CHAPTER 3

COVERING $L-$ SEMI UNIFORM SPACES

3.1 Introduction

In the first chapter, a sketch on development of uniform spaces on the various category of fuzzy topological spaces and it was pointed that important notion covering $L-$ semi-uniform space not developed and remained unexplored in the category of **C-TOP**. For filling up this void in the development of theory of generalisation of uniform spaces in fuzzy topological spaces. In this chapter developed covering $L-$semi uniform space (In short CLS-uniform spaces) in the category of **C-TOP** by generalising covering $L-$uniform space.

In the first section introduced the notion of CLS-uniform space and build up this notion in terms of various properties. In second section the study of uniformly continuous function in the context of CLS-Uniform spaces, and then studied its properties. In the last section, condition for metrization in the same is presented.

3.2 CLS-uniform spaces

This section introduced a CLS-uniform structure and studied essential notions such as interior operator, topological interior operator, and some significant results were obtained.

Definition 3.2.1. A non-empty family $\mathfrak{S}$ of $L-$covers of L^X is said to be covering semi-uniform spaces (in short CLS-uniform space) if it satisfies the following conditions:-

(SC1) $\mathscr{A} \preccurlyeq \mathscr{B}, \mathscr{A} \in \mathfrak{S} \Rightarrow \mathscr{B} \in \mathfrak{S}$.

(SC2) For every $\mathscr{A}, \mathscr{B} \in \mathfrak{S}, \mathscr{A} \bigcap \mathscr{B} \in \mathfrak{S}$.

A CLS-uniform space will be denoted by $(L^X, \mathfrak{S})$.

Definition 3.2.2. A non-empty sub-family $\mathfrak{B}$ of $\mathfrak{S}$ is called base for CLS-uniform space on L^X if for any $\mathscr{S} \in \mathfrak{S}$, there is $\mathscr{B} \in \mathfrak{S}$ such that $\mathscr{B} \preccurlyeq \mathscr{S}$.

Example 3.2.1. Let $X = \{a, b, c\}$ with $L = [0, 1]$. Then clearly $\mathscr{A} = \{\{a\}, \{b\}, \{a, b\}, \{c\}\}$ and $\mathscr{B} = \{\{a\}, \{a, b\}, \{b, c\}, \{c\}\}$ are $L-$ covers then $\mathfrak{B} = \{\mathscr{A}, \mathscr{B}\}$ is a CLS-uniform space on L^X.

Lemma 3.2.1. *Let $(L^X, \mathfrak{S})$ be an CLS-uniform space. The mapping $int : L^X \to L^X$ defined by*

$$int(A) = \bigcup\{x_\alpha : st(x_\alpha, \mathscr{C}) \subseteq A,\ for\ some\ \mathscr{C} \in \mathfrak{S}\}$$

is an interior operator on L^X

Proof. **(IO1)** Clearly, $int(\underline{1}) = \underline{1}$ and

(IO2) $int(A) \subseteq A$

(IO3) By (SC2) we have $int(A \bigcap B) = int(A) \bigcap int(B)$.

$\square$

Every CLS-uniform space generates an interior space. Generated interior space of CLS-Uniform space but not topological interior for this an example is given below,

Example 3.2.2. Let $X = \{a, b, c\}$ and $L = [0, 1]$.

Then $\mathscr{A} = \{\{a, b\}, \{b, c\}\}, \mathscr{B} = \{a, b, c\}\}$ are L-covers of X.

The collection $\mathfrak{S} = \{\mathscr{A}, \mathscr{B}\}$ forms a base for CLS-Uniform space.

Now considering, $A = \{a, b\}$,

Then

$$\begin{aligned} int(A) &= \bigcup\{x_\alpha : st(x_\alpha, \mathscr{C}) \subseteq A,\ for\ some\ \mathscr{C} \in \mathfrak{S}\} \\ &= \{a\} \\ \Rightarrow int(int(A)) &= \bigcup\{x_\alpha : st(x_\alpha, \mathscr{C}) \subseteq int(A),\ for\ some\ \mathscr{C} \in \mathfrak{S}\} \\ &= \underline{0} \end{aligned}$$

