CHAPTER 2

PRELIMINARIES

In this chapter, the basic definitions and results are utilized in the subsequent chapters. Most of the definitions and results can be found in [10, 33, 48]. Some brief notions were used such as nbhd means neighbourhood and iff means if and only if.

2.1 Lattice Structures

Definition 2.1.1. Let P be a set and " $\leq$ " is a relation on P. Then P is called partially ordered set (briefly poset) with respect to the relation " $\leq$ ", if it satisfies the following conditions

PO1. Reflexive: $a \leq a$ for all $a \in P$

PO2. Antisymmetric: For any $a, b \in P$ with $a \leq b$ and $b \leq a$ implies $a = b$.

PO3. Transitive: For any $a, b, c \in P$ with $a \leq b$ and $b \leq c$ implies $a \leq c$.

Definition 2.1.2. Let L be a poset and $A \subseteq L$.

1. An element $x \in L$ is called he join of A, denoted by $\bigvee A$ or $\sup A$, if,

   (a) x is upper bound of A,

   (b) if another y is upper bound for A, then $x \leq y$

   If A is finite, we shall call $\bigvee A$(if it exists) a finite join. For two elements a and b the join is denoted by $a \bigvee b$.

2. An element $x \in L$ is called the meet of A, is denoted by $\bigwedge A$ or $\inf A$, if,

   (a) x is a lower bound of A

   (b) if y is lower bound for A, then $y \leq x$

   If A is finite, is call $\bigwedge A$(if it exists) a finite meet. For two elements a and b the meet is denoted by $a \bigwedge b$.

Definition 2.1.3. Let L be a poset. Then,

1. **Join-semilattice**: Every join for a finite subset of L exists; particulary, the smallest element exists as the join of non-empty subset.

2. **Meet-semilattice**: Every meet for a finite subset of L exists; particulary, the largest element exists as the meet of non-empty subset.

3. **Lattice**: It is both join-semilatice and meet-semilattice

Remark 2.1.1. A lattice will always be non-empty. Every lattice is always assumed to possess at least two elements, the smallest element 0_L and largest element 1_L.

Definition 2.1.4. Let L be a poset. Then,

1. **Complete join-semilattice**: Every join for arbitrary subset of L exists, i.e., smallest element exists as the join of non-empty subset.

2. **Complete meet-semilattice**: Every meet for arbitrary subset of L exists, i.e., largest element exists as the meet of non-empty subset.

3. **Complete lattice**: It is both complete join-semilattice and complete meet-semilattice.

Proposition 2.1.1. *Let L be a poset, then the following conditions are equivalent:*

1. *L is a complete lattice.*

2. *L has the smallest element and $\forall\ A \subseteq L,\ A \neq \phi,\ \bigvee A$ exists in L.*

3. *L has the largest element and $\forall\ A \subseteq L,\ A \neq \phi,\ \bigwedge A$ exists in L.*

4. *L is a lattice and $\forall A \subseteq L, A \neq \phi, \bigvee A$ exists in L*

5. *L is a lattice and $\forall A \subseteq L, A \neq \phi, \bigwedge A$ exists in L*

Definition 2.1.5. Let L be a complete lattice. Then L is called infinitely distributive if it satisfies the following conditions

IFD1. For all $a \in L,\ \forall\ B \subseteq L,\ a \bigwedge \bigvee B = \bigvee_{b \in B}(a \bigwedge b)$.

IFD2. For all $a \in L,\ \forall\ B \subseteq L,\ a \bigvee \bigwedge B = \bigwedge_{b \in B}(a \bigvee b)$

where **IFD1** and **IFD2** is called 1st infinitely distributive law and 2nd infinitely distributive law respectively.

Remark 2.1.2. The 1st infinitely distributive law is not equivalent to 2nd infinitely distributive law.

Proposition 2.1.2. *Let L be a complete lattice. Then,*

1. *L satisfies the **IFD1** if and only if $\forall A, B \subseteq L$, $\bigwedge A \bigvee \bigwedge B = \bigvee_{a\in A, B\in B}(a \bigwedge b)$.*

2. *L satisfies the**IFD2** if and only if $\forall A, B \subseteq L$, $\bigvee A \bigwedge \bigvee B = \bigwedge_{a\in A, b\in B}(a \bigvee b)$.*

Definition 2.1.6. Let L be a complete lattice. Then L is called completely distributive lattice if it satisfies the following conditions:

$$\forall\{\{a_{i,j}|j \in J_i\}, i \in I\} \subseteq \mathscr{P}(L) \setminus \phi, I \neq \phi$$

CD1. $\bigwedge_{i\in I}(\bigvee_{j\in J_j} a_{i,j}) = \bigvee_{\Phi\in\prod_{i\in I} J_i}(\bigwedge_{i\in I} a_{i,\Phi(i)})$

CD2. $\bigvee_{i\in I}(\bigwedge_{j\in J_j} a_{i,j}) = \bigwedge_{\Phi\in\prod_{i\in I} J_i}(\bigvee_{i\in I} a_{i,\Phi(i)})$

where **CD1** and **CD2** are called completely distributive law.

Remark 2.1.3. For $I = \{0, 1\}$, $J_0 = \{0\}$ in **CD1** and **CD2**. So **CD1** implies **IFD1** and **CD2** implies **IFD2** and hence completely distributive lattice is infinitely distributive. But converse is not true in general.

Theorem 2.1.3. *A complete lattice satisfies **CD1** if and only if **CD2**.*

Proposition 2.1.4. *Let $\{P_i : i \in I\}$ be a family of posets then the relation "$\leq$" on $\prod_{i\in I} P_i$ defined by*

$$\alpha, \beta \in \prod_{i\in I} P_i, \quad \alpha \leq \beta \Leftrightarrow \forall i \in I, \alpha(i) \leq \beta(i)$$

is reflexive, antisymmetric and transitive.

