Chapter-2

PRELIMINARIES

In this chapter cited some definitions and results that will required in the subsequent chapter. All the results and definition cited in this chapter can be found in (G. J.Klir, B. Yuan, 1997: Hsieh and Tzeng, 2004: Kim, Y, 2013a: Lee, G, 2013: Shukla, Garg & Agarwal 2014)

2.1 BASIC OF FUZZY SET

In the real world, people often use concepts which are quite vague. For example, we say that a woman is young or very young, an object is costly or cheap, a mango is yellow and rip, a number is big or small, a bike is slow or fast and so on. Let us take *young* as an illustration. Consider *A* is a 25-year-old woman. Maybe we think *A* is certainly young. Now comes a woman *B* only one day younger than *A*. Of course, *A* is still young. Then how about a woman only one day younger than A. Continuing in this way, we shall find it difficult to determine an exact age beyond which a woman will be young. As a matter of fact, there is no sharp line between young and younger. The transition from one concept to the other is gradual. This gradualness results in the vagueness of the concept *young*, which in return makes the boundary of the set of all young women unclear.

Zadeh in (1965 and 1975) introduced Fuzzy set s which is a generalization of conventional set theory. A fuzzy set assigns to each possible individual in the universe of discourse, a value representing its grade of membership in the set. It is concerned with the degree to which events occur rather than the likelihood of their occurrence. Fuzzy logic is most successful in situations with very complex models, where understanding is strictly limited and where human reasoning, human perception, human decision making are inextricably involved. Fuzzy sets assume a significant job in human thinking, especially in the areas examples of pattern recognition, correspondence of data, decision making and abstraction. Applications of fuzzy sets in various fields are discussed in George J. Klir and Bo Yuan

In general set theory an element is either a member of a set or not. We can express this reality with the characteristic function for the elements of a given universe to have a place with specific subset of this universe. We call such a set a *crisp set*.

Characteristic function:

The classical set A in the Universal Set X, A⊆X is normally characterized by the function μ_A which takes the value 1 and 0, indicating whether or not x ∈X is a member of A?

$$\mu_A : X = \begin{cases} 1 & iff\ x \in A \\ 0 & iff\ x \notin A \end{cases}$$

Here $\mu_A(x) \in \{0, 1\}$. The function $\mu_A(x)$ takes only the values 1 or 0. Now, assumed that the function $\mu_A(x)$ may take values in the interval [0, 1]. In this way the concept of membership is not crisp any more, but becomes fuzzy in the sense of representing partial belonging or degree of membership.

A fuzzy set is defined and denoted by

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) / x \in U, \mu_{\tilde{A}}(x) \in [0,1]\}$$

Where $\mu_{\tilde{A}}(x)$ is called the membership function; $\mu_{\tilde{A}}(x)$ defines the grade or

degree to which any elements in $\tilde{A}$ belongs to the fuzzy set $\tilde{A}$

When X finite set or is countable, a fuzzy set $\tilde{A}$ on X is expressed as

$$\tilde{A} = \sum_{x_i \in X} \mu(x_i) / x_i$$

When X is a finite set whose elements are x_1, x_2, ………., x_n, a fuzzy set $\tilde{A}$ on x is expressed as

$$\tilde{A} = \{(x_1, \mu_{\tilde{A}}(x)), (x_2, \mu_{\tilde{A}}(x)), ...(x_n, \mu_{\tilde{A}}(x))\}$$

When X is uncountable set and an infinite, a fuzzy set $\tilde{A}$ on X is expressed as

$$\tilde{A} = \int_x \mu(x) / x$$

This means that expression is the grade of x is $\mu_{\tilde{A}}(x)$ and the operations '/' is not a algebraic 'quotient but a delimiter, '$\sum$' is not algebraic 'add' but unions and

'$\int$' do not refer to integral but a continuous function.

2.1.1 Definitions:

Support

The support of a fuzzy set A is defined as the crisp set that contains all the elements with nonzero membership grades in A of X within a universal set X.

Support of A is denoted as Supp (A) = $\{x \in X / \mu_{\tilde{A}}(x) > 0\}$

Core

The **core** of a fuzzy set A is the set of all points with unit membership degree in A and is represented as core (A) = $\{x \in X / \mu_{\tilde{A}}(x) = 1\}$

Normal

A fuzzy set A of X is called **normal** if there exists at least one element x in X such that $\mu_{\tilde{A}}(x) = 1$ A fuzzy set A is normal if its core is nonempty. A fuzzy set that is not normal is called subnormal.

Height

The height, of a fuzzy set A is defined as the largest membership grade in A which is obtained by any element in that set. Height (h) is represented as

$$h(A) = Max \mu_{\tilde{A}}(x), \forall x \text{ in X}$$

Alpha Cut

Alpha cut is the most important concepts of fuzzy sets.

