CHAPTER 6

COMPLETENESS AND COMPACTNESS

6.1 Introduction

In the previous chapter 5, the notion of covering $L-$locally uniform space is introduced and then various important results of uniform spaces concerning characterisation of $L-$topology, weakly uniform continuous functions and problem of metrization is also considered in the same context. It is become pertinent to investigate the possible role of notions of completeness, compactness and totally boundedness in the context of covering $L-$locally uniform space. For this the notions of Cauchy filters, weakly Cauchy Filter in the context of covering $L-$locally uniform spaces introduced, and then studied convergence structures such as strongly completeness, hereditary property, and isomorphic. Also established the equivalency of the compactness and completeness in the context of covering $L-$locally uniform spaces, and then the uniqueness property in the same.

6.2 Completeness

In this section, the notions of Cauchy filters, weakly Cauchy Filter, in the context of covering $L-$locally uniform spaces introduced, and then studied strongly completeness, hereditary property, and isomorphic in the context of covering $L-$locally uniform spaces.

Definition 6.2.1. Let $(L^X, \mathfrak{U})$ be a covering $L-$locally uniform spaces, then a filter $\mathscr{F}$ is called cauchy filter for each $\mathscr{A} \in \mathfrak{U}$, there exists $F \in \mathscr{F}$ and $A \in \mathscr{A}$ such that $F \subseteq A$

Definition 6.2.2. A filter $\mathscr{F}$ to be weakly Cauchy if each $\mathscr{A} \in \mathfrak{U}$, there is a filter $\mathscr{G}$ containing $\mathscr{F}$ and $G \in \mathscr{G}$ such that $G \subset A$, for some $A \in \mathscr{A}$.

Clearly, Cauchy filters are weakly Cauchy filters.

Definition 6.2.3. A covering $L-$locally uniform space $(L^X, \mathfrak{U})$ is said to be (strongly) complete if every (weakly) Cauchy filter in $(L^X, \mathfrak{U})$ converges.

Definition 6.2.4. Let $f^{\rightarrow} : (L^X, \mathfrak{U}) \rightarrow (L^Y, \mathfrak{V})$ be a function, then $f^{\rightarrow}$ is said to be weakly uniform isomorphism iff $f^{\rightarrow}$ is bijective and both $f^{\rightarrow}$ and $f^{\leftarrow}$ are weakly uniform continuous.

Proposition 6.2.1. *Let $\mathscr{A}$ and $\mathscr{B}$ be $L-$covers and let $A \in L^X$ be $L-$fuzzy subset, then we have $st(A, \mathscr{A} \bigcap \mathscr{B}) \subseteq st(A, \mathscr{A}) \bigcap st(A, \mathscr{B})$*

Lemma 6.2.2. *Let $(L^X, \mathfrak{U})$ be a covering $L-$locally uniform spaces, then the collection*

$\{st(x_\alpha, \mathscr{A}) : \mathscr{A} \in \mathfrak{U}\}$ is the family of all Q-nbhd at x_α in $(L^X, \mathfrak{U}(\mathbb{F}))$.

Proof. By regularity for $x_\alpha \in L^X$ and $\mathscr{A} \in \mathfrak{U}$, there exists an $L-$fuzzy open set G such that $x_\alpha \subseteq G \subseteq cl(G) \subseteq st(x_\alpha, \mathscr{A})$. Which implies $x_\alpha \in cl(G)$ and then by Theorem 2.2.16, $x_\alpha q G \subseteq st(x_\alpha, \mathscr{A})$. By Definition 2.2.16, we have $st(x_\alpha, \mathscr{A})$ is $Q-$nbhd at x_α in $(L^X, \mathbb{F}(\mathfrak{U}))$. □

Theorem 6.2.3. *Convergent filter in covering $L-$locally uniform spaces is weakly Cauchy filter.*