Theorem 2.1.5. *Let $\{L_i : i \in I\}$ be a family of posets. Then $\prod_{i\in I} L_i$ is a completely distributive lattice if and only if $\forall i \in I$, L_i is completely distributive lattice.*

Definition 2.1.7. Let L be a lattice, $\alpha \in L$. α is called join-irreducible, if $\alpha < 1_L$ and $\forall a, b \in L, \alpha = a \bigvee b \Rightarrow \alpha = a$ or $\alpha = b$.
A join-irreducible element of L is called molecule in L.
The set of all molecules in L is denoted by $M(L)$.

Remark 2.1.4. Every element in a completely distributive lattice can be represented as a join of molecules.

Definition 2.1.8. Let L be a complete lattice. Define a relation $\preceq$ in L as follows: For all $a, b \in L, a \preceq b$ iff $\forall S \subseteq L, b \leq \bigvee S \Rightarrow \exists s \in S$ such that $a \leq s,\ \forall\ a \in L$, denoted by $\beta_L(a) = \{b \in L : b \preceq a\}$, $\beta^*_L(a) = M(\beta_L(a))$ or denoted by them $\beta(a)$ and $\beta^*(a)$ in briefly.
$\forall a \in L, D \subseteq \beta(a)$ is called a minimal set of a if $\bigvee D = a$.

Theorem 2.1.6. *Let L be a complete lattice,then the following are equivalent:*

1. *L is completely distributive lattice.*
2. $\forall a \in L$, $\beta(a)$ *is minimal set of* a*.*
3. $\forall a \in L$, $\beta^*(a)$ *is minimal set of* a*.*

Theorem 2.1.7. *Let L be a complete lattice. Then*

1. $\beta : L \to \mathscr{P}(L)$ *is an arbitrary join-preserving mapping, i.e., for every* $A \subseteq L$*,*

$$\beta(\bigvee A) = \bigvee_{a \in A} \beta(a)$$

2. $\beta^* : L \to \mathscr{P}(M(L))$ *is an arbitrary join-preserving mapping, i.e., for every* $A \subseteq L$*,*

$$\beta^*(\bigvee A) = \bigvee_{a \in A} \beta^*(a)$$

where $\mathscr{P}(L)$ and $\mathscr{P}(M(L))$ are power sets of L and $M(L)$ respectively.

Definition 2.1.9. A set D equipped with a relation " $\leq$ " is called down directed, if for any finite set $D_0 \subseteq D$, there exists $d_0 \in D$ such that $d_0 \leq d$ for all $d \in D$.

Definition 2.1.10. Let L be a lattice, then a map $' : L \to L$ is called order reversing involution if

$$\forall a, b \in L, \quad a \leq b \Rightarrow b' \leq a' \text{ and } (a')' = a$$

Proposition 2.1.8. *Let L be a lattice with order reversing involution $'$. Then*

1. $(\bigvee_i a_i)' = \bigwedge_i a_i'$.

2. $(\bigwedge_i a_i)' = \bigvee_i a_i'$

Definition 2.1.11. A completely distributive lattice L is called a fuzzy lattice or an $F-$lattice (in briefly), if L has an order reversing involution $' : L \to L$.

2.2 $L-$fuzzy sets and $L-$Topological Spaces

Throughout the thesis, the notion $(L, \leq, \bigwedge, \bigvee, ')$ as a fuzzy lattice with order reversing involution $'$; $\inf L = 0_L$ and $\sup L = 1_L$ (in briefly L for the simplicity). The basic definition and results of $L-$fuzzy sets and $L-$topological space can be found in [26, 34, 35, 48]

Definition 2.2.1. Let X be an arbitrary set and L a complete lattice. Let L^X will denote the collection of all mapping $A : X \to L$. Then any member of L^X is called an $L-$fuzzy set. L^X is called an $L-$fuzzy space.

Definition 2.2.2. The $L-$fuzzy sets $x_\alpha : X \to L$, where $\alpha \in L$ defined by

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } x = y \\ \\ 0_L & \text{if } x \neq y \end{cases}$$

are called the $L-$fuzzy points. The set of all fuzzy point of X is denoted by $Pt(L^X)$.

Remark 2.2.1. The mapping $A : X \to L$ and $B : X \to L$ defined by $A(x) = 1_L,\ \forall x \in X$ and $B(x) = 0_L$,$\forall x \in X$ are denoted by $\underline{1}$ and $\underline{0}$ respectively. Clearly, $\underline{1}$ and $\underline{0}$ will act as the largest and smallest element respectively on the fuzzy lattice L^X.

Definition 2.2.3. Let X be a non-empty crisp set and L be a fuzzy lattice with order reversing involution $'$. Let $' : L \to L$ be an operation on L^X is called pseudo complementary operation, defined by

$$A'(x) = (A(x))', \forall x \in X, \forall A \in L^X$$

.

Proposition 2.2.1. *Let X be a crisp set and L be a fuzzy lattice with order reversing involution $'$. Then the pseudo-complementary operation $' : L^X \to L^X$ is an order reversing involution.*

Definition 2.2.4. For any $A, B \in L^X$, then

1. A union B as $A \bigcup B$ defined as $A \bigcup B(x) = A(x) \bigvee B(x),\ \forall x \in X$.

2. A intersection B as $A \bigcap B$ defined as $A \bigcap B(x) = A(x) \bigwedge B(x),\ \forall x \in X$

3. A is subset of B as $A \subseteq B$ and defined as $A \subseteq B$ iff $A(x) \leq B(x)$.

4. Complement of A denoted by A' defined as $A'(x) = A(x)'$.

Remark 2.2.2. For any $x_\alpha \in Pt(L^X)$ and $A \in L^X$, $x_\alpha \in A$ iff $\alpha < A(x)$ and $x_\alpha \subseteq A$ iff $\alpha \leq A(x)$. Particularly for any $y_\beta \in Pt(L^X), x_\alpha \subseteq y_\beta$ iff $x = y$ and $\alpha \leq \beta$.