For a given fuzzy set A and set of all $\alpha \in [0,1]$ the alpha cut (α – cut) represented as

$$A_\alpha = \{x \in X / \mu_A(x) \geq \alpha\}$$

The strong α -cut $A_{\alpha^+} = \{x \in X / \mu_A(x) > \alpha\}$

The α -cut set is a crisp set. This threshold cut restricts the domain of the fuzzy set. Two main reasons why α -cuts are important are (i) The alpha level set describes a power or strength that is used by fuzzy models to decide whether or not a truth value should be considered equivalent to zero. This is a facility that controls the execution of fuzzy rules as well as intersection of multiple fuzzy sets, (ii) the strong α -cut at zero defines the support set for a fuzzy set. Figure 2.1 illustrates the regions in the universe compromising the core, support, crossover points and alpha cut of a typical fuzzy set.

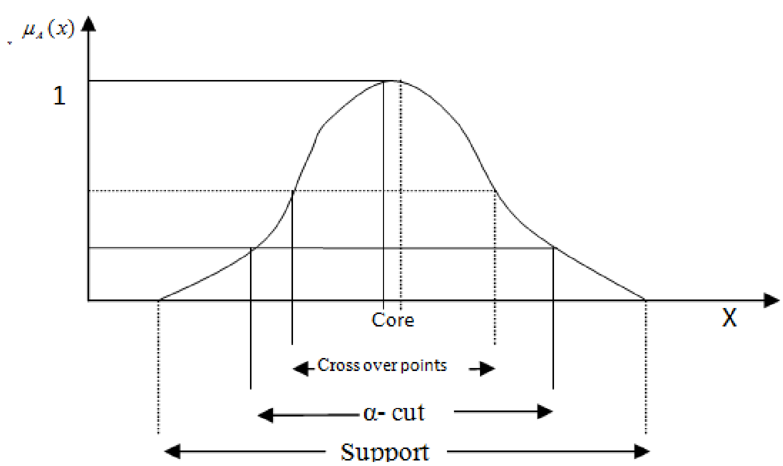


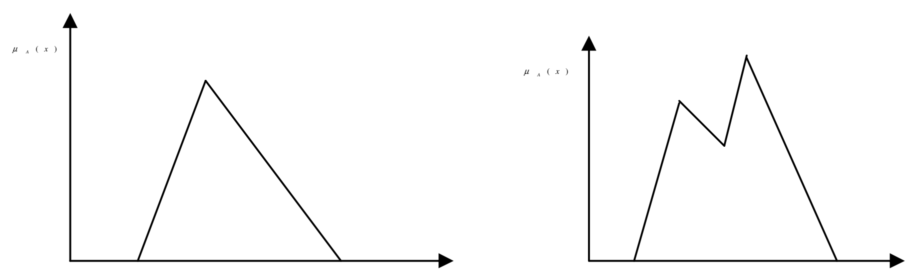
Figure 2.1

2.1.2 Convex fuzzy set

A convex fuzzy set is defined as a membership function fuzzy set A whose values are strictly monotonically increasing, or whose membership values are strictly monotonically decreasing or whose membership values are strictly monotonically increasing then strictly monotonically decreasing with increasing values for elements in the universe. Fig 1.2 represents a convex and a non convex fuzzy set.

In other words for x, y, z in the fuzzy set A with x<y<z implies

$\mu_A(y) =$ min[$\mu_A(x)$, $\mu_A(z)$] then A is said to be a convex fuzzy set.



(a) (b)

Figure: 2.2 (a) Convex, (b) Non convex

2.1.3 Representation of fuzzy set:

The method of identifying fuzzy attributes and drafting the fuzzy set is an important technique. While dealing with uncertainties, decision makers are commonly provided,with information characterized by vague linguistic descriptions such as "low risk", "high profit", "high annual interest rate" etc. The objective of fuzzy set theory is primarily concerned with the quantification of such vague descriptions. The more the object fits the vague predicate, the larger is its grade of membership. The membership function may be viewed as representing an opinion poll of human thought or an expert's opinion.

2.2 FUZZY NUMBERS

A fuzzy number A is a fuzzy set in real line which satisfies the conditions of both convexity and normality. In the literature most of the fuzzy sets use to satisfy the conditions of normality and convexity. Therefore fuzzy numbers are considered to be the most basic type of fuzzy sets.

A fuzzy set $\tilde{A}$ on the set of real number R is defined to fuzzy number if the membership functions $\tilde{A}$: R $\rightarrow$ [0, 1] satisfy the following properties:

(i) $\tilde{A}$ Must be a normal set.

