CHAPTER 1

INTRODUCTION

1.1 Introduction

Several essential concepts of topological properties such as uniform continuity, uniform convergence, and completeness can't be expressed in the theory of topological spaces. There have been serious efforts by mathematicians to overcome this problem. Many solutions have been offered for the problem by different mathematicians, such as uniform structures, proximity structures, contiguity structures, syntopogeneous structures, and merotopic structures. In 1937 Weil[1] introduced a uniform structure with a set of axioms. However, Weil's axiomatization is not convenient and was soon succeeded in two different approaches such as **Entourages** introduced by Bourbaki [2] and **Covering** introduced by Tukey [3]. Uniform structures were equipped with extreme conditions, which is well known for extensions of completely regular topological spaces. Several researchers thought it necessary to weaken the vital needs to develop more general spaces wherein results in uniform structures could possibly develop. As a result, various generalisations of uniform structures developed, such as - Quasi-uniform spaces, Locally uniform spaces, Locally Quasi-uniform spaces, Semi-uniform spaces.

1.2 Significance of Covering over Entourage

The concept of metric spaces leads naturally to the idea of uniform spaces. There are two types of approaches in the study of uniform structure, namely Entourage approach and covering approach. Entourage uniformity is defined in the power set of $X \times X$ and covering uniform spaces is defined in the power set of X itself. Peters [84] shows the covering uniformity has the advantage over Entourage uniformity of a non-empty set, the cardinality of its covers, uniform covering of the form $\{S(x, 2^{-n}) : x \in M\}$ where M is a metric space with measure distance, usually $S(x, \epsilon)$ is considered a sphere with a radius ϵ, for each integer n provides a uniform measure of nearness throughout the entire spaces also a much easier approach for us to use for the mathematical developments as well as other applications in other branches.

1.3 Generalised uniform spaces

In 1951 Morita in[4] considered a generalisation of uniform space first time, the concept of covering uniform spaces by Tukey [3]. Morita analyses that the basic set is topological and considers collections of open covers for the generalisation, which plays a vital role in the completion and extension theory of topological spaces.

In 1950, The study of quasi-uniformities started with L.Nachbin's investigation recorded in [7], quasi-uniformity $\mathcal{U}$ and whose topology is induced by associated $\mathcal{U} \vee \mathcal{U}^{-1}$. The further development of the theory of quasi-uniform spaces is recorded in[61].

In 1959, E.$\check{C}$ech in [9] introduced through Weil's approach semi-uniform spaces for a non-empty set X is a filter $\mathcal{V}$ on $X \times X$ such that for each $V \in \mathcal{V}$, $\Delta \subseteq V$ and $V^{-1} \in \mathcal{V}$. Further development of theory is found in [9, 58]

Isbell [6] expresses a predilection for the use of covers in the study of uniform spaces, and the works serves as bolster this attitude; families of covers constitute the only tool that works for frame. They even work (suitably modified) in non-symmetric situations. Gantner and Steinlage[18] characterized quasi-uniform spaces in terms of conjugate covers, but their formulation of these covers is somewhat cumbersome. Later Frith [20] formulated a family $\mathfrak{U}$ of conjugate covers of a biframe B is a quasi-uniformity on B and then obtained significant results of uniform properties in the same context.

In 1972, J. William in[16] introduced Locally uniform space by localizing the triangular axiom through the entourage approach i.e., for x and entourage U there is another entourage V such that $V \circ V[x] \subseteq U[x]$. William showed there is a one-one correspondence between local uniformity with regular topological spaces. Further development of the theory of locally uniform is carried out in [17, 44].