Hence $int(A) \neq int(int(A))$

Lemma 3.2.2. *For every $L-$covers $\mathscr{A}$ and for each $A \in L^X$, we have*

$$st(A, \mathscr{A}) = \bigcap\{B \mid st(B', \mathscr{A}) \subseteq A'\}$$

Proof. It follows from the fact that for any $B \in L^X$, $B \subseteq st(A, \mathscr{A})$ if and only if $A \subseteq st(B, \mathscr{A})$ as $A \bigcap B \neq \underline{0}$ if and only if $B \bigcap A \neq \underline{0}$. $\square$

Lemma 3.2.3. *Let $(L^X, \mathfrak{S})$ be a CLS-uniform space and $cl : L^X \to L^X$ be a mapping such that for any $A \in L^X$,*

$$cl(A) = \bigcap\{st(A, \mathscr{A}) : \mathscr{A} \in \mathfrak{S}\}$$

Then $cl(A) = int(int(A'))', \forall A \in L^X$.

Proof. For any $A \in L^X$, we have

$$\begin{aligned} int(A') &= \bigcup\{x_\alpha \in L^X \mid st(x_\alpha, \mathscr{A}) \subseteq A' \text{ for some } \mathscr{A} \in \mathfrak{S}\}. \\ &= \bigcup\{\bigcup\{x_\alpha \in L^X \mid st(x_\alpha, \mathscr{A}) \subseteq A'\}, \mathscr{A} \in \mathfrak{S}\}. \\ &= \bigcup\{st(A, \mathscr{A})' \mid \mathscr{A} \in \mathfrak{S}\} \quad \text{[By Lemma 3.2.2]}. \\ \text{Hence, } int(A')' &= \bigcap\{st(A, \mathscr{A}) \mid \mathscr{A} \in \mathfrak{S}\}. \\ &= cl(A). \end{aligned}$$

□

Theorem 3.2.4. *Let* $(L^X, \mathfrak{S})$ *be an CLS-uniform space. Then the required condition for generated interior space to be topological interior space is* $\forall \mathscr{A} \in \mathfrak{S}$ *and* $\forall x_\alpha$ *there exists* $\mathscr{B} \in \mathfrak{S}$ *such that* $\forall y_\beta \in st(x_\alpha, \mathscr{B})$ *there corresponds* $\mathscr{C} \in \mathfrak{S}$ *with* $st(y_\beta, \mathscr{C}) \subseteq st(x_\alpha, \mathscr{A})$.

Proof. Let $x_\alpha \in int(A)$, then there exist some $\mathscr{A} \in \mathfrak{S}$ such that $st(x_\alpha, \mathscr{A}) \subseteq A$. Let $\mathscr{B} \in \mathfrak{S}$ such that for any $y_\beta \in st(x_\alpha, \mathscr{B})$ there is $\mathscr{C} \in \mathfrak{S}$ such that $st(y_\beta, \mathscr{C}) \subseteq st(x_\alpha, \mathscr{A})$. Since $x_\alpha \in st(x_\alpha, \mathscr{B})$ we may choose $\mathscr{C} \in \mathfrak{S}$ such that $st(x_\alpha, \mathscr{C}) \subseteq st(x_\alpha, \mathscr{A})$. This implies $x_\alpha \in int(int(A))$ and since the other inclusion follows by $(IO2)$ in Lemma 3.2.1, we have $int(A) = int(int(A))$.

□

Theorem 3.2.5. *An CLS- uniform spaces with the condition in Theorem 3.2.4 generates an* $L-$*topological spaces.*

Proof. It follows from Lemma 3.2.1 and Theorem 3.2.4. □

The $L-$topology induced by an CLS-uniform spaces is denoted by $\mathbb{F}(\mathfrak{S})$. It may conclude that the topological CLS-uniform spaces lies between CLS-uniform space and covering $L-$uniform spaces.

Theorem 3.2.6. *Let $(L^X, \mathfrak{S}_1)$ and $(L^X, \mathfrak{S}_2)$ be two CLS-uniform spaces. If $\mathfrak{S}_1 \subseteq \mathfrak{S}_2$, then $\mathbb{F}(\mathfrak{S}_1) \subseteq \mathbb{F}(\mathfrak{S}_2)$.*

Proof. Straight forward. □

Theorem 3.2.7. *Let $(L^X, \mathfrak{S})$ be an CLS-uniform spaces, then $\{st(x_\alpha, \mathscr{A}) : \mathscr{A} \in \mathfrak{S}\}$ is a base for nbds(=neighbourhoods) of x_α in interior spaces*

Proof. Suppose $G \in L^X$ is open and $x_\alpha \in G$. Since $int(G) = G$, there exists $\mathscr{A} \in \mathfrak{S}$ such that $st(x_\alpha, \mathscr{A}) \subseteq G$. Thus $\{st(x_\alpha, \mathscr{A}) : \mathscr{A} \in \mathfrak{S}\}$ is a base for nbds of x_α. □

3.3 Semi-Uniformly Continuous

In this section, semi-uniformly continuous function in the context of CLS-Uniform spaces is introduced, and then studied its properties.