(ii) There exists at least one x ∈ R with $\mu_{\tilde{A}}(x) = 1$

(iii) $\mu_{\tilde{A}}(x)$ Is piecewise continuous

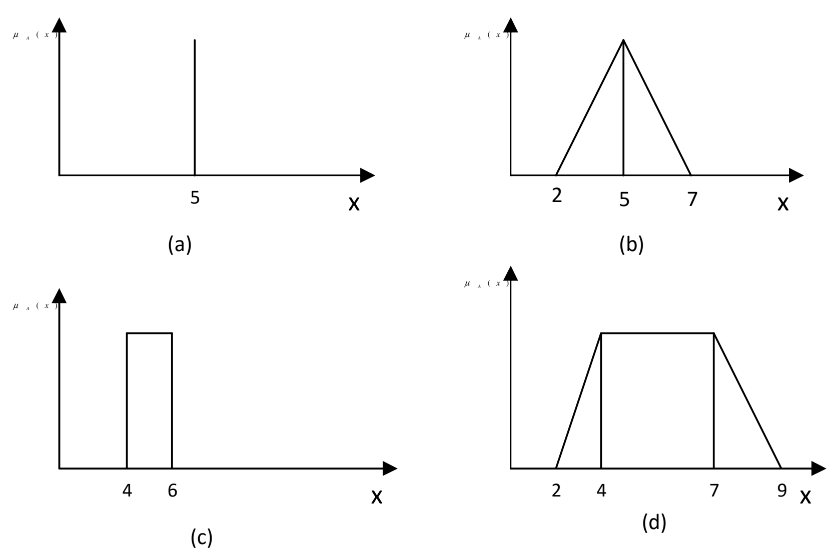


Figure 2.3 Real Number and Fuzzy Number

A fuzzy number (fuzzy set) represents a real number interval whose boundary is fuzzy. Membership functions of fuzzy numbers need be symmetrical. Fuzzy sets can have a variety of shapes. However, a triangle or a trapezoid often provides an adequate representation of the expert knowledge, simultaneously simplifying the process of computation to a significant level. So the triangular and trapezoidal shapes of membership functions are used most often for representing fuzzy numbers.

Special cases of real numbers include ordinary real numbers and interval of real numbers. Fig 1.3 (a) is a real number 5, Fig 1.3 (c) is a closed interval [2,7], Fig 1.3 (b) and Fig 1.3 (d) are a triangular fuzzy number and a trapezoidal fuzzy number expressing the concept "numbers close to 5", respectively.

2.2.1 Triangular fuzzy number

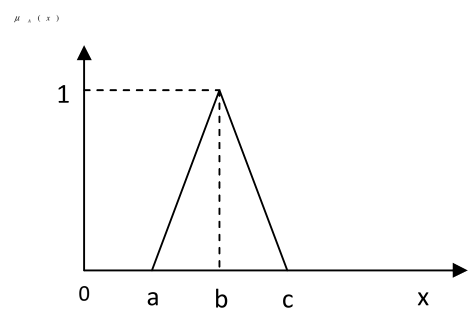


Figure 2.4 Triangular Fuzzy Number

A Triangular Fuzzy Number can be defined by a triplet (a, b, c) (with $a<b<c$) whose membership function are given by

$$\mu(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ \dfrac{c-x}{c-b} & b \le x \le c \\ 0 & c > x \end{cases}$$

2.2.2 Trapezoidal Fuzzy Number

A Trapezoidal Fuzzy Number (TFN) is a fuzzy number (a, b, c, d) (with a<b<c<d)and its membership function is defined by

$$\mu(x)=\left\{\begin{array}{cc} 0 & x<a \\ \dfrac{x-a}{b-a} & a\le x\le b \\ 1 & b\le x\le c \\ \dfrac{d-x}{d-c} & c\le x\le d \\ 0 & d<x \end{array}\right\}$$

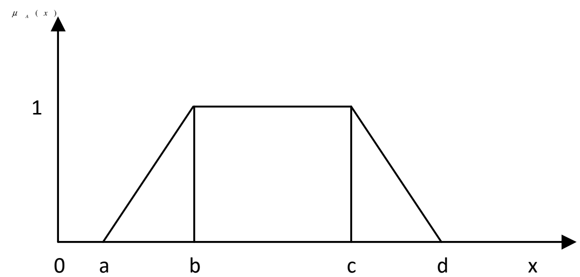


Figure 2.5 Trapezoidal Fuzzy Number

Although the triangular and trapezoidal shapes of membership function are used most often for representing fuzzy numbers, other shapes like bell shapes may be used in other applications.

2.3 LINGUISTIC VARIABLES

The idea of a fuzzy number assumes an essential job in detailing quantitative fuzzy variables. Fuzzy variables are variable whose details are fuzzy numbers. When likewise the fuzzy numbers representing linguistic ideas, for example, exceptionally small, small, and medium, etc, as translated in a specific idea, the subsequent developments are normally called linguistic variables.

Each linguistic variables, the conditions of which are communicated by linguistic term deciphered as a particular fuzzy numbers is defined in terms of base variable, the estimation of which are real number inside which a specific range. A base variable is a variable in the old style sense exemplified by any physical variable (for example temperature, Pressure, Speed and so on) just as some other numerical variable (for example Age, execution, compensation, stature and so on.) in linguistic variable phonetic terms speaking to surmised estimation of base variable, are capture by approximate suitable fuzzy numbers.