Proposition 2.2.2. *Let L^X be an $L-$fuzzy space. Then*

1. *L^X is a complete lattice for any $\mathscr{A} \subseteq L^X$, we have*

$$\forall x \in X \ (\bigvee_{A \in \mathscr{A}} A)(x) = \bigvee_{A \in \mathscr{A}} A(x)$$

$$\forall x \in X \ (\bigwedge_{A \in \mathscr{A}} A)(x) = \bigwedge_{A \in \mathscr{A}} A(x)$$

2. *L is distributive iff L^X is distributive.*

3. *L satisfies **IDF1** iff L^X satisfies **IDF1**.*

4. *L satisfies **IDF2** iff L^X satisfies **IDF2**.*

5. *L is completely distributive iff L^X is completely distributive.*

Definition 2.2.5. For any ordinary mapping $f : X \to Y$, the induced L-fuzzy mapping $f^{\to} : L^X \to L^Y$ and its L-fuzzy reverse mapping $f^{\leftarrow} : L^Y \to L^X$ respectively are defined as:

$f^{\to}(A)(y) = \bigvee\{A(x) \mid x \in X,\ f(x) = y\},\ \forall A \in L^X,\ \forall y \in Y.$

$f^{\leftarrow}(B)(x) = B(f(x)),\ \forall B \in L^Y,\ \forall x \in X.$

Symbol $f^{\to}$ and $f^{\leftarrow}$ always denote $f^{\to}$ to be the L-fuzzy mapping induced from an ordinary mapping f and $f^{\leftarrow}$ is the L-fuzzy reverse mapping of $f^{\to}$. Both the L-fuzzy mappings $f^{\to}$ and $f^{\leftarrow}$ are order preserving.

Theorem 2.2.3. *Let L^X and L^Y be $L-$fuzzy spaces, $f : X \to Y$ an ordinary mapping. Then*

1. $f^{\rightarrow}$ *is injective iff* f *is injective.*

2. $f^{\rightarrow}$ *is surjective iff* f *is surjective.*

3. $f^{\rightarrow}$ *is bijective iff* f *is bijective.*

Theorem 2.2.4. *Let* L^X, L^Y *and* L^Z *be* $L-$ *fuzzy space,* $f : X \to Y$ *and* $g : Y \to Z$ *are ordinary mappings. we have*

1. $g^{\rightarrow} f^{\rightarrow} = (gf)^{\rightarrow}$.

2. $f^{\leftarrow} g^{\leftarrow} = (gf)^{\leftarrow}$.

Theorem 2.2.5. *Let* L^X *and* L^Y *be* $L-$ *fuzzy space,* $f : X \to Y$ *an ordinary mapping. Then*

1. $A \subseteq f^{\leftarrow} f^{\rightarrow}(A),\ \forall A \in L^X$.

2. $f^{\rightarrow} f^{\leftarrow}(B) \subseteq B,\ \forall B \in L^Y$.

3. $f^{\rightarrow}(A) = f^{\rightarrow} f^{\leftarrow} f^{\rightarrow}(A),\ \forall A \in L^X$.

4. $f^{\leftarrow} f^{\rightarrow} f^{\leftarrow}(B) = f^{\leftarrow}(B),\ \forall B \in L^Y$.

Proposition 2.2.6. *Let* X *and* Y *be two non-empty sets,* L *be a fuzzy lattice and* $f : X \to Y$ *be an ordinary mapping. Then* $\forall A \in L^X$, $(f^{\rightarrow}(A))' \subseteq f^{\rightarrow}(A')$ *and* $f^{\leftarrow}(A') = (f^{\leftarrow}(A))'$. *Where* $'$ *is the pseudo complementary operation.*

Theorem 2.2.7. *Let* L^X *and* L^Y *be* $L-$*fuzzy spaces,* $f : X \to Y$ *an ordinary mapping. then* $f^{\leftarrow}$ *is bijective iff* $f^{\leftarrow} \circ f^{\rightarrow} = id_{L^X}$, $f^{\rightarrow} \circ f^{\leftarrow} = id_{L^Y}$.

Definition 2.2.6. Let $\mathbb{F}$ be a subset of L^X, then F is called $L-$fuzzy topology on L^X (in briefly $L-$topology), if F is closed under finite intersection and arbitrary union. The members of $\mathbb{F}$ is called $L-$fuzzy open sets and its complements are called the $L-$fuzzy closed sets. The pair $(L^X, \mathbb{F})$ is called $L-$topological space.

Remark 2.2.3. Let $\mathbb{F}_1$ and $\mathbb{F}_2$ are two $L-$topologies on L^X. The $\mathbb{F}_2$ is called finer than $\mathbb{F}_1$, if $\mathbb{F}_1 \subseteq \mathbb{F}_2$.

Definition 2.2.7. Let $(L^X, \mathbb{F})$ be an $L-$topological space. Then a non-empty sub-family $\mathscr{B}$ of $\mathbb{F}$ is called base for $\mathbb{F}$ if

$$\mathbb{F} = \{\bigcup \mathscr{A} | \mathscr{A} \subseteq \mathscr{B}\}.$$

Definition 2.2.8. Let $(L^X, \mathbb{F})$ be an $L-$topological spaces and let $A \in L^X$.