Proof. Let $(L^X, \mathfrak{U})$ be a covering $L-$locally uniform space and $\mathscr{F}$ be a filter such that for some $x_\alpha \in L^X, \mathscr{F} \to x_\alpha$ in $(L^X, \mathbb{F}(\mathfrak{U}))$. Let $\mathscr{Q}(x_\alpha) = \{st(x_\alpha, \mathscr{A}) : \mathscr{A} \in \mathfrak{U}\}$, then by Lemma 6.2.2, $\mathscr{Q}(x_\alpha)$ is $Q-$nbhd in $\mathbb{F}(\mathfrak{U})$. Since $\mathscr{F}$is convergent then by Definition 2.3.6, for any $U = st(x_\alpha, \mathscr{B}) \in \mathscr{Q}(x_\alpha)$, there exists $F \in \mathscr{F}$ such that $F \subseteq U$. Now let $\mathscr{G} = \{st(U, \mathscr{A}) : \mathscr{A} \in \mathfrak{U}\}$, then $\mathscr{G} \neq \underline{0}$ and as by Proposition 6.2.1, $st(U, \mathscr{A} \bigcap \mathscr{B}) \subseteq st(U, \mathscr{A}) \bigcap st(U, \mathscr{B})$. Again by Definition 5.2.1, $(\mathscr{A} \bigcap \mathscr{B}) \in \mathfrak{U}$, so $st(U, \mathscr{A} \bigcap \mathscr{B}) \in \mathscr{G}$, implies $\mathscr{G}$ is base for a filter. Also $F \subseteq U = st(x_\alpha, \mathscr{A}) \subseteq st(U, \mathscr{A})$ implies $\mathscr{F}$ is weakly Cauchy filter.

□

Definition 6.2.5. Let $(L^X, \mathfrak{U})$ be a covering $L-$locally uniform spaces and $A \in L^X$. Let for each $\mathscr{B} \in \mathfrak{U}$ define $\mathfrak{U}_A = \{A \bigcap B : B \in \mathscr{B} \in \mathfrak{U}\}$. Then $\mathfrak{U}_A$ is a covering $L-$locally uniform spaces on A which we call a sub covering $L-$locally uniform spaces on A and $(A, \mathfrak{U}_A)$ said to be the subspace. $\mathfrak{U}_A$ is open or closed sub covering $L-$uniform spaces according to $A \in \mathbb{F}(\mathfrak{U})$ or $A' \in \mathbb{F}(\mathfrak{U})$.

Proposition 6.2.4. *Let $\mathscr{F}$ be a filter on a subspace of $(A, \mathfrak{U}_A)$, then $\mathscr{F}$ is also filter on $(L^X, \mathfrak{U})$.*

Proof. Let $(L^X, \mathfrak{U})$ be a covering $L-$locally uniform space and let $A \in L^X$ be a

$L-$fuzzy subset, then by Definition 6.2.5, $(A, \mathfrak{U}_A)$ is a subspace. Suppose $\mathscr{F}$ be filter on $(A, \mathfrak{U}_A)$, then for any $F \in \mathscr{F} \subset L^A$ implies $F \in \mathscr{F} \subset L^X$ as $A \in L^X$, so $\mathscr{F}$ is also filter on L^X. □

Lemma 6.2.5. *Let $\mathscr{F}$ be a weakly Cauchy filter in a covering $L-$locally uniform space $(L^X, \mathfrak{U})$ and let $A \in L^X$. Then $\mathscr{F}_A = \{A \bigcap F : F \in \mathscr{F}\}$ is weakly Cauchy filter in $(A, \mathfrak{U}_A)$.*

Proof. Let $\mathscr{F}$ be a weakly Cauchy in a covering $L-$locally uniform spaces $(L^X, \mathfrak{U})$. Also let $A \in L^X$, then $\mathscr{F}_A = \{A \bigcap F : F \in \mathscr{F}\}$. Since $\mathscr{F}$ is weakly Cauchy filter there exists another filter $\mathscr{G}$(say) containing $\mathscr{F}$. So, $\mathscr{G}_A = \{A \bigcap G : G \in \mathscr{G}\}$ and $A \bigcap F \subseteq A \bigcap G$, as $F \subseteq G$, implies $\mathscr{G}_A$ is filter containing $\mathscr{F}_A$, therefore $\mathscr{F}_A$ is weakly Cauchy filter in $(A, \mathfrak{U}_A)$. □

Theorem 6.2.6. *Every closed subspace in strongly complete covering $L-$locally uniform spaces is strongly complete.*