Each of fuzzy variable is fully characterized by a quintuple (v, T, X, g, m) in which v is the name of the variable, T is the set of linguistic variable that refer to base variable whose values ranges over a universal set X, g is a syntactic rule that assigns to each linguistic term $t_\in T$ its meaning m(t) which is a fuzzy set on X (i.e., m: T $\rightarrow$ F(X))

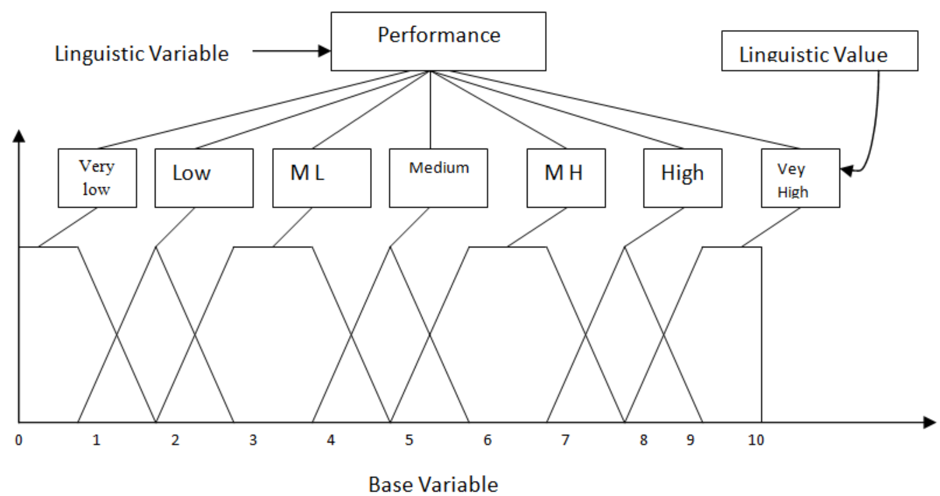


Figure 2.6 Linguistic Variable Examples

The example of linguistic variable is shown in the figure 1.6 this variable is express in the form of performance of a goal entity. The entity is given in the base variable i.e seven linguistic term very low, low , medium low, Medium, Medium high, High, Very High. Each of the linguistic term is assigned by one of the seven fuzzy number defined by figure 1.6 the fuzzy number whose membership value have the trapezoidal shape are defined by the interval [1, 10]

2.4 OVERVIEW OF DECISION MAKING AND ITS SUPPORT

The life of each person is filled with alternatives. From the moment of conscious thought to a venerable age, from morning awakening to nightly sleeping, a person meets the need to make a decision of some sort. This necessity is associated with the fact that any situation may have two or more mutually exclusive alternatives and it is necessary to choose one among them. The process of decision-making, in the majority of cases, consists of the evaluation of alternatives and the choice of the most preferable from them. Making the "correct" decision means choosing such an alternative from a possible set of alternatives, in which, by considering all the diversified factors and contradictory requirements, an overall value will be optimized (Pospelov and Pushkin, 1972); that is, it will be favorable to achieving the goal sought to the maximal degree possible

2.4.1 Fuzzy Multi criteria Decision-Making:

Models, Methods and Applications If the diverse alternatives, met by a person, are considered as some set, then this set usually includes at least three intersecting subsets of alternatives related to personal life, social life, and professional life. The examples include, for instance, deciding where to study, where to work, how to spend time on a weekend, who to elect, and so on. At the same time, if we speak about any organization, it encounters a number of goals and achieves these goals through the use of diverse types of resources (material, energy, financial, human, etc.) and the performance of managerial functions such as organizing, planning, operating, controlling, and so on, To carry out these functions, managers engage in a continuous decision-making process. Since each decision implies a reasonable and justifiable choice made among diverse alternatives, the manager can be called a decision maker (DM). DMs can be managers or expert at various levels, from a technological process manager to a chief executive officer of a large company, and their decision problems can vary in nature. Furthermore, decisions can be made by individuals or groups (individual decisions are usually made at lower managerial levels and in small organizations and group decisions are usually made at high managerial levels and in large organizations). The examples include, for instance, deciding what to buy, when to buy, when to visit a place, who to employ, and so on. These problems can concern logistics management, customer relationship management, marketing, and production planning.

2.4.2 Decision making in a fuzzy environment:

Decision making is one of the fundamental activities of human being. All of us are facing our daily life with varieties of alternative action available to us and we have to decide which action to be taken.

Making a decision is defined as the process of choosing alternatives to archive a goal. There are three distinct stages in the process of selection among available alternatives.

1. The past, in which problems developed, information accumulated and the need for a decision was perceived.
2. The present, in which alternatives are found and the choice is made and
3. The future, in which decision will be carried out and evaluated

In classical decision theory, a decision can be characterized by

- A set of decision alternatives
- A set of states of nature
- A relation indicating the state or outcome to be expected from each alternatives action finally,
- A utility or objective function which order the outcomes according to their desirability.