1. The interior of A, denoted by $int(A)$ is defined as $int(A) = \{G : G \in \mathbb{F} \text{ and } G \subseteq A\}$.

2. The closure of A, denoted by $cl(A)$ is defined as $cl(A) = \bigcap\{F : F' \in \mathbb{F} \text{ and } A \subseteq F\}$.

Theorem 2.2.8. *Let $(L^X, \mathbb{F})$ be an $L-$topological spaces. Then*

1. $int(\underline{0}) = \underline{0}, \quad int(\underline{1}) = \underline{1}$ *and* $cl(\underline{0}) = \underline{0}, \quad cl(\underline{1}) = (\underline{1})$.

2. $int(A) \subseteq A$ *and* $A \subseteq cl(A) \quad \forall A \in L^X$.

3. $int(int(A)) = int(A)$ *and* $cl(cl(A)) = cl(A) \quad \forall A \in L^X$.

4. $A \subseteq B$, *then* $int(A) \subseteq int(B)$ *and* $cl(A) \subseteq cl(B)$, $\forall A, B \in L^X$.

5. $int(A \bigcap B) = int(A) \bigcap int(B)$ *and* $cl(A \bigcup B) = cl(A) \bigcup cl(B) \quad \forall A, B \in L^X$.

6. $int(A)' = cl(A'), int(A') = cl(A)', int(A) = cl(A')'$ *and* $cl(A) = int(A')' \quad \forall A \in L^X$.

Proposition 2.2.9. *Let X be a non-empty set and L^X be a fuzzy lattice. Let $cl : L^X \to L^X$ is called closure operator if it satisfies the following properties*

CO1. $cl(\underline{0}) = \underline{0}$

CO2. $A \subseteq cl(A)$ $, \forall A \in L^X$.

CO3. $cl(A \bigcup B) = cl(A) \bigcup cl(B), \forall A, B \in L^X$

Then the pair (L^X, cl) is called closure space.

Proposition 2.2.10. *The closure space is topological if it satisfies* ***CO1,CO2, CO3*** *and the following*

CO4. $cl(cl(A)) = cl(A), \ \forall A \in L^X$

then it is called topological closure operator. Also $\mathbb{F} = \{A \in L^X : cl(A') = A'\}$ is an $L-$topology.

Proposition 2.2.11. *Let X be a non-empty set and L^X be a fuzzy lattice. Let $int : L^X \to L^X$ is called interior operator if*

IO1. $int(\underline{1}) = \underline{1}$

IO2. $int(A) \subseteq A$ $, \forall A \in L^X$.

IO3. $int(A \bigcap B) = int(A) \bigcup int(B), \forall A, B \in L^X$

Then the pair (L^X, int) is called interior space.

Proposition 2.2.12. *The interior space is topological if it satisfies **IO1,IO2,IO3** and the following*

IO4. $int(int(\underline{1})) = int(\underline{1}),\ \forall A \in L^X$.

then it is called topological interior operator. Also $\mathbb{F} = \{A \in L^X : int(A) = A\}$ *is an* $L-$*topology.*

Definition 2.2.9. Let $(L^X, \mathbb{F}_1)$ and $(L^Y, \mathbb{F}_2)$ be two $L-$topological spaces. The mapping $f^{\rightarrow} : L^X \rightarrow L^Y$ is called $L-$ fuzzy continuous mapping iff for any $V \in \mathbb{F}_2$ implies $f^{\leftarrow}(V) \in \mathbb{F}_1$.

Definition 2.2.10. Let $(L^X, \mathbb{F}_1)$ and $(L^Y, \mathbb{F}_2)$ be two topological spaces and $f^{\rightarrow} : L^X \rightarrow L^Y$ be a mapping. Then

1. $f^{\rightarrow}$ is called open map, if for any $G \in \mathbb{F}_2$ we have $f^{\rightarrow}(G) \in \mathbb{F}_2$.

2. $f^{\rightarrow}$ is called closed map, if for any $F \in \mathbb{F}_1'$ we have $f^{\rightarrow}(F) \in \mathbb{F}_2'$.

Definition 2.2.11. Let $(L^X, \mathbb{F}_1)$ and $(L^Y, \mathbb{F}_2)$ be two topological spaces, then $f^{\rightarrow} : L^X \rightarrow L^Y$ is called an $L-$fuzzy homeomorphism, if it is bijective, continuous and open mapping.

Definition 2.2.12. Let $\{(L^{X_t}, \mathbb{F}_t) : t \in \Lambda\}$ be a family of $L-$topological spaces, where Λ is the index set. Denote $X = \prod_{t \in \Lambda} X_t$. For every $t \in \Lambda$, let $\pi_t : X \rightarrow X_t$ be an ordinary projection define the projection from L^X to L^Y as

$$\pi_t^{\rightarrow} : L^X \rightarrow L^Y$$

The product topology of $L-$topologies $\{\mathbb{F}_t : t \in \Lambda\}$ on X is denoted by $\prod_{t \in \Lambda} \mathbb{F}_t$, as the $L-$ topology $\mathbb{F}$ on L^X generates by the subbase

$$\{\pi_t^{\leftarrow}(G_t) : G_t, t \in \Lambda\}$$

and called the $L-$topological spaces $(L^X, \mathbb{F})$, the product space of $L-$topological spaces $\{(L^{X_t}, \mathbb{F}) : t \in \Lambda\}$, denoted by $\prod_{t\in\Lambda}(L^{X_t}, \mathbb{F}_t)$.

Definition 2.2.13. Let $(L^X, \mathbb{F})$ be an $L-$topological space. Then $(L^X, \mathbb{F})$ is said to be regular if for every $G \in \mathbb{F}$ and $x_\alpha \subseteq G$, there is $A \in \mathbb{F}$ such that $x_\alpha \subseteq A \subseteq cl(A) \subseteq G$.