Proof. Let $(L^X, \mathfrak{U})$ be a strongly complete covering $L-$locally uniform space. Let A is closed subset of L^X, and then by Definition 6.2.5, $(A, \mathfrak{U}_A)$ be a closed subspace of $(L^X, \mathfrak{U})$. Let $\mathscr{F}$ be a weakly Cauchy filter on $(A, \mathfrak{U}_A)$, and then by proposition 6.2.4, $\mathscr{F}$ is weakly Cauchy filter on $(L^X, \mathfrak{U})$. Since $(L^X, \mathfrak{U})$ is strongly complete, so $\mathscr{F} \to x_\alpha \in L^X$. Since A is closed subset of L^X so we must have $x_\alpha \in A$. So, $\mathscr{F}$ is converges in $(A, \mathfrak{U}_A)$. Hence $(A, \mathfrak{U}_A)$ is strongly complete. □

Theorem 6.2.7. *Let $(L^X, \mathfrak{U})$ and $(L^Y, \mathfrak{V})$ be covering $L-$locally uniform spaces and $f^{\to} : (L^X, \mathfrak{U}) \to (L^Y, \mathfrak{V})$ be weakly uniform continuous. If $\mathscr{F}$ is weakly filter in $(L^X, \mathfrak{U})$, then $f^{\to}(\mathscr{F})$ is weakly cauchy filter in $(L^Y, \mathfrak{V})$.*

Proof. Let $\mathscr{F}$ be a weakly Cauchy filter in $(L^X, \mathfrak{U})$ and let $\mathscr{C} \in \mathfrak{V}$. Since $f^{\rightarrow} : (L^X, \mathfrak{U}) \rightarrow (L^Y, \mathfrak{V})$ be weakly uniform continuous, therefore $f^{-1}(\mathscr{C}) \in \mathfrak{U}$, where $f^{-1}(\mathscr{C}) = \{f^{\leftarrow}(C) : C \in \mathscr{C}\}$. As $\mathscr{F}$ is a weakly Cauchy filter on $(L^X, \mathfrak{U})$, then by Definition 6.2.2, there exists a filter $\mathscr{G}$ containing $\mathscr{F}$ such that $G \subseteq f^{\leftarrow}(A)$ for some $f^{\leftarrow}(A) \in f^{-1}(\mathscr{C}) \Rightarrow f^{\rightarrow}(A) \in \mathscr{C}$.Since $f^{\rightarrow}$ is order preserving and hence $f^{\rightarrow}(G) \subseteq f^{\rightarrow}(A)$. Hence $f^{\rightarrow}(\mathscr{G})$ is a filter containing $f^{\rightarrow}(\mathscr{F})$, with $f^{\rightarrow}(G) \subseteq f^{\rightarrow}(A)$. Which implies $f^{\rightarrow}(\mathscr{F})$ is weakly Cauchy filter on $(L^Y, \mathfrak{V})$.

□

Theorem 6.2.8. *Let $(L^X, \mathfrak{U})$ and $(L^Y, \mathfrak{V})$ be two covering $L-$locally uniform spaces and $f^{\rightarrow} : (L^X, \mathfrak{U}) \rightarrow (L^Y, \mathfrak{V})$ be weakly uniform isomorphism, then $(L^X, \mathfrak{U})$ is strongly complete iff $(L^Y, \mathfrak{V})$ is strongly complete.*

Proof. $(\Rightarrow)$ Let $(L^Y, \mathfrak{V})$ be a strongly complete and let $\mathscr{F}$ be a weakly Cauchy filter on $(L^X, \mathfrak{U})$. Then by Theorem 6.2.7, $f^{\rightarrow}(\mathscr{F})$ is weakly Cauchy filter on $(L^Y, \mathfrak{V})$. Therefore $f^{\rightarrow}(\mathscr{F})$ is converges to $x_\alpha \in L^Y$ being $(L^Y, \mathfrak{V})$ be a strongly complete i.e., $f^{\rightarrow}(\mathscr{F}) \rightarrow x_\alpha \in L^Y$. Also $f^{\rightarrow} : (L^X, \mathfrak{U}) \rightarrow (L^Y, \mathfrak{V})$ being weakly uniform isomorphism, then by Definition 6.2.4, $f^{\leftarrow}$ is weakly uniform continuous, so $f^{\leftarrow}(f^{\rightarrow}(\mathscr{F}))$ is weakly Cauchy filter on $(L^X, \mathfrak{U})$. Since $f^{\rightarrow}$ is an $L-$fuzzy homeomorphism being weakly uniform isomorphism and so, $f^{\leftarrow}(f^{\rightarrow}(\mathscr{F})) \rightarrow f^{\leftarrow}(x_\alpha)$ and being $f^{\rightarrow}$ bijective as weakly uniform isomorphism, therefore by Theorem 2.2.7, $f^{\leftarrow}(f^{\rightarrow}(\mathscr{F})) = \mathscr{F} \rightarrow f^{\leftarrow}(x_\alpha) \in L^X$ and hence $(L^X, \mathfrak{U})$ is strongly complete.
$(\Leftarrow)$ It follows the other way implication.
Hence the theorem. □

6.3 Compactness and Totally boundedness

In this section, the study equivalency of compactness and completeness in the context of covering $L-$locally uniform spaces, and then uniqueness of covering $L-$locally uniform spaces is investigated.