Definition 3.3.1. Let $(L^X, \mathfrak{S}_1)$ and $(L^Y, \mathfrak{S}_2)$ be two CLS-uniform spaces. Then $f^\rightarrow : L^X \rightarrow L^Y$ is called semi-uniformly continuous if $f^\leftarrow(\mathscr{B}) \in \mathfrak{S}_1$ for each $\mathscr{B} \in \mathfrak{S}_2$, where $f^\leftarrow(\mathscr{B}) = \{f^\leftarrow(B) : B \in \mathfrak{B}\}$.

Theorem 3.3.1. *Let $f^\rightarrow : (L^X, \mathfrak{S}_1) \rightarrow (L^Y, \mathfrak{S}_2)$ be semi-uniformly continuous is continuous. Then $f^\rightarrow : (L^X, \mathbb{F}(\mathfrak{S}_1)) \rightarrow (L^Y, \mathbb{F}(\mathfrak{S}_2))$ is continuous function.*

Proof. Let $f^\rightarrow : (L^X, \mathfrak{S}_1) \rightarrow (L^Y, \mathfrak{S}_2)$ be a semi-uniformly continuous function. Let $A \in L^Y$, then $int(A) = \bigcup\{x_\alpha : st(x_\alpha, \mathscr{A}) \subseteq A\}$ for some $\mathscr{A} \in \mathfrak{S}_2$.

Since $f^{\leftarrow}$ is arbitrary join preserving, then by Theorem 2.1.17(i) in [48], we have

$$f^{\leftarrow}(int(A)) = \bigcup\{f^{\leftarrow}(x_\alpha) : st(x_\alpha, \mathscr{A}) \subseteq A \ for \ some \ \mathscr{A} \in \mathfrak{S}_2\} \tag{3.3.1}$$

Also, $f^{\leftarrow}$ is order preserving, we have

$$st(x_\alpha, \mathscr{A}) \subseteq A \Rightarrow f^{\leftarrow}(st(x_\alpha, \mathscr{A})) \subseteq f^{\leftarrow}(A). \tag{3.3.2}$$

By the Proposition 2.4.2(6) and line (3.3.2) we have

$$st(f^{\leftarrow}(x_\alpha), f^{\leftarrow}(\mathscr{A})) \subseteq f^{\leftarrow}(st(x_\alpha, \mathscr{A})) \subseteq f^{\leftarrow}(A) \tag{3.3.3}$$

Again from line (3.3.1), we have

$$f^{\leftarrow}(int(A)) = \bigcup\{f^{\leftarrow}(x_\alpha) : (st(f^{\leftarrow}(x_\alpha), f^{\leftarrow}(\mathscr{A})) \subseteq f^{\leftarrow}(A) \ for \ some \ \mathscr{A} \in \mathfrak{S}_2\} \tag{3.3.4}$$

Since $f^{\rightarrow}$ is semi-uniformly continuous, so $f^{\leftarrow}(\mathscr{A}) \in \mathfrak{S}_1$, therefore by line (3.3.4) $f^{\leftarrow}(int(A)) \subseteq int(f^{\leftarrow}(int(A)))$ implies $f^{\leftarrow}(int(A)) \in \mathbb{F}(\mathfrak{S}_1)$. Hence the theorem. □

Theorem 3.3.2. *Let $f^{\rightarrow} : (L^X, \mathfrak{S}_1) \rightarrow (L^Y, \mathfrak{S}_2)$ and $g^{\rightarrow} : (L^Y, \mathfrak{S}_2) \rightarrow (L^Z, \mathfrak{S}_3)$ be two semi-uniformly continuous. Then $(g \circ f)^{\rightarrow}$ is semi-uniformly continuous.*