2.4.3 Classification of MCDM:

Since the 1960s, numbers of MCDM techniques have been created and used to describe a set of technique that can be applied by considering different criteria for the most part; they can be characterized into the accompanying groups (Hajkowicz and Collins, 2007):

1. Multi-attribute utility and value functions: The objective of these techniques is to characterize an articulation for the decision maker's inclinations using utility/esteem capacities. In light of this, all criteria are changed into a typical dimensionless scale

(Linkov et al., 2004). Mainstream techniques incorporate MAUT (multi-attribute utility theory) and MAVT (multi-attribute esteem theory), which have a compensatory nature. This infers the lackluster showing of one foundation (for example high loss of lives) can be remunerated by the better execution of another (for example money related expense). In spite of the fact that MAUT and MAVT have entrenched hypothetical establishments, the inclination elicitation can be psychologically testing and tedious (Schuwirth et al., 2012);

2. Pairwise comparisons: This methodology includes contrasting sets of criteria by asking the amount more significant one is than the other as indicated by a predefined scale. Pairwise comparisons are especially important when it is absurd to expect to characterize utility capacities. Basic procedures incorporate AHP (analytic hierarchy process), ANP (analytic network process) and MACBETH (measuring attractiveness by a categorical based evaluation technique). Because of its effortlessness and adaptability, AHP is the most useful MCDM tool. Nevertheless, AHP has a constraint when managing reliance among the criteria as it accepts that they are autonomous (Li et al., 2011). In addition, only a limited number of alternatives can be considered at the same time;

3. Distance to ideal point methods: The alternatives are assessed and ordered based on their distance from the ideal point, which represents a supposed alternative that best suits the decision makers' objectives. Consequently, the elective that is nearest to the perfect point is the best arrangement (Malczewski, 1999).

Well-known methods include (technique for order preference by similarity to an ideal solution) TOPSIS, (compromise programming) CP and (vlsekriterijumska optimizacija i kompromisno resenje) VIKOR. The main attribute and advantage of this family of approaches is the ability to consider an un-limited number of alternatives and criteria;

Other methods: there are a big number of various techniques such as ELECTRE (elimination et choix traduisant la realité), (Preference ranking organization method for enrichment of evaluations) PROMETHEE and ORESTE (organization, rangement et synthese de donnes relationnelles) and tailored methods.

2.5 METHODS

2.5.1 Fuzzy Analytic Hierarchy Process

AHP was developed in 1980 and since then it has been commonly used in various decision making situations. The quality of AHP are (1) viewing a unpredictable problem as a simple hierarchical structure with alternatives and decision attributes, (2) having the capability of recognizing the relative weight of the factors and the total values of each alternative weight in multiple attribute problems based on using a series of pairwise comparisons, and (3) calculating the consistency index (CI) of pairwise comparisons, which is the ratio of the decision makers DM's inconsistency. Since the real world is full of ambiguity and vagueness, orthodox MCDM cannot handle problems with imprecise information. In addition, incorporation of imprecise information and vagueness is an unavoidable requirement of fine multi criteria decision-making models. MCDM techniques need to tolerate vagueness; however, conventional MCDM, like traditional AHP, does not contain uncertainty in its pair wise comparisons (Yu, 2002). To deal with this kind of problem, Zadeh (1965) proposed using fuzzy set theory for complicated systems that are hard to define. Thus, fuzzy multiple criteria decision-making (FMCDM) was developed. As mentioned earlier, in decision-making, there is a fuzzy concept in comparisons as well. That is why there are many FAHP methods that are proposed by different authors in the literature (Buckley, 1985; Chang, 1996; Cheng, 1997; Deng, 1999; Mikhailov, 2004; Van Laarhoven al et., 1983). Furthermore, it is easier to know and can effectively handle both qualitative and quantitative data (Liao, 2011). In this study, the FAHP approach that was introduced by Chang (1996) was preferred. In his approach, for the pairwise comparison scale, triangular fuzzy numbers (TFNs) can be used. This is simpler and needs less complicated calculations compared with other proposed FAHP methods.

2.5.1 Chang 1996 Extent Analysis Method

The extent analysis method proposed by Chang (1996) has been widely used to obtain scrip weights from a fuzzy comparison matrix. In the method, every criteria or alternative is evaluated by linguistic variables and then the extent analysis is performed.

The algorithm of the extent analysis method can be summarized as follows:

Step 1: The pairwise comparison matrix $\widetilde{D} = [\tilde{f}_{ij}]$ is a set as follows:

$$\widetilde{D} = \begin{bmatrix} (1,1,1) & \tilde{f}_{12} & \cdots & \tilde{f}_{1n} \\ \tilde{f}_{21} & (1,1,1) & \cdots & \tilde{f}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{f}_{n1} & \tilde{f}_{n2} & \cdots & (1,1,1) \end{bmatrix} \tag{2.1}$$

Where $\tilde{f}_{ij} x \tilde{f}_{ji} \approx 1,$ i, j =1,2, …, n and all $\tilde{f}_{ij} = (\tilde{l}_{ij}, \widetilde{m}_{ij}, \tilde{u}_{ij})$, i, j =1, 2, …, n are triangular fuzzy numbers.