Definition 6.3.1. Let $(L^X, \mathfrak{U})$be a covering $L-$locally uniform space then it is said to be totally bounded if for all $\mathscr{A} \in \mathfrak{U}$ there is a finite $L-$fuzzy set $F \in Pt(L^X)$ such that $st(F, \mathscr{A}) = \underline{1}$.

Lemma 6.3.1. *If $f^{\rightarrow}$ is weakly uniform continuous then $\mathscr{A} \in f^{\rightarrow}(\mathfrak{U})$ iff $f^{\leftarrow}(\mathscr{A}) \in \mathfrak{U}$.*

Proof. Straight forward. □

Theorem 6.3.2. *Let $f^{\rightarrow} : (L^X, \mathfrak{U}_1) \rightarrow (L^Y, \mathfrak{U}_2)$, weakly uniformly continuous. If $(L^X, \mathfrak{U}_1)$ compact then $(L^Y, f^{\rightarrow}(\mathfrak{U}_1))$ is compact.*

Proof. Let $\mathscr{C}$ be an open covering of L^Y. Then $f^{\leftarrow}(\mathscr{C}) = \{f^{\leftarrow}(C) : C \in \mathscr{C}\}$ is open covering of L^X as $f^{\rightarrow}$ is weakly uniformly continuous. By compactness there exists a finite subcover of $f^{\leftarrow}(\mathscr{C})$, i.e., $\bigcup_{i=1}^{n} f^{\leftarrow}(C_i) = \underline{1}, \quad n \in \mathbb{N}$ where $f^{\leftarrow}(C_i) \in f^{\leftarrow}(\mathscr{C})$. Now, by Lemma 6.3.1, $\bigcup_{i=i}^{n} C_i = \underline{1}$. Hence $(L^X, f^{\rightarrow}(\mathfrak{U}_1))$ is compact.

□

Theorem 6.3.3. *Every compact covering $L-$locally uniform spaces is totally bounded.*

Proof. Let $(L^X, \mathfrak{U})$ be a compact space. Then for any $\mathscr{A} \in \mathfrak{U}$, the collection

$$\{st(x_\alpha, \mathscr{A}) : x_\alpha \in Pt(L^X)\}$$

is an open cover of $\underline{1}$.

Now, since $\underline{1}$ is closed. Therefore, by compactness, there exists finite subcover of

$$\{st(x_\alpha, \mathscr{A}) : x_\alpha \in Pt(L^X)\}$$

For some finite $L-$fuzzy points x_{α_i}, $1 \leq i \leq n,\ n \in \mathbb{N}$ such that $\bigcup_{i=1}^{n} st(x_{\alpha_i}, \mathscr{A}) = \underline{1}$ $\Rightarrow st(\bigcup_{i=1}^{n} x_{\alpha_i}, \mathscr{A}) = \underline{1} \Rightarrow st(F, \mathscr{A}) = \underline{1}$, where $\bigcup_{i=1}^{n} x_{\alpha_i} = F$. Since F is finite and hence $(L^X, \mathfrak{U})$ is totally bounded. □

Theorem 6.3.4. *In covering $L-$locally uniform space every ultra-filter is a weakly Cauchy filter.*

Proof. Let $\mathscr{F}^*$ be ultra-filter on $(L^X, \mathfrak{U})$. Then the collection $\mathscr{B} = \{st(x_\alpha, \mathscr{A}) : x_\alpha \in Pt(L^X)\}$ is base for $\mathfrak{U}$ by totally boundedness there is finite $L-$fuzzy points x_{α_i}, $1 \leq i \leq n,\ n \in \mathbb{N}$ such that $\bigcup_{i=1}^{n} st(x_{\alpha_i}, \mathscr{A}) = \underline{1}$. But $\underline{1} \in \mathscr{F}^*$, then by Definition 2.3.5, $F \in \mathscr{F}^*$ such that $F \subseteq st(x_\alpha, \mathscr{A})$ for some $x_\alpha \in Pt(L^X)$. Hence $\mathscr{B}$ is filter base containing $\mathscr{F}^*$, so $\mathscr{F}^*$ is a weakly Cauchy filter. □

Remark 6.3.1. Every $L-$fuzzy filter has the finite intersection property.