Proof. Let $f^{\rightarrow} : (L^X, \mathfrak{S}_1) \rightarrow (L^Y, \mathfrak{S}_2)$ and $g^{\rightarrow} : (L^Y, \mathfrak{S}_2) \rightarrow (L^Z, \mathfrak{S}_3)$ be two semi-uniformly continuous functions. Let $\mathscr{C} \in \mathfrak{S}_3$, by the Theorem 2.1.23 (ii) in [48] we have $(g \circ f)^{\leftarrow}(\mathscr{C}) = f^{\leftarrow}(g^{\leftarrow}(\mathscr{C}))$. Since $g^{\rightarrow}$ is semi-uniformly continuous $g^{\leftarrow}(\mathscr{C}) \in \mathfrak{S}_2$. Also $f^{\leftarrow}$ is uniformly continuous implies $f^{\leftarrow}(g^{\leftarrow}(\mathscr{C})) \in \mathfrak{S}_1$. Hence $(g \circ f)^{\rightarrow}$ is semi-uniformly continuous. □

3.4 L-Semi-Pseudo-Metrization

The problem of metrization has occupied an important place in the study of uniform spaces. Having developed the theory of CLS-uniform spaces, the study of metrization(semi-pseudo-metrization) in the same context.

Theorem 3.4.1. *Every $L-$semi-pseudo metric generates a CLS-uniform space.*

Proof. let (L^X, P) be an $L-$semi-pseudo-metric space and for any $s > 0$, let $\mathscr{U}_s$ be an L-cover of L^X such that $\mathscr{U}_s = \{B_\epsilon(x_\alpha) : x_\alpha \in L^X\}$. Then clearly $\mathscr{U}_{\frac{1}{2}s} \preccurlyeq \mathscr{U}_s$ and $\mathscr{U}_s \bigcap \mathscr{U}_t \preccurlyeq \mathscr{U}_{\max[s,t]}$. Therefore, $\psi(P) = \{\mathscr{U}_s : s > 0\}$ is a base for CLS-uniformity. □

Definition 3.4.1. We shall say that a CLS-uniform space $(L^X, \mathfrak{S})$ is $L-$semi-pseudo-metrizable if there is an $L-$semi-pseudo-metric that generates $\mathfrak{S}$.

Definition 3.4.2. A CLS-uniform space is $L-$semi-pseudo-metrizable if it is induced by a $L-$semi-pseudo-metric.

Lemma 3.4.2. *Let $(L^X, \mathfrak{S})$ be a CLS-uniform space. For $\mathscr{C} \in \mathfrak{S}$ define a mapping $\psi(\mathscr{C}) : L^X \to L^X$ such that $[\psi(\mathscr{C})](A) = st(A, \mathscr{C})$. Then*

$$[\psi(\mathscr{C})](\bigcup_i A_i) = st(\bigcup_i A_i, \mathscr{C}) = \bigcup_i [\psi(\mathscr{C})](A_i)$$

Theorem 3.4.3. *A CLS-uniform space is $L-$semi-pseudo-metrizable if it has a countable base.*

Proof. Let $\{\mathscr{C}_n : n \in N\}$ be a base for CLS-uniform space $(L^X, \mathfrak{S})$.Without lost of generality we can assume $\mathscr{C}_{n+1} \preccurlyeq \mathscr{C}_n$ for each $n \in \mathbb{N}$. For any $r > 0$, let $\psi_r : L^X \to L^X$ be a mapping defined by

$$\forall A \in L^X, \text{ if } \tfrac{1}{2^n} < r \leq \tfrac{1}{2^{n-1}}, \text{ then } [\psi_r(\mathscr{C}_n)](A) = st(A, \mathscr{C}_n)$$

and if $[\psi_r(\mathscr{C}_n)](A) = \underline{1}$ or $\underline{0}$ according $A \neq \underline{0}$ or $A = \underline{0}$.

For every $r > 0$, let $\mathscr{F}_r : L^X \to L^X$ be a mapping defined by

$\mathscr{F}_r = \{\psi_{r_k} : \sum_{i=0}^{k} r_k = r, \forall i \leq k, r_i > 0, k < \omega\}$

Obviously $\{\mathscr{F}_r : r > 0\}$ is a base for $\mathfrak{S}$ and also define

$\mathscr{F}_r(A) = \bigcup_{x_\alpha \in A} st(x_\alpha, \mathscr{F}_r)$ for all $A \in L^X$.

Let $P : L^X \times L^X \to [0, \infty]$ be a mapping defined by

$P(A, B) = \bigwedge\{r : B \subseteq \mathscr{F}_r(A)\}$, where we assume that $\bigwedge \Phi = +\infty$.