Theorem 2.2.13. *Product of regular $L-$topology is regular iff each of $L-$ topology is regular.*

Definition 2.2.14. [48] Let $(X, \mathbb{F})$ be an $L-$ topological space.$(L^X, \mathbb{F})$ is regular, if every $U \in \mathbb{F}$, there exists $\mathscr{V} \subseteq \mathbb{F}$ such that $\bigcup \mathscr{V} = U$ and $cl(U) \subseteq U$ for every $V \in \mathscr{V}$.

Definition 2.2.15. For any $x_\alpha, A, B \in L^X$, x_α is said to be quasi-coincident with A, denoted as $x_\alpha q A$ if $x_\alpha \nsubseteq A'$, i.e., $\alpha \nleq A'(x)$.

A is called quasi-coincident with B at y if $A(y) \neq B'(y)$, denoted by $A\hat{q}B$, if A is quasi-coincident with B at some $y \in X$.

Corollary 2.2.14. *If A and B are quasi coincident at x, both $A(x)$ and $B(x)$ are not zero and hence A and B intersect x.*

Definition 2.2.16. Let $x_\alpha \in Pt(L^X)$. Then an $L-$fuzzy set U is said to be a quasi-coincident neighbourhood (Q-nbhd) at x_α in an $L-$topological space $(L^X, \mathbb{F})$, if there is $G \in \mathbb{F}$ such that $x_\alpha q G \subseteq U$.

The family of all Q-nbhd at x_α in an $L-$topological space $(L^X, \mathbb{F})$ is denoted by $\mathscr{Q}(x_\alpha)$

Definition 2.2.17. A subfamily $\mathscr{A} \subseteq \mathscr{Q}(x_\alpha)$ is called a Q-nbhd base of x_α, if for every $U \in \mathscr{Q}(x_\alpha)$, there exists $V \in \mathscr{A}$ such that $V \subseteq U$.

Theorem 2.2.15. *Let $(L^X, \mathbb{F})$ be an $L-$topological space. Then for any $x_\alpha \in M(L^X)$, $\mathscr{Q}(x_\alpha)$ is a down-directed set in L^X and $\underline{0} \notin \mathscr{Q}(x_\alpha)$.*

Theorem 2.2.16. *Let $(L^X, \mathbb{F})$ be an L-topological space and $A \in L^X$. Then an L-fuzzy point $x_\alpha \in \bar{A}$ iff each $Q-$nbhd at x_α is quasi-coincident with A.*

Definition 2.2.18. [52] A family $\mathscr{A}$ of L^X has the finite intersection property if the intersection of the members of each finite subfamily of $\mathscr{A}$ is non-empty.

Definition 2.2.19. [48] An $L-$topology $(L^X, \mathbb{F})$ is called compact, if every open cover of $(L^X, \mathbb{F})$ has a finite subcover.

Theorem 2.2.17. *[35] Let$(L^X, \mathbb{F})$ be an $L-$topology, then $(L^X, \mathbb{F})$ is compact iff*

1. *Every open cover $\mathscr{C}$ of closed subset A of L^X has a finite subcover.*
2. *Every closed collection with finite intersection property has non-empty intersection.*

2.3 Convergence structures in $L-$ Topology

In this section, some basic definitions and results regarding net and filters in $L-$topology that will necessary in our subsequent chapters. All definition and results can be found in [35, 36, 37, 48].

Definition 2.3.1. An $L-$fuzzy space L^X, a net $S : D \to Pt(L^X)$ a net in L^X, where D a directed set . In particular a net $S : D \to M(L^X)$ a molecule net in L^X.
For $A \subseteq Pt(L^X)$ and a net $S = \{S(n) : n \in D\}$ such that $S(n) \in A \forall\ n \in D$, we say S consists of points in A.

Definition 2.3.2. Let S be a net in an $L-$topological space $(L^X, \mathbb{F})$ and $x_\alpha \in Pt(L^X)$. Then S is said to be convergent to x_α, denoted by $S \to x_\alpha$, if for any $U \in \mathscr{Q}(x_\alpha)$ there is $n_0 \in D$ such that $S(n)\widehat{q}U \forall n \geq n_0$.

Definition 2.3.3. A non-empty sub collection $\mathscr{F}$ of L^X is said to be a filter in an L-topological space, if

(F1) $\underline{0} \neq \mathscr{F}$.

(F2) $U_1, U_2 \in \mathscr{F} \Rightarrow U_1 \bigcap U_2 = \mathscr{F}$.

(F3) $U \in \mathscr{F}$ and $V \in L^X$ such that $U \subseteq V$ then $V \in \mathscr{F}$.

$\mathscr{F}$ is said to be proper in $(L^X, \mathbb{F})$, if $\mathscr{F} \neq L^X$.

Definition 2.3.4. A subfamily $\mathscr{B}$ of L^X is called a filter base in an $L-$topological spaces, if

(B1) $\underline{0} \notin \mathscr{B}$

(B2) for any $U, V \in \mathscr{B}$, there exists $W \in \mathscr{B}$ such that $W \subseteq U \bigcap V$.

Definition 2.3.5. If $\mathscr{F}^*$ be a $L-$fuzzy ultrafilter on L^X, then

(U1) For every $A \in L^X$, either $A \in \mathscr{F}^*$ or $A' \in \mathscr{F}^*$.

(U2) $A \bigcup B \in \mathscr{F}^*$ implies that either $A \in \mathscr{F}^*$ or $B \in \mathscr{F}^*$.

An $L-$fuzzy filter $\mathscr{F}$ is $L-$fuzzy ultrafilter iff every $A \in L^X$, either $A \in \mathscr{F}$ or $A' \in \mathscr{F}$.