Step 2: The value of fuzzy synthetic extent with respect to the criteria S_i is defined as

$$S_i = \sum_{j=1}^{n} \tilde{f}_{ij} \left[\sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{f}_{ij}\right]^{-1} \tag{2.2}$$

In the equation (2.1) $\sum_{j=1}^{n} \tilde{f}_{ij}$ and $\left[\sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{f}_{ij}\right]^{-1}$ are calculated by using the fuzzy addition operation of n extent analysis for a fuzzy pairwise comparison matrix as follows:

$$\sum_{j=1}^{n} \tilde{f}_{ij} = \left(\sum_{j=1}^{n} l_j , \sum_{j=1}^{n} m_j , \sum_{j=1}^{n} u_j \right) \tag{2.3}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{f}_{ij} = \left(\sum_{i=1}^{n} l_i , \sum_{i=1}^{n} m_i , \sum_{i=1}^{n} u_i \right) \tag{2.4}$$

$$\left[\sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{f}_{ij} \right]^{-1} = \left(\frac{1}{\sum_{i=1}^{n} u_i}, \frac{1}{\sum_{i=1}^{n} m_i}, \frac{1}{\sum_{i=1}^{n} l_i} \right) \tag{2.5}$$

The principles for the comparison of fuzzy numbers were introduced to get the weight vectors of all essentials for each level of hierarchy with the use of fuzzy synthetic values.

Step 3: To compare the fuzzy numbers, the degree of possibility of $M_2 \geq M_1$ is calculated as

$$V(M_2 \geq M_1) = \sup_{y \geq x}\left[\min\left(\mu_{M_1(x)}, \mu_{M_2(y)}\right)\right] = hgt(M_1 \cap M_2) = \mu_{M_2(d)}$$

$$= \begin{cases} 1, & \text{if } m_2 \geq m_1 \\ 0, & l_1 \geq u_2 \\ \dfrac{(l_1 - u_2)}{(m_2 - u_2) - (m_1 - l_1)}, & \text{otherwise} \end{cases}$$

Where $M_1 = (l_1, m_1, u_1)$ and $M_2 = (l_2, m_2, u_2)$ and d is the ordinate of the highest intersection point D between μ_{M_1} *and* μ_{M_2} To compare M_1 and M_2, both $V(M_2 \geq M_1)$ and $V(M_1 \geq M_2)$ are needed

Step 4: The degree of possibility for a fuzzy number to be greater than k fuzzy number S_i (i=1,2,…,k) can be defined by

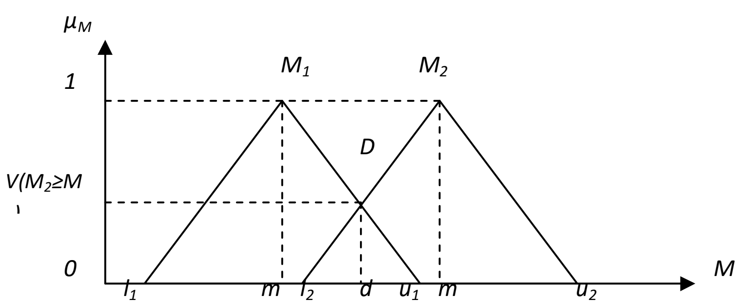


Figure 2.7 the intersection between M_1 and M_2

Assume that

$$d'(A_i) = minV(S_i \geq S_k), \text{ i, k=1,2,…,n: } k \neq i \qquad (2.6)$$

Then, the weight vector is given by

$$W' = (d'(A_1), d'(A_2), \ldots, d'(A_n))^T \quad (2.7)$$

Step 5: Via normalization, the normalized non fuzzy weight vector is computed as

$$W = (d(A_1), d(A_2), \ldots, d(A_n))^T$$

2.5.2 VIKOR method

In Opricovic 1998 (Opricovic and Tzeng 2004), developed the Serbian name VIKOR stands for 'VlseKriterijumska Optimizacija I Kompromisno Resenje', means multi-criteria optimization and compromise solution This method applies on ranking and selecting the best from a set of alternatives, which are related with multi-conflicting criteria. Moreover, it makes it easy for the decision makers to reach the final decision by finding the compromise solution (closest to the ideal) of a problem. The basic principle of VIKOR is determining the positive-ideal solution as well as the negative-ideal solution in the first place (Wu and Liu, 2011). The positive ideal solution is the best value of alternatives under the measurement criteria, and the negative -ideal solution is the worst value of alternatives under measurement criteria. In the end, arrange the precedence of the schemes based on the closeness of the alternatives assessed value to the ideal scheme. Therefore, the VIKOR method is popularly known as a multi-criteria decision making method based on the ideal point technique (Opricovic and Tzeng, 2007).

For compromise ranking of multi-criteria measurement, VIKOR adopted a following form of LP- metric aggregate function (Yu, 1973)

$$L_{P_i} = \left\{\sum_{j=1}^{n}[w_j(f_j^* - f_{ij})/(f_j^* - f_j^-)^P]\right\}^{1/P} \quad (2.8)$$

Here $1 \leq P \leq \infty$; j=1, …,n, with respect to criteria and the variable i=1, …, m, represent the number of alternatives such as A_1, A_2,, .., A_m. For alternatives Ai, the evaluated value of the *j*th criterion is denoted by f_{ij}, and n is the number of criteria. The measure L_{Pi} shows the distance between the alternative Ai and the positive-ideal solution. Within the VIKOR method L_{1i} (as Si in Equation (D)) and $L_{\infty I}$ (as Ri in equation(E) has been used to formulate the ranking measure. The value obtained by minimum S_i is with maximum group utility ('majority' rule) and the solution obtained by minimum R_i is with a minimum individual regret of the 'opponent' (Sanagyei et al., 2010). Then, the compromise ranking algorithm of the traditional VIKOR method has been the following steps (Chang, 2010; Samantra C. et al., 2009)

