V > 0$ is satisfied if $K^n T_* > 0$

Thus both the conditions of thermo dynamic stability are satisfied. So the Polytropic gas is thermo dynamically stable.

9.3 Conclusion

In this chapter, we have studied the thermo dynamical behavior of the Polytropic gas. We consider the value of $n = -3.4$ to study the whole work done in this work. Some important results are given below:

(i) As we have considered $n = -3.4$, the pressure goes more and more negative.

(ii) In the sense of thermodynamics, the deceleration parameter is studied and our study reveals that the Universe is decelerated and accelerated at the early and late stage of it respectively.

(iii) Both the conditions of the thermodynamic stability of the Polytropic gas are studied and our analysis shows that the Polytropic gas is thermodynamically stable.