We claim that P is the required $L-$semi-pseudo-metric that generates $\mathfrak{S}$

(SEM1) By Theorem 2.5.4, P fulfils (SEM1).

(SEM3) By Theorem 1.3.24(ii) in [48] we have $\beta^*[\mathscr{F}_r] = \bigcup_{A \in \mathscr{F}_r} \beta^*(A)$.

Now for any arbitrary $A, B \neq \underline{0}$ by assumption $P(A, B) < r \Rightarrow B \subseteq \mathscr{F}_r(A)$.

$\Rightarrow \forall x_\alpha \in \beta^*(B), x_\alpha \in \beta^*(\mathscr{F}_r(A))$.

$\Rightarrow \forall x_\beta \in \beta^*(B), \exists y_\beta \in \beta^*(A), x_\alpha \in \beta^*(\mathscr{F}_r(y_\beta))$

$\Rightarrow x_\alpha \in \beta^*(B), \exists y_\beta \in \beta^*(A), P(y_\beta, x_\alpha) < r$

$\Rightarrow \bigcup_{x_\alpha \in \beta^*(B)} \bigcap_{y_\beta \in \beta^*(A)} P(y_\beta, x_\alpha) < r$.

Again suppose that $\bigcup_{x_\alpha \in \beta^*(B)} \bigcap_{y_\beta \in \beta^*(A)} P(y_\beta, x_\alpha) < r$

$\Rightarrow \forall x_\alpha \in \beta^*(B), \exists y_\beta(x_\alpha) \in \beta^*(A), P(y_\beta(x_\alpha), x_\alpha) < r$,

Where $y_\beta(x_\alpha)$ is an $L-$fuzzy point corresponding to the $L-$fuzzy point x_α.

$\Rightarrow x_\alpha \in \beta^*(B), \exists y_\beta(x_\alpha) \in \beta^*(A), x_\alpha \subseteq \mathscr{F}_r(y_\beta(x_\alpha))$

$\Rightarrow B = \bigcup \beta^*(B) \subseteq \bigcup_{x_\alpha \in \beta^*(B)} \mathscr{F}_r(y_\beta(x_\alpha))$

$\Rightarrow \mathscr{F}_r(\bigcup_{x_\alpha \in \beta^*(B)} y_\beta(x_\alpha)) \subseteq \mathscr{F}_r(\bigcup \beta^*(A)) = \mathscr{F}_r(A)$.

Which implies $P(A, B) < r$.

Hence $A, B \neq \underline{0} \Rightarrow P(A, B) = \bigcup_{x_\alpha \in \beta^*(B)} \bigcap_{y_\beta \in \beta^*(A)} P(y_\beta, x_\alpha)$.

(SEM4) Suppose $\mathscr{F}_r(A) \subseteq B \Leftrightarrow \bigcup\{C : P(A,C) < r\} \subseteq B$

$$\Leftrightarrow P(A,C) \Rightarrow C \subseteq B$$

$$\Leftrightarrow P(B',D) < r \Rightarrow D \subseteq A'$$

$$\Leftrightarrow \bigcup\{D : P(B',D) < r\} \subseteq A'$$

$$\Leftrightarrow \mathscr{F}_r(B') \subseteq A'$$

Which implies "$P(A,C) < r \Rightarrow C \subseteq B$" $\Leftrightarrow$ "$P(B',D) < r \Rightarrow D \subseteq A'$"

For any $x_\alpha \in L^X$, they have same nbhd at x_α viz, $\{\psi_r[\mathscr{C}_n](x_\alpha)r > 0\} = \{\mathscr{F}_r(x_\alpha) : r > 0\}$. which implies they induced the same interior operator. Conversely, Let $\mathfrak{S}$ is CLS-uniform spaces generated by $L-$semi-pseudo-metric P. For any $s > 0$, let $\mathscr{U}_s$ be an L-cover of L^X such that $\mathscr{U}_s = \{B_\epsilon(x_\alpha) : x_\alpha \in L^X\}$. Then clearly $\mathscr{U}_{\frac{1}{2}s} \preccurlyeq \mathscr{U}_s$ and $\mathscr{U}_s \bigcap \mathscr{U}_t \preccurlyeq \mathscr{U}_{\max[s,t]}$. Therefore, $\psi(P) = \{\mathscr{U}_s : s > 0\}$ is a base for CLS-uniformity. $\square$

******