The fuzzy VIKOR method is briefly review as steps follows:

Step 1: Forming matrix evaluation alternatives in term of criteria as follows:

$$\tilde{D} = \left\|\tilde{x}_{ijl}\right\| \tag{2.9}$$

Where $\tilde{x}_{ij}$ is the fuzzy performance rating of alternatives (A_i, i=1,2,..,n) with respect to criteria C_j (j=12, ..m) Evaluated by k decision maker DM_l (l=1,.., k) $\tilde{x}_{ij} = (a_{ij}, b_{ij}.c_{ij})$ is a linguistic variable denoted by TFNS

Step 2: Construct the aggregated fuzzy performance decision matrix as follows:

$$\tilde{F} = \begin{bmatrix} \tilde{f}_{11} & \tilde{f}_{12} & \vdots & \tilde{f}_{1n} \\ \tilde{f}_{21} & \tilde{f}_{22} & \vdots & \tilde{f}_{2n} \\ \cdots & \cdots & \ddots & \vdots \\ \tilde{f}_{m1} & \tilde{f}_{m2} & \cdots & \tilde{f}_{mn} \end{bmatrix} \tag{2.10}$$

Where

$$\tilde{f}_{ij} = \frac{1}{k}\sum_{l=1}^{k} x_{ijl} \qquad \forall\, i, j$$

Step 3: Determine the fuzzy best value (FBV) and fuzzy worst value (FWV)

$$\tilde{f}_j^* = max\tilde{f}_{ij} , \forall\, i \qquad \tilde{f}_j^- = min\tilde{f}_{ij}, \forall\, i \tag{2.11}$$

Step 4: Calculate the normalized fuzzy distance d_{ij} , i=1,2,m, j=1,2,...n

$$d_{ij} = \frac{d\left(\tilde{f}_j^* - \tilde{f}_{ij}\right)}{d(\tilde{f}_j^* - \tilde{f}_j^-)} \tag{2.12}$$

Step 5: Compute the value of S_i and R_i (i=1,…, m) by using the relations:

$$S_i = \sum_{j=1}^{n} w_j * d_{ij} \tag{2.13}$$

$$R_i = \max_j \left(w_j * d_{ij}\right) \tag{2.14}$$

S_i is the aggregate value of i[th] alternatives with a maximum group utility and R_i is the aggregated value of i[th] alternatives with a minimum individual regret of 'opponent'.

Step 6: Compute the value of Q_i for I = 1, ..., m with the relation,

$$Q_i = \frac{V(S_i - S^*)}{(S^- - C)} + \frac{(1-v)(R_i - R^*)}{(R^- - R^*)} \qquad (2.15)$$

Here, $S^* = \min_{i=1,\dots,m} S_i$, $S^- = \max_{i=1,\dots,m} S_i$, $R^* = \min_{i=1,\dots,m} R_i$, $R^- = \max_{i=1,\dots,m} R_i$ and *v* is a weight for the strategy of maximum group utility, and v = 0.5 where as 1-v is the weight of individual regret. The compromise can be selected with 'voting by majority' (v> 0.5), with 'consensus' (v = 0.5) with 'veto' (v<0.5)

Step 7: Rank the alternatives by sorting each S, R and Q values in ascending order.

Step 8: If the two conditions are satisfied simultaneously, then the system with minimum value of Q in ranking is measured the optimal compromise solution. Such as,

C1. The alternative Q ($A^{(1)}$) has an *acceptable advantage*; in other words,

$Q(A^{(2)}) - Q(A^{(1)}) \geq 1/(m-1)$.

Where, $A^{(2)}$ is the alternative with the second position in the ranking list by and *m* is the number of alternatives.

C2. The alternative $Q(A^{(1)})$ is *stable within the decision making process*; in other words, it is also best ranked in S_i and R_i.

If condition C1 is not satisfied, that means $Q(A^{(m)}) - Q(A^{(1)}) < 1/(m-1)$, then alternatives $A^{(1)}, A^{(2)} \dots A^{(m)}$ all are the same compromise solution, there is no comparative advantage of $A^{(1)}$ from others. But the case of maximum value, the corresponding alternative is the compromise (closeness) solution. If condition C2 is not satisfied, the stability in decision making is deficient while $A^{(1)}$ has a comparative advantage. Therefore, $A^{(1)}$ and $A(2)$has the same compromise solution.

Step 9. Select the best alternative by choosing $Q(A^{(m)})$ as a best compromise solution with the minimum value of Q_i and must have to satisfy with the above conditions (Park *et al.* 2011).

2.5.3 Fuzzy TOPSIS

Chen (2000) and Li (2007) extended The Order Preference by Similarity to Ideal Solution (TOPSIS) method to present methods that handles fuzzy multi- criteria group decision making problems under fuzzy environment. Consider that there are k decision- makers DM_1, DM_2, …, DMk. m alternatives and n criteria C_1, C_2, …, C_n. Chen's method of fuzzy multi criteria-decision making is reviews as follows:

Step 1: The decision makers apply the linguistic variable to evaluate the importance of the criteria

Step 2: The decision-makers apply the linguistic rating variables to estimate the rating of the alternatives with respect to each criteria.