Definition 2.3.6. Let $x_\alpha \in L^X$ and $\mathscr{F}$ be a filter in $L-$topology, then $\mathscr{F}$ is said to be convergent to x_α, denoted by $\mathscr{F} \to x_\alpha$ if for any $U \in \mathscr{Q}(x_\alpha)$ there exists $F \in \mathscr{F}$ such that $F \subseteq U$, i.e., $\mathscr{Q}(x_\alpha) \subseteq \mathscr{F}$.

An $L-$fuzzy point x_α is called a cluster point of $\mathscr{F}$, denoted by $\mathscr{F} \propto x_\alpha$ if every $U \in \mathscr{Q}(x_\alpha)$ and $F \in \mathscr{F}$, $U \bigcap F \neq \underline{0}$

2.4 Covering $L-$uniform spaces

In this section, the basic definitions and results of the theory of covering $L-$uniform spaces which requires in the subsequent chapters. Most of the definition and results can be found in [59, 60, 75].

Definition 2.4.1. Let X be a non-empty set. Let $\mathscr{A} \subseteq L^X$ is called $L-$fuzzy cover of X if and only if $\bigcup \mathscr{A} = \underline{1}$.

Remark 2.4.1.

1. The set of all $L-$covers of X is denoted by $L - Cov(X)$, a pre-ordered set.
2. Let $\mathscr{A}, \mathscr{B}$ be $L-$covers of X, then $\mathscr{A} \bigcup \mathscr{B} = \{A \bigcup B : A \in \mathscr{A} \text{ and } B \in \mathscr{B}\}$.
3. Let $\mathscr{A}, \mathscr{B}$ be $L-$covers of X, then $\mathscr{A} \bigcap \mathscr{B} = \{A \bigcap B : A \in \mathscr{A} \text{ and } B \in \mathscr{B}\}$.

Definition 2.4.2. For any $\mathscr{A}, \mathscr{B} \subseteq L^X$. Then $\mathscr{A}$ is refinement of $\mathscr{B}$ denoted by $\mathscr{A} \preccurlyeq \mathscr{B}$ if for all $A \in \mathscr{A}$, there exists some $B \in \mathscr{B}$ such that $A \subseteq B$.

Proposition 2.4.1. *Let $\mathscr{A}, \mathscr{B}$ be $L-$covers of X, then*

1. $\mathscr{A} \bigcap \mathscr{B} \preccurlyeq \mathscr{A}$ *and* $\mathscr{A} \bigcap \mathscr{B} \preccurlyeq \mathscr{B}$
2. $\mathscr{A} \bigcup \mathscr{B} \preccurlyeq \mathscr{A}$ *and* $\mathscr{A} \bigcup \mathscr{B} \preccurlyeq \mathscr{B}$

Definition 2.4.3. Let $x_\alpha, A \in L^X$ and $\mathscr{A} \subseteq L^X$ be a $L-$cover, then

1. star of A with respect to cover $\mathscr{A}$ is denoted by $st(A, \mathscr{A})$ and defined as

$$st(A, \mathscr{A}) = \bigcup\{B \in \mathscr{A} : A \bigcap B \neq \underline{0}\}$$

2. star of x_α with respect to cover $\mathscr{A}$ is denoted by $st(x_\alpha, \mathscr{A})$ and defined as

$$st(x_\alpha, \mathscr{A}) = \bigcup\{A \in \mathscr{A} : x_\alpha \in A\}$$

3. star of $L-$cover $\mathscr{A}$ is denoted by $st(\mathscr{A})$ and defined as

$$st(\mathscr{A}) = \{st(A, \mathscr{A}) : A \in \mathscr{A}\}$$

Proposition 2.4.2. *Let $\mathscr{A}, \mathscr{B} \subseteq L^X$ and $A, B \in L^X$, then*

1. *If $\mathscr{A}$ is $L-$ cover then $A \subseteq st(A, \mathscr{A})$ and consequently, $\mathscr{A} \preccurlyeq st(\mathscr{A})$.*

2. *If $A \subseteq B$, then $st(A, \mathscr{A}) \subseteq st(B, \mathscr{A})$.*

3. *If $\mathscr{A} \preccurlyeq \mathscr{B}$, then $st(A, \mathscr{A}) \subseteq st(A, \mathscr{B})$.*

4. *$st(\bigcup \mathscr{B}, \mathscr{A}) = \bigcup_{B \in \mathscr{B}} st(B, \mathscr{A})$.*

5. *If $\mathscr{A}$ is an $L-$cover then $st(st(A, \mathscr{A}), \mathscr{A}) \subseteq st(A, st(\mathscr{A}))$.*

6. *$st(A \bigcap B, \mathscr{C}) \subseteq st(A, \mathscr{C}) \bigcap st(B, \mathscr{C})$.*

7. *Let $f : X \to Y$, $\mathscr{B} \subseteq L^Y$, $f^{-1}(\mathscr{B}) = \{f^{\leftarrow}(B) : B \in \mathscr{B}\}$ and let $C \in L^Y$, then $st(f^{\leftarrow}(C), f^{-1}(\mathscr{B})) \subseteq f^{\leftarrow}(st(C, \mathscr{B}))$.*

Remark 2.4.2. Let $\mathscr{A}$ and $\mathscr{B}$ be two $L-$covers of L^X such that $st(\mathscr{A}) \preccurlyeq \mathscr{B}$, then $st(A, st(\mathscr{A})) \subseteq st(A, \mathscr{B})$ for all $A \in L^X$.

Definition 2.4.4. Let X be a non-empty set. Let $\mathfrak{U}$ be a family of $L-$covers of X is called covering $L-$uniform space if it satisfies the following conditions

C1. $\mathscr{A} \preccurlyeq \mathscr{B}, \mathscr{A} \in \mathfrak{U} \Rightarrow \mathscr{B} \in \mathfrak{U}$.