In 1973 Steiner et al. in [19], rediscover Mortia's regular $T-$ uniformities and introduced semi-uniform space to the pair $(X, \mathcal{S})$, where S is family of covers of set X and $\mathscr{U}_1$ and $\mathscr{U}_2$ in $\mathcal{S}$, then $\mathscr{U}_1$ is locally star-refine to $\mathscr{U}_2$. First semi-uniform space is not consider as topological in nature, and they applied this structure to obtain a compatible regular topology [i.e., semi-uniform spaces is topological iff for each $x \in U$, there exists $\mathscr{U} \in \mathcal{S}$ such that $st(x, \mathscr{U}) \subseteq U$.] to also shown that semi-uniform space is regular T-uniformity of Morita. Further development of semi-uniform spaces are found in [21, 43, 27]

In 1974, Pu et al., in [23], introduced a semi-quasi- uniform space to be pair $(X, \mathscr{U})$ consisting of a set X and filter $\mathscr{U}$ in $X \times X$ such that $(x, x) \in U$ for each $x \in X$ and $U \in \mathscr{U}$. A semi-quasi-uniform space $(X, \mathscr{U})$ determines a closure space by defining the nbd filter of $x \in X$ to $\{U[x] : U \in \mathscr{U}\}$ by assuming $cl(A) = \{x \in X : U[x] \bigcap A \neq \phi\}$. A necessary and sufficient condition in to be topological space of a Semi-quasi-uniform spaces. Subsequently, various results uniform spaces have been generalised for semi-quasi-uniform spaces in [24].

In 1974, Lindgren et al. in [22] introduced the concept of local-quasi-uniformity and necessary and sufficient conditions have been investigated to admit local-quasi-uniformity with a countable base. Hicks et al. in [31] consider the other aspects of the theory of local-quasi uniform spaces and prove that a topological space is uniquely locally quasi-uniformisable if and only topological space is finite.

In 1978, Vasudevan et al. in [30] characterised the notion of William locally uniform space using covering weakening the covering uniform space.

1.4 Fuzzy set and Fuzzy Topology

Zadeh in [8] introduced fuzzy set in the year 1965 to deal with uncertainties, Crispness, Vagueness, Fuzziness. For a set X and for each $A \subseteq X$ has a degree of membership function $\mu : A \to [0, 1]$, then (A, μ) is called fuzzy set. Since then, fuzzy set theory has entered into many disciplines of science and technology as well as many branches of humanities to become a highly fruitful and fast-growing interdisciplinary area of research and wide applications. Goguen [11] who was a student of Zadeh generalised fuzzy set to $L-$fuzzy set in a set X as a map $X \to L$ such that $(L^*, \leq_{L^*})$ is a complete lattice, where $L^* = \{(x_1, x_2) \in [0,1]^2, x_1 + x_2 \leq 1\}$ and $(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$. In 1968, Chang in [12] used the fuzzy set to generalised the general topology and developed the basic notions for such spaces. Since then, the intensity of research in the areas of fuzzy topology has increased sharply, resulting in several thousands of research papers. After Chang, several other definitions of a fuzzy topological space were the attempt of either generalising the space or an attempt to remove several pathologies exists in the prevailing space. Chang's definition was later generalised to $L-$fuzzy topological spaces. Other important approaches are due to Lowen, Shostak, et al. [25, 46].

It has been nearly more than five decades since the existence of fuzzy topology. Several advantages have been obtained in theory as well as real-life applications:

★ The fuzzy topological space categories have been able to fill in numerous important gaps in the crisp topological space categories [54].

★ Various techniques are the use of the stratification method improves the effectiveness of proofs in fuzzy topological spaces.

★ Results in $L-$topological spaces have given characterisation of completely distributive lattice leads to applications in algebra and analysis.

★ Emerging research areas such as Digital topology, Flood prediction, and Object extraction were also discovered to have applications and it is recorded in [65, 80, 86].

Various notions of fuzzy topology given crisp topology have not been easy. Due to gradation structures, it is often difficult to establish a theory in fuzzy topology. For the same cause, one single notion of crisp topology usually generates different counterparts in fuzzy topology that have their advantages and deficiency. Various attempts have been made to bring disparate ideas together, with different levels of success.

1.5 Uniform structure in fuzzy topology

Uniform structures in the category of fuzzy topological spaces through the entourage approach have been studied by viz., Hölle, Koetz, Hutton, Katsaras, Lowen, Cheng Ming et al. in [26, 29, 38, 40, 45]. The study of generalised uniform structure through entourage was carried out by different authors.

Mitra et al. in [63] introduced $L-$local quasi-uniformity in terms of the entourage approach by extending the $L-$ Semi-quasi-uniformity and which is a generalisation of Hutton's quasi-uniformity in the category of **L-TOP** and several essential results were in [73].