Lemma 6.3.5. *Let $(L^X, \mathfrak{U})$ be a compact $L-$locally uniform spaces, then every filter has a cluster point.*

Proof. Let $(L^X, \mathfrak{U})$ be compact and $\mathscr{F}$ be a filter. Then $\mathscr{G} = \{cl(F) : F \in \mathscr{F}\}$ is closed filter. Then by Theorem 2.2.17 and Remark 6.3.1, $\mathscr{G}$ has a finite intersection property with non-empty intersection, i.e., $\bigcap \mathscr{G} \neq \underline{0}$. This implies there exists $x_\alpha \in L^X$ such that $x_\alpha \in \bigcap cl(F) \Rightarrow x_\alpha \in cl(F)$, with $F \in \mathscr{F}$. Hence x_α is a cluster point of $\mathscr{F}$. □

Theorem 6.3.6. *Every compact covering $L-$locally uniform spaces is a strongly complete.*

Proof. Let $(L^X, \mathfrak{U})$ be a compact covering $L-$locally uniform spaces. Let $\mathscr{F}$ be a weakly Cauchy filter on $(L^X, \mathfrak{U})$, then for each $\mathscr{A} \in \mathfrak{U}$ there exists a filter $\mathscr{G}$ containing $\mathscr{F}$ with $G \in \mathscr{G}$ such that $G \subset A$ for some $A \in \mathscr{A}$. Also for $A \in \mathscr{A}$ there exists

$F \in \mathscr{F}$ such that $F \subseteq A$ as $\mathscr{F}$ is Cauchy filter. By Theorem 6.3.5, $\mathscr{G}$ is has a cluster point, i.e., there exists $x_\alpha \in L^X$ such that $x_\alpha \in \bigcap cl(G), G \in \mathscr{G}$. This implies $x_\alpha \in cl(G)$, then by Theorem 5.2.9, $st(x_\alpha, \mathscr{A})\widehat{q}G$ [Since the family of Q-nbhd at x_α is $\mathscr{Q}(x_\alpha) = \{st(x_\alpha, \mathscr{A}) : \mathscr{A} \in \mathfrak{U}\}$]. Also $st(x_\alpha, \mathscr{A}) \bigcap G \neq \underline{0}$, there exists $y_\beta \in L^X$ such that $y_\beta \in st(x_\alpha, \mathscr{A}) \bigcap G \Rightarrow y_\beta \in st(x_\alpha, \mathscr{A})$ and $y_\beta \in G$. Now also we have, $y_\beta \in st(y_\beta, st(\mathscr{A})) \Rightarrow G \subseteq st(y_\beta, st(\mathscr{A})) \subseteq (x_\alpha, \mathscr{A})$ [as $y_\beta \in st(x_\alpha, \mathscr{A})$]. Now since $\mathscr{F}$ is weakly Cauchy filter for each $F \in \mathscr{F}$ with $F \subseteq G$, $F \subseteq G \subseteq st(x_\alpha, \mathscr{A}) \in \mathscr{Q}(x_\alpha)$. Which implies $\mathscr{F}$ converges at x_α. Hence strongly complete.

□

Theorem 6.3.7. *Let $(L^X, \mathfrak{U})$ be a covering $L-$locally uniform space, then the space is compact iff*

1. *$(L^X, \mathfrak{U})$ is totally bounded, and*

2. *$(L^X, \mathfrak{U})$ is strongly complete.*

Proof. ($\Rightarrow$) Let $(L^X, \mathfrak{U})$ be a compact covering $L-$locally uniform spaces then by Theorem (6.3.3) (i) $(L^X, \mathfrak{U})$ is totally bounded and
Theorem (6.3.6) (ii) $(L^X, \mathfrak{U})$ is strongly complete.
($\Leftarrow$) Let $(L^X, \mathfrak{U})$ be a totally bounded and strongly complete covering $L-$locally uniform space. Let $\mathscