# CHAPTER 4

# PROXIMALLY COARSE CLS-UNIFORM SPACES

## 4.1 Introduction

In the previous chapter, the notion CLS-uniform spaces had developed in the category of **C-TOP**. Various important results concerning interior space, topological interior space, uniformly continuous function is developed and also carried out problem of metrizablity. It is however, observed that the generating spaces in the above context are interior spaces, which are generalisation of $L-$topological spaces. On the other hand $L-$fuzzy basic proximity spaces(In short $L-$fbps) generates interior spaces. So, it becomes pertinent to investigate whether there is a relation between CLS-uniform spaces and $L-$fbps in $L-$topological spaces.

The relation between $L-$ fuzzy proximity spaces and CLS-uniform spaces is obtained and it shows that every $L-$fuzzy basic proximity space (in short $L-$fbps) induces CLS-uniform space, for a given $L-$fuzzy basic proximity there exists a uniformly coarsest one which will be called the proximally coarse CLS-uniform spaces. It turns out that CLS-uniformity is totally bounded if $L-$fbps induce it.

## 4.2 $L-$fbps in the context of CLS-Uniform spaces

In this section, we study the relation between the CLS-uniform space and $L-$fbps, and then the relation them is obtained. It is found that every $L-$fbps is induced by totally bounded CLS-uniformity.

**Theorem 4.2.1.** *Let $(L^X, \mathfrak{S})$ be a CLS-uniform spaces. Define a relation $\delta$ on $L^X$ such that*

$$A\delta_{\mathfrak{S}}B \text{ iff } st(A,\mathscr{C})\bigcap st(B,\mathscr{C}) \neq \underline{0} \ A, B \in L^X, \mathscr{C} \in \mathfrak{S}$$

*Then $\delta_{\mathfrak{S}}$ is an $L-$fbps on $L^X$.*

*Proof.* **(PB1)** Let $A, B \in L^X$, $\mathscr{C} \in \mathfrak{S}$ then

$$\begin{aligned} A\delta B &\Rightarrow st(A,\mathscr{C})\bigcap st(B,\mathscr{C}) \neq \underline{0} \\ &\Rightarrow st(B,\mathscr{C})\bigcap st(A,\mathscr{C}) \neq \underline{0} \\ &\Rightarrow B\delta A \end{aligned}$$

**(PB2)** Let $A, B, C \in L^X$, $\mathscr{C} \in \mathfrak{S}$ then

$$\begin{aligned} A\delta(B\bigcup C) &\Leftrightarrow st(A,\mathscr{C})\bigcap st(B\cup C,\mathscr{C}) \neq \underline{0} \\ &\Leftrightarrow st(A,\mathscr{C})\bigcap [st(B,\mathscr{C}) \cup st(C,\mathscr{C})] \neq \underline{0} \text{ by Proposition2.4.2(4)} \\ &\Leftrightarrow st(A,\mathscr{C})\bigcap st(B,\mathscr{C}) \neq \underline{0} \text{ or } st(A,\mathscr{C})\bigcap st(C,\mathscr{C}) \neq \underline{0} \\ &\Leftrightarrow A\delta B \text{ or } A\delta C \end{aligned}$$

**(PB3)** Since $st(\underline{0}, \mathscr{C}) = \underline{0}$ which implies $\underline{0}\not\delta A$ for every $A \in L^X$;

**(PB4)** $A$ and $B$ are $L-$quasi-coincident then by Corollary 2.2.14 we have $A \cap B \neq \underline{0}$ and Let $\mathscr{C} \in \mathfrak{S}$, then by proposition 2.4.2 we have

$$st(A \cap B, \mathscr{C}) \subseteq st(A, \mathscr{C}) \cap st(B, \mathscr{C}) \neq \underline{0}$$

Hence $A\delta B$.

□

Now from Definition (2.6.3) we have

**Corollary 4.2.2.** *Let $\delta_{\mathfrak{C}}$ is a $L-$fbps induced by a CLS-uniform space $(L^X, \mathfrak{S})$. Then $A \in \mathscr{N}(B)$ iff there is an $L-$cover of $\mathscr{C} \in \mathfrak{S}$ such that $st(A, \mathscr{C}) \bigcap st(B', \mathscr{C}) = \underline{0}$.*

Now we tackle up the problem how can one construct a CLS-uniform spaces when $L-$fbps is given.