Step 3: Convert the linguistic evaluation into triangular fuzzy numbers to construct the fuzzy decision matrix $\widetilde{D}$

$$\widetilde{D} = \begin{array}{c} \\ X_1 \\ X_2 \\ \vdots \\ X_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1n} \\ \tilde{x}_{21} & \tilde{x}_{22} & \cdots & \tilde{x}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{x}_{m1} & \tilde{x}_{m2} & \cdots & \tilde{x}_{mn} \end{bmatrix} \end{array}$$

(2.16)

Where$\tilde{X}_{ij} = \frac{1}{k}\left[\tilde{x}_{ij}^{1}\ (+)\tilde{x}_{ij}^{2}\ (+)\ ...\ (+)\ \tilde{x}_{ij}^{p}\ (+)\ ...\ (+)\tilde{x}_{ij}^{k}\right]$, $\tilde{x}_{ij}^{p}$ denotes the rating of the pth decision maker with respect to criteria C_j to the alternatives x_i, where $1 \le p \le k$, $1 \le j \le m$ and $1 \le j \le n$. Construct fuzzy weighting vector W, shown as follows:

$$\widetilde{W} = [\widetilde{w}_1, \widetilde{w}_2, ..\widetilde{w}_n], \tag{2.17}$$

Where the weight of criteria C_i is denoted by$\widetilde{w}_j$, that is

$\widetilde{W}_{ij} = \frac{1}{k}\left[\widetilde{w}_{ij}^1 \,(+)\, \widetilde{w}_{ij}^2 \,(+) \ldots (+)\, \widetilde{w}_{ij}^p \,(+) \ldots (+)\, \widetilde{w}_{ij}^k\right]$, $\widetilde{w}_j^p$ denotes the importance weight of decision maker DM_p with respect to C_i where 1≤p≤k, 1≤j≤n.

Step 4: Construct the normalized fuzzy decision matrix

$$\tilde{R} = \left[\tilde{r}_{ij}\right]_{mxn}, \tag{2.18}$$

where 1≤i≤m, 1≤j≤n.

Step 5: Construct the weighted normalized fuzzy decision matrix

$$\tilde{V} = \left[\tilde{v}_{ij}\right]_{mxn}, \tag{2.19}$$

where $\tilde{v}_{ij} = \tilde{r}_{ij}(\otimes)\, \widetilde{w}_{ij}$ and $(\otimes)$ denotes the multiplicative operator between fuzzy numbers.

Step 6: Determine the fuzzy positive ideal solution F^* and fuzzy negative ideal solution F^-, where $F^* = (\tilde{v}_1^*, \tilde{v}_2^*, \ldots, \tilde{v}_n^*)$ and $F^- = (\tilde{v}_1^-, \tilde{v}_2^-, \ldots, \tilde{v}_n^-)$

Step 7: Calculate the distance d_i^* between the alternative between the alternative $x_i = (\tilde{v}_{1j}^-, \tilde{v}_{2j}^-, \ldots, \tilde{v}_{mj}^-)$ and the fuzzy positive ideal solution F^* where 1≤i≤m. Calculate the distance d_i^- between the alternative $x_i = (\tilde{v}_{1j}^-, \tilde{v}_{2j}^-, \ldots, \tilde{v}_{mj}^-)$ and the fuzzy negative ideal solution F^- where 1≤ i ≤ m, shown as follows:

$$d_i^* = \sum_{j=1}^{n} d(\tilde{v}_{ij}, \tilde{v}_j^*) \tag{2.20}$$

$$d_i^- = \sum_{j=1}^{n} d(\tilde{v}_{ij}, \tilde{v}_j^-) \tag{2.21}$$

Where d_i^* denotes the distance between alternative x_i and the fuzzy positive ideal solution F^* Where d_i^- denotes the distance between alternative x_i and the fuzzy negative ideal solution F^- $d(\tilde{v}_{ij}, \tilde{v}_j^*)$ denotes the distance between the fuzzy numbers $\tilde{v}_{ij}$ and $\tilde{v}_j^*$, $d(\tilde{v}_{ij}, \tilde{v}_j^-)$ denotes the distance between the fuzzy numbers $\tilde{v}_{ij}$ and $\tilde{v}_j^-$, 1≤ i ≤ m and 1≤ j ≤ n

Step 8: Calculate the closeness coefficient CCi of alternative x_i,

where 1≤ i ≤ m, shown as follows:

$$CC_i = \frac{d_i^-}{d_i^* + d_i^-}, \tag{2.22}$$

Where d_i^* denotes the distance between alternative x_i and the fuzzy positive ideal solution F^*, d_i^* denotes the distance between alternative x_i and the fuzzy negative ideal solution F^- , 1≤ i ≤m.

Step 9: Based on the closeness coefficient of each alternative calculated in Step 8, the ranking order of the alternatives is obtained. The larger the value of the closeness coefficient CC_i of alternative x_i, 1≤ i ≤m.