C2. For every $\mathscr{A}, \mathscr{B} \in \mathfrak{U}, \mathscr{A} \bigcap \mathscr{B} \in \mathfrak{U}$

C3. For each $\mathscr{A} \in \mathfrak{U}$, there exists $\mathscr{B} \in \mathfrak{U}$ such that $st(\mathscr{B}) \preccurlyeq \mathscr{A}$.

the pair $(L^X, \mathfrak{U})$ is called covering $L-$uniform space.

Definition 2.4.5. A base for the covering $L-$uniformity $\mathfrak{U}$ is any sub-collection of $\mathfrak{U}$ from which $\mathfrak{U}$ can be obtained by applying condition **C1.**

Proposition 2.4.3. *Let* $(L^X, \mathfrak{U})$ *be a covering* $L-$*uniform spaces, then* $\{A \in L^X : A = \bigcup\{B \in L^X : st(B, \mathscr{A}) \subseteq A \text{ for some } \mathscr{A} \in \mathfrak{U}\}\}$ *is* $L-$*topology on* L^X *and denoted by* $\mathbb{F}(\mathfrak{U})$*.*

Definition 2.4.6. Let $(L^X, \mathfrak{U}_1)$ and $(L^Y, \mathfrak{U})$ be a covering $L-$uniform spaces. A mapping $f^{\rightarrow} : (L^X, \mathfrak{U}_1) \rightarrow (L^X, \mathfrak{U}_2)$ is called covering uniformly continuous if $f^{-1}(\mathscr{C}) \in \mathfrak{U}_1$ for every $\mathscr{C} \in \mathfrak{U}_2$

2.5 Fuzzy Metric Spaces

In this section, a few basic definition and results of Fuzzy metric spaces that are used in the subsequent chapters. Most of the these can be found in [32, 49, 51, 56].

Definition 2.5.1. A mapping $P : L^X \times L^X \rightarrow [0, +\infty]$ is called an $L-$pseudo quasi-metric on L^X, if P satisfies the following conditions

EM1. $A \neq \underline{0} \Rightarrow P(\underline{0}, A) = +\infty$

$P(A, \underline{0}) = P(A, A) = 0.$

EM2. $P(A, B) \leq P(A, C) + P(C, B)$

EM3. 1. $A \subseteq B \Rightarrow P(B, C) \leq P(A, C)$

2. $P(A, \bigcup_{\lambda\in\Lambda} B_\lambda) = \bigcup_{\lambda\in\Lambda} P(A, B_\lambda)$

EM4. If $P(A_\lambda, C) < r \Rightarrow C \subseteq B$ for every $\lambda \in \Lambda$, then for every $D \in L^X$: $P(\bigcup_{\lambda\in\Lambda} A_\lambda, D) < r \Rightarrow D \subseteq B$.

Definition 2.5.2. A pointwise pseudo-metric on L^X is a map $d : M(L^X) \times M(L^X) \to [0, +\infty]$ satisfying:

M1. For all $A \in M(L^X), d(A, A) = 0$;

M2. For all $A, B, C \in M(L^X)$, $d(A, C) \leq d(A, B) + d(b, c)$;

M3. For all $A, B \in L^X$, $d(A, B) = \bigcap_{C<<D} d(a, c)$;

M4. For all $A, B, C \in M(L^X)$, $A \subseteq B$ implies $d(A, C) \leq d(B, c)$;

M5. Given $A, B \in M(L^X)$, there exists a point $x_\alpha \not\leq A'$ such that $d(x_\alpha, B) < r$ iff there exists $x_\gamma \not\leq B'$ such that $d(x_\gamma, A) < r$

A pointwise pseudo metric d is said to be pointwise metric if d satisfies

M6. $d(A, B) = 0$ iff $A \subseteq B$.

Theorem 2.5.1. *An Erceg's pseudo metric on L^X is equivalent to a family of maps $\{D_r : D_r : L^X \to L^X, r > 0\}$ satisfying the following conditions:*

D1. $\forall A \in L^X, D_r(A) \supseteq A$;

D2. $D_r(\bigcup_{i\in\Lambda} A_i) = \bigcup_{i\in\Lambda} D_r(A_i)$

D3. $D_r \circ D_s \subseteq D_{r+s}$

D4. $D_r = \bigcup_{s<r} Ds$

D5. $D^{-1} = D$.

Theorem 2.5.2. *If (L^X, d) is a pointwise pseudo metric spaces, then $\mathbb{F}(d) = d(\mathbb{F})$.*

Theorem 2.5.3. *Let $(L^X, \mathbb{F})$ be a countable base, then $(L^X, \mathbb{F})$ is pointwise metrizable if and only if it is a regular Space.*

Theorem 2.5.4. *[48] A mapping $P : L^X \times L^X \to [0, +\infty]$ is an $L-$pseudo quasi-metric($L-$pseudo metric, respectively) on L^X if P satisfies the following conditions (SEMI1)-(SEM3)((SEM1)-(SEM4), respectively:*

(SEM1) $B \subseteq A \Rightarrow P(A, B) = \underline{0}$, $B \neq \underline{0} \Rightarrow P(\underline{0}, B) = +\infty$

(SEM2) $P(A, B) \leq P(A, C) + P(C, B)$

(SEM3) $A, B \neq \underline{0} \Rightarrow P(A, B) = \bigcup_{x_\alpha \in \beta^*(B)} \bigcap_{y_\beta \in \beta^*(A)} P(y_\beta, x_\alpha)$.