**Definition 4.2.1.** Let $(L^X, \delta)$ be a $L-$fbps. Then an $L-$covering $\mathscr{U} = \{U_i : i \in \Lambda\}$ of $L^X$ is called as $\delta-$cover of $L^X$ iff there exists a refinement of $\mathscr{U}$ of $\mathscr{V} = \{V_i : i \in \Lambda\}$ such that for each $i$, $V_i$ is a $\delta-$nbhd of $U_i$ and in that case we called $\mathscr{V}$ a $\delta-$refinement of $\mathscr{U}$.

**Theorem 4.2.3.** *Let $(L^X, \delta)$ be a $L-$fbps then the collection of finite $\delta$-covers forms a base for CLS-uniformity on $L^X$*

*Proof.* Let $(L^X, \delta)$ be an $L-$fbps, the collection $\mathfrak{U}$ of all finite $\delta-$covers of $L^X$. We claim that $\mathfrak{U}$ is a base for CLS-uniformity.

**(SC1)** It follows from Definition 4.2.1.

**(SC2)** Let $\mathscr{U} = \{U_i : 1 \leq i \leq n\}, \mathscr{V} = \{V_i : 1 \leq i \leq n\}$, are finite $\delta-$covers of $L^X$. Without loss of generality consider a finite $L-$covering $\mathscr{W}_i = \{W_i : 1 \leq i \leq n\}$ such that $W_i$ is $\delta-$nbhd of $U_i$ and $W_i$ is $\delta-$nbhd of $U_i$ for each $i$. Then by Theorem (2.6.2) we have $W_i$ is $\delta-$nbhd of $U_i \bigcap V_i$ which further implies $\mathscr{U} \bigcap \mathscr{V}$ is finite $\delta-$covering of $L^X$

□

**Theorem 4.2.4.** *Let $(L^X, \mathfrak{S})$ be a CLS-uniform space, then $\mathfrak{S}_\delta$ is coarser than $\mathfrak{S}$.*

*Proof.* Let $\mathscr{U} = \{U_i : 1 \leq i \leq n\}$ is called $\delta-$cover of the $L-$fbps space $L^X$ . Then by Definition 4.2.1 there exits another $L-$covering $\mathscr{V} = \{V_i : 1 \leq i \leq n\}$ such that $\mathscr{V}$ refinement of $\mathscr{U}$. Hence the theorem. □

**Theorem 4.2.5.** *Let $(L^X, \mathfrak{S}_\delta)$ be a CLS-uniform space generate by $(L^X, \delta)$ be an $L-$fbps, denote $\delta^* = \delta_{\mathfrak{S}_\delta}$, then $\delta^* = \delta$.*

*Proof.* Let us suppose that $A\delta B$, then for any $\mathscr{A} \in \mathfrak{S}_\delta$, $st(A, \mathscr{A}) \bigcap st(B, \mathscr{A}) \neq \underline{0}$, which implies $A\delta^* B$.

Conversely suppose that $A\delta^* B$ then for any finite $\delta^*-$cover $\mathscr{B}$ of $(L^X, \delta^*)$ we have $st(A, \mathscr{B}) \bigcap st(B, \mathscr{B}) \neq \underline{0}$. Since $\mathscr{B}$ is finite $\delta^*-$cover, so we can consider $\mathscr{B} = \{B_i : 1 \leq i \leq n\}$, there exits another finite $L-$cover $\mathscr{C} = \{C_i : 1 \leq i \leq n\}$ such that $\mathscr{C}$ a $\delta-$refinement of $\mathscr{B}$ which implies $st(A, \mathscr{C}) \bigcap st(B, \mathscr{C}) \neq \underline{0}$ and hence $A\delta B$. □

We now proceed to find the relation between the collection of all $L-$fbps and the collections of CLS-uniform spaces on $L^X$. Given a $L-$fbps on $L^X$, CLS-uniformity constructed in Theorem 4.2.3 will denoted by $\mathfrak{S}_\delta$

**Definition 4.2.2.** Let $(L^X, \mathfrak{S})$ be a CLS-uniformity, then it is called totally bounded if for each $\mathscr{A} \in \mathfrak{S}$ there exists a finite $A \subseteq Pt(L^X)$ such that $st(A, \mathscr{A}) = \underline{1}$.