Mitra et al. [70] introduced $L-$local uniformity by extending the $L-$local quasi–uniformities containing the inverse of each of its members. The theory of $L-$local uniformity also generalises Hutton's uniformity, and many vital uniformity results were acquired . Further development of the theory of $L-$local uniformity was found in [88]

Hazarika et al. in [69] introduced the concept of stratified $L-$locally uniform spaces and studied some relation with stratified $L-$topology and also established that every typically T_1 regular $L-$ topology of stratified and shown that every separable $L-$locally uniformity is stratified.

Mitra et al., in [71] generalised Hutton's quasi-uniformity to L-semi-quasi-uniformity and L-semi-uniformity through the entourage approach in the category **L-TOP** and several important results of uniformity have been established; more results are in [72]. Abbas in [74] introduced the notions of stratified (L, M)-fuzzy quasi-uniform spaces and investigate the images of preimages of developed notion and study the relationship between advanced concept and stratification of it.

Ghareeb in [76] introduced the concept of $L-$ double uniform spaces and studied the natural relationship between $L-$double uniform spaces, $L-$double fuzzy topologies, and $L-$double topogeneous structures were considered. The author also shows that the family of all double uniform spaces is compatible with $L-$double topogeneous.

Ramadan et al. in [87] studied $L-$nbhd spaces and investigated the topological properties of $L-$uniform spaces and obtained $L-$fuzzy topology, $L-$nbhd space induced by $L-$uniformity in complete residuated lattices.

Garica et al. in [93] studied functional behaviour and its relations with respect to Lowen's functors and Katsaras's functors and established a relationship between the categories of probabilistic uniform spaces and Hutton's quasi-uniformities with the category of the classical concept.

Pedraza et al. [94]studied the functions between partially ordered sets which preserves some special sets such as filters and characterised those functions aggregating bases of $L-$probabilistic quasi-uniform spaces.

Uniform structures through covering approach Soetens et al. [47], Chandrika [50, 55] and Garcia et al. [59]. The study of uniform structures through the covering approach is carried out by different authors.

Soetens et al. [47]and Chandika [50, 55] used Tukey classical definition to defined fuzzy uniform spaces using fuzzy covering and fuzzy star-refinement of covers in fuzzy topological space. Further the fuzzy topology is uniformizable iff fuzzy topology is completely regular spaces.

Garcia et al. in [59] introduced covering uniform spaces in the category of $L-$ topological spaces by generalising the notion of classical definition of Tukey and it is found to be equivalent to the Hutton uniform spaces of the Entourage approach. Authors, also studied various properties like uniform continuous functions, uniform isomorphism and various application in Algebra found in [66].

Hashem et al. in [57] introduced the fuzzy $TL-$ uniform spaces for each continuous triangle norm T. Later Hashem in [78] characterised the fuzzy $TL-$ uniform spaces in term of covering approach. Hashem also shown that for covering $TL-$uniform space $(X, \mathscr{K})$ interior operator defined the Fuzzy topology, C-uniformly continuous functions between covering $TL-$uniform spaces and isomorphism between the category of $TL-$uniform spaces. Author also studied $\alpha-$level covering for a covering $TL-$uniform spaces and the relation between them.

Yanghoubi in [77] introduced a different type of fuzzy uniformity in term of $T-$covers of a frame L that is called $T-$valued uniformity and studied various uniform structures. Further development of $T-$valued uniformity is found in [89].

Abbas et al,[82] introduced a notion of double L-fuzzy C-uniformity through the covering approach and investigated their uniform properties and relationship be double $L-$fuzzy C-uniform spaces and double $L-$fuzzy uniform spaces.

Tiwari in [81] introduced α^*- uniformity on a non-empty set X using α^* covers of X in $L-$fuzzy set and equivalent conditions fuzzy topology and α^*-uniformity and various properties of uniformity have been obtained.

1.6 Motivation of the Work

The study of generalised fuzzy uniform space has been studied through the entourage approach. In the review of literature, it is seen that none of Author's studied generalisations of covering fuzzy uniform structures. It is found that various generalisations of fuzzy uniform structures through the covering are still unexplored. This motivates us to generalised the uniform structure through the covering approach for the further development of theory and applications of uniform structures.

1.7 Objectives

To develop and study of some generalisation uniformity and related structures in the category **C-TOP**. To study various other topological notions such as compactness, completeness, proximity relations and other notions in the context of uniform spaces in L-topological spaces.

In brief, this study shall make a concerted attempt develop a theory of generalised uniform structures and related concepts in the setting of uniform spaces the category **C-TOP**

1.8 Overview of the thesis

The structure of the thesis is organized as follows:

Chapter 1: In this chapter, the introductory in which brief outline of the background of the work includes surveys(work carried out by different authors), significance aspects, primary objectives and outline of the work carried out also included in this chapter.

Chapter 2: In this chapter, the basic definitions and results that are used in the subsequent chapters provide as preliminaries.

Chapter 3: In this chapter, the study of covering L-semi-uniform space (In short CLS-Uniform spaces) carried out by generalising covering $L-$ uniform space introduced by Garcia, at al. [59] in the category **C-TOP**. Various important results of uniform spaces including the problem of metrization are developed in the context of CLS-uniform spaces. Example of CLS-uniform spaces that is not covering $L-$uniform space is provided. A sufficient condition under which the generating interior space will be $L-$topological is obtained. Generalised uniform continuous are shown as continuous with respect to induced interior space in the same context. Further, it has been established that every CLS-uniform spaces with countable base is $L-$semi-pseudo-metrizable.

A part of the work is accepted for publication in the journal *Annals of Fuzzy Mathematics and Informatics* entitled with "A study on CLS-uniform space".

Chapter 4: In this chapter, the study of relation between $L-$fuzzy basic proximity space introduced by Granim in [42] (in short $L-$fbps) and CLS-Uniform space define in chapter 3 has been considered and then the relation is obtained. Relation between CLS-uniform space and $L-$fbps is obtained. Further, proximally coarse CLS-uniform space and some important such as hereditary and closed under arbitrary products, uniqueness of proximally coarse CLS-uniformity proximally continuous are obtained and it is turn out that proximally coarse CLS-uniform spaces is equivalent to totally bounded CLS-uniform spaces.

A part of the work is under communication for publication in the journal.

Chapter 5: In this chapter, the study of covering $L-$locally uniform spaces by generalising covering $L-$ uniform spaces in the category of **C-TOP**. Covering $L-$locally uniform spaces is stronger than CLS-uniform space introduced in Chapter 3. The study is made compatibility of covering $L-$locally uniform space and $L-$topological space and it has been shown that every regular $L-$topology generates covering $L-$locally uniform space and conversely every CLS-uniform space generates regular $L-$topology. Further, the notion of weakly uniformly continuous function is introduced and it is shown that every weakly uniform continuous functions are continuous in $L-$topology and also shown that every covering $L-$locally uniform spaces with countable base is pseudo-metrizable.

A part of the work of this chapter is published in the journal entitled as "Covering $L-$locally uniform spaces", Journal of Mathematical and computational Science **9**(6) 2019, 678–691.ISSN: 1927-5307, DOI:10.28919/jmcs/4196.

Chapter 6: In this chapter, the study of notions such as Cauchy filters, weakly Cauchy filter in the context of covering $L-$locally uniform space introduced in chapter 5, and then, convergence structures in terms of strongly completeness, hereditary property and isomorphic in covering $L-$locally uniform spaces. The result of equivalency of the compactness and completeness is obtained, and also condition for uniqueness of covering $L-$locally uniform spaces in $L-$topology.

A part of the work of this chapter is published in the journal entitled as , " A note on covering $L-$locally uniform spaces", Advances in Mathematics: Scientific Research **9**(9)2020, 7137-–7148. ISSN: 1857-8365 (printed); 1857-8438 (electronic) , DOI:10.37418/amsj.9.9.64.

Chapter 7: In this last chapter, a brief observation, in the form of a conclusion is presented. The directions of future research scopes that can be approach from the thesis are briefly outlined.

A bibliography that includes all the references is given at the end of the thesis.

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