(SEM4) "$P(A, C) < r \Rightarrow C \subseteq B$" $\Leftrightarrow$ "$P(B', D) < r \Rightarrow D \subseteq A'$"

Definition 2.5.3. Let P be a $L-$semi-pseudo-metric on L^X. The for any $x_\alpha \in L^X$ and $\epsilon > 0$, $B_\epsilon(x_\alpha) = \bigcup\{y_\beta : d(x_\alpha, y_\beta) < \epsilon$ is a fuzzy set, which is called $\epsilon-$open ball of x_α.

2.6 Fuzzy Proximity spaces

In this section, a few basic definition and results of Fuzzy proximity spaces that are used in the subsequent chapters. Most of these can be found in [41, 42].

Definition 2.6.1. [41] Let δ be binary relation on L^X, i.e., $\delta \subseteq L^X \times L^X$. The fact that $(A,B) \in \delta, (A,B) \notin \delta$ are denoted by the symbols $A\delta B, A\not\delta B$ respectively. A binary relation δ on L^X is called an $L-$fuzzy proximity iff

(LFP1) $A\delta B \Rightarrow B\delta A$;

(LFP2) A and B are $L-$quasi-coincident $\Rightarrow A\delta B$;

(LFP3) $A\delta B, A \subseteq C, C \subseteq D \Rightarrow C\delta B$;

(LFP4) $\underline{0}\not\delta\underline{1}$;

(LFP5) $A\not\delta C, B\not\delta C \Rightarrow (A \bigcup B)\not\delta C$;

(LFP6) $A\not\delta B$ there exists a $C \in L^X$ such that $A\not\delta C$ and $B\not\delta C'$

The Pair (L^X, δ) is called an $L-$ fuzzy proximity space.

Remark 2.6.1. The condition **(LFP6)** equivalent to the following condition:

(LFP6') $A\not\delta B$ there exists a $\delta-$nbhd C of A and D of B such that $C \bigcap D = \underline{0}$

Definition 2.6.2. A binary relation δ on L^X is said to be an $L-$fuzzy basic proximity if it satisfies the four conditions

(PB1) $A\delta B \Rightarrow B\delta A$;

(PB2) $A\delta(B \bigcup C)$ iff $A\delta B$ or $A\delta C$;

(PB3) $\underline{0}\not\delta A$ for every $A \in L^X$;

(PB4) A and B are $L-$quasi-coincident $\Rightarrow A\delta B$

Remark 2.6.2. Every $L-$fuzzy proximity spaces is $L-$fuzzy basic proximity space. But converse is not be true.

Definition 2.6.3. Let (L^X, δ) be an $L-$fbps and let $A, B \in L^X$, then B is said to be a $\delta-$neighbourhood of A if $A\not\delta B'$. The set of all $\delta-$neighbourhoods of A is denoted by $\mathscr{N}(A)$. If $A = \underline{0}$, then $\mathscr{N}(A) = L^X$.

Theorem 2.6.1. *If (L^X, δ) be an $L-$fbps, then*

(N1) $B \in \mathscr{N}(A) \Rightarrow A \subseteq B$;

(N2) $B \in \mathscr{N}(A) \Leftrightarrow A' \in \mathscr{N}(B')$

(N3) $\mathscr{N}(A \bigcup B) = \mathscr{N}(A) \bigcup \mathscr{N}(B)$

Theorem 2.6.2. *Let X be a non-empty set and let there be assigned to each $A \in L^X$ a subset $\mathscr{N}(A)$ of L^X satisfying (N1),N(2) and (N3). Then:*

(N4) $B, C \in \mathscr{N}(A)$ *implies* $B \bigcap C \in \mathscr{N}(A)$

(N5) $A \subseteq B \Rightarrow \mathscr{N}(A) \subseteq \mathscr{N}(B)$

(N6) $B \in \mathscr{N}(A)$, $B \subseteq C \Rightarrow C \in \mathscr{N}(A)$

(N7) $\mathscr{N}(\underline{1}) = \underline{1}$

(N8) $\underline{1} \in \mathscr{N}(A)$ *for all* $A \in L^X$.

(N9) $\mathscr{N}(\underline{0}) = L^X$

(N10) $\mathscr{N}(A \bigcap B) \subseteq \mathscr{N}(A) \bigcap \mathscr{N}(B)$

Theorem 2.6.3. *Let X be a non-empty set and let there be assigned to each $A \in L^X$ a subset $\mathscr{N}(A)$ of L^X satisfying (N1),N(2) and (N3). Then the binary relations δ on L^X defined by*

$$A \not\delta B \Leftrightarrow B' \in \mathscr{N}(A)$$

is an $L-$fbp on X.

Definition 2.6.4. Let (L^X, δ_1) and (L^X, δ_2) be two $L-$fuzzy proximity spaces. A function $f^{\rightarrow} : (L^X, \delta_1) \rightarrow (L^X, \delta_2)$ is said to be proximally continuous iff

$$A\delta_1 B \Rightarrow f^{\rightarrow}(A)\delta_2 f^{\rightarrow}(B)$$

Theorem 2.6.4. *A function $f^{\rightarrow} : (L^X, \delta_1) \rightarrow (L^Y, \delta_2)$ is proximally continuous iff for every $A, B \in L^Y$, $A \not\delta_2 B$ implies $f^{\leftarrow}(A) \not\delta_1 f^{\leftarrow}(B)$.*

Theorem 2.6.5. *Let (L^X, δ) be an $L-$fbps. Define a map $u_\delta : L^X \rightarrow L^X$ as follows:*

$$\text{for } A \in L^X, u_\delta = \bigcap\{B \in L^X : A \not\delta B'\} = \bigcap\{B \in L^X : B \in \mathscr{N}(A)\}$$

Then u_δ is an $L-$fuzzy closure operator on L^X.

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