**Theorem 4.2.6.** *For any $L-$fbps $\delta$, the CLS-uniform space $\mathfrak{S}_\delta$ is totally bounded.*

*Proof.* Let $\mathscr{A} \in \mathfrak{S}_\delta$ then $\mathscr{A}$ is a finite $\delta-$cover. So, $\mathscr{A} = \{A_i : 1 \leq i \leq n, n \in \mathbb{Z}\}$. For each $A_i$ we collect one $x_{\alpha_i}$ fuzzy point and denote it by $A^* = \{x_{\alpha_i} : 1 \leq i \leq n\}$. Then $A^* \subseteq Pt(L^X)$ is finite [since $\mathscr{A}$ is finite $\delta-$cover] also $st(A^*, \mathscr{A}) = \underline{1}$ since $\bigcup \mathscr{A} = \underline{1}$. Hence $\mathfrak{S}_\delta$ is totally bounded. □

Then by the Theorem 4.2.3 we have the following corollary

**Corollary 4.2.7.** *Every $L-$fbps is induced by totally bounded CLS-uniform space.*

From above we may conclude that "There is a one-one correspondence between $L-$fbps and the collection of all totally bounded CLS-uniform spaces."

## 4.3 Proximally coarse CLS-Uniform spaces

In this section, the study of proximally CLS-uniform spaces considered, and then it is found that for a given $L-$fbps has a uniformly coarsest CLS-uniform space that will be called the proximally coarse CLS-uniform space. It turns out that proximally coarsest CLS-uniform spaces are totally bounded CLS-uniform spaces.

**Theorem 4.3.1.** *$\mathfrak{S}_\delta$ is the only totally bounded CLS-uniformity giving the $L-$fbps $\delta$.*

*Proof.* It is sufficient to show that the finite $\delta-$covers form a base for any totally bounded CLS-uniformity $\mathfrak{S}$ which generates $\delta$. Let $\mathscr{U}$ be a any finite uniform cover, $\mathscr{V} \preccurlyeq \mathscr{U}$ and for each $V \in \mathscr{V}$, let us consider $U_v \in \mathscr{U}$ such that $V \subseteq U_v$. Then $\mathscr{U}_0 = \{U_v : V \in \mathscr{V}\}$ is a finite cover which refines $\mathscr{U}$ so it suffices to show that $\mathscr{U}_0$ is a $\delta-$cover. If $\mathscr{W}$ is refinement $\mathscr{V}$, then $st(V, \mathscr{W}) \bigcap st(U_v', \mathscr{W}) = \underline{0}$; it follows that $V \not\delta U_v$, i.e., $V$ is $\delta-$nbhd of $U_v$. □

**Corollary 4.3.2.** *Let a CLS-uniform space $\mathfrak{S}$ on $L^X$ induces $L-$fbps $\delta$ on $L^X$. Then $\mathfrak{S}$ is the uniformly coarsest CLS-uniform space inducing $\delta$ iff the finite $\delta-$covers of $\mathfrak{S}$ form a base for $\mathfrak{S}$ i.e., if $\mathfrak{S}$ and $\mathfrak{S}^*$ are two CLS-uniform spaces induces same $L-$fbps such that finite $\delta-$cover form a base for $\mathfrak{S}$, then $\mathfrak{S} \subset \mathfrak{S}'$*

**Definition 4.3.1.** A CLS-uniform space $\mathfrak{S}$ will be called proximally coarse iff finite $\delta-$covers form a base for $\mathfrak{S}$.

**Theorem 4.3.3.** *Every $L-$fbps is induced by a CLS-uniformity. Among all the CLS-uniformity inducing a given $L-$fbps $\delta$ there exists unique proximally coarse CLS-uniformity $\mathfrak{S}$.*

*Proof.* Follows from Theorem 4.2.3 and Corollary 4.3.2. □

**Theorem 4.3.4.** *Every proximally coarse CLS-uniformity is totally bounded.*

*Proof.* Let $\mathfrak{S}$ be a proximally coarse CLS-uniform space for $L^X$ and let $\mathfrak{S}^*$ be a collection of finite $\delta-$covers of $\mathfrak{S}$. If $\mathscr{U} \in \mathfrak{S}$, then $\mathscr{V} \preccurlyeq \mathscr{U}$ for some $\mathscr{V} \in \mathfrak{S}^*$ where $\mathscr{V}$ is finite $L-$cover. Now if $A \subseteq Pt(L^X)$ is finite $L-$fuzzy set with $A \bigcap V_i \neq \underline{0}$ and hence clearly $st(A, \mathscr{V}) = \underline{1}$. □

**Theorem 4.3.5.** *The class of all proximally coarse CLS-uniform spaces is hereditary and closed under arbitrary products*

*Proof.* If $L^X \subset L^Y$ and $\mathscr{U}$ is finite $\delta-$covers of $L^X$, then $L^Y \bigcap \mathscr{U}$ is a finite $\delta-$ cover of $L^Y$ and therefore every relative of proximally coarse CLS-uniform space proximally coarse.

If $(L^X, \mathfrak{S})$ product of family $\{(L^{X_\alpha}, \mathfrak{S}_\alpha\}$, where $L^X = \prod L^{X_\alpha}$ and $\mathfrak{S} = \prod \mathfrak{S}_\alpha$ and $\mathscr{U}_\alpha \in \mathscr{S}_\alpha$ is a finite $\delta-$cover, then $\prod \mathscr{U}_\alpha$ is also $\delta-$cover of $\prod L^{X_\alpha}$. This shows that $\mathfrak{S}$ is proximally coarse. □

**Theorem 4.3.6.** *If $f^{\rightarrow} : (L^X, \mathfrak{S}_1) \rightarrow (L^Y, \mathfrak{S}_2)$ be a semi-uniformly continuous, then $f^{\rightarrow} : (L^X, \delta_{\mathfrak{S}_1}) \rightarrow (L^Y, \delta_{\mathfrak{S}_2})$ is proximally continuous.*

*Proof.* Let $f^{\rightarrow} : (L^X, \mathfrak{S}_1) \rightarrow (L^Y, \mathfrak{S}_2)$ is semi-uniformly continuous. Then $f^{\leftarrow}(\mathscr{B}) \in \mathfrak{S}_1$ for each $\mathscr{B} \in \mathfrak{S}_2$, where $f^{\leftarrow}(\mathscr{B}) = \{f^{\leftarrow}(B) : B \in \mathfrak{B}\}$. Let $A, B \in L^Y$ and suppose $A \not\delta_2 B$ then there exists $\mathscr{C} \in \mathfrak{S}_2$ such that $st(A, \mathscr{C}) \bigcap B = \underline{0}$, Since $f^{\rightarrow}$ is semi-uniformly continuous, so $f^{\leftarrow}[st(A, \mathscr{C}) \bigcap B] = f^{\leftarrow}(\underline{0})$ implies that $st(f^{\leftarrow}(A), f^{\leftarrow}(\mathscr{C}) \bigcap f^{\leftarrow}(B) = \underline{0}$ which further implies that $f^{\leftarrow}(A) \not\delta_1 f^{\leftarrow}(B)$. Hence the Theorem 2.6.4, $f^{\rightarrow} : (L^X, \delta_{\mathfrak{S}_1}) \rightarrow (L^Y, \delta_{\mathfrak{S}_2})$ is proximally continuous.

□

**Theorem 4.3.7.** *If $(L^X, \mathfrak{S})$ be a proximally coarse CLS-uniform space, then every proximally continuous mapping of CLS-uniform spaces in to $(L^X, \mathfrak{S})$ is semi-uniformly continuous.*

*Proof.* Suppose $f^{\rightarrow}$ is a proximally continuous mapping of a CLS-uniform space $(L^X, \mathfrak{S}_1)$ into proximally coarse CLS-uniform space $(L^Y, \mathfrak{S})$. To prove $f^{\rightarrow}$ is semi-uniformly continuous it sufficient to find a sub-base $\mathfrak{B}$ for $\mathfrak{S}$ such that $f^{\leftarrow}(\mathscr{B}) \in \mathfrak{S}_1$ for each $\mathscr{B} \in \mathfrak{S}$. Considering $\mathfrak{B}$ is collection of finite $\delta-$cover for the $L-$fbps induced by $\mathfrak{S}$. Since $f^{\rightarrow}$ is proximally continuous for any $A, B \in L^Y$ $A \not\delta_2 B$ implies $f^{\leftarrow}(A) \not\delta_1 f^{\leftarrow}(B)$. Then by assumption there exists $\mathscr{B} \in \mathfrak{B}$ such that

$$st(A, \mathscr{B}) \bigcap B \neq \underline{0}$$
$$\Rightarrow st(f^{\leftarrow}(A), f^{\leftarrow}(\mathscr{B})) \bigcap f^{\leftarrow}(B) \neq \underline{0}$$

Hence $f^{\leftarrow}(\mathfrak{B})$ is finite $\delta_1$ cover of $L^X$ where $\delta_1$ is $L-$fbps induced by $\mathfrak{S}_1$. Hence the theorem. □

From above theorem it has the following characteristic for proximally coarse CLS-uniform spaces.

**Theorem 4.3.8.** *A CLS-uniform space* $\mathfrak{S}$ *is proximally coarse iff every proximally continuous mapping of CLS-uniform space into* $\mathfrak{S}$ *is semi-uniformly continuous.*

*Proof.* Immediate consequence of the Theorem 4.3.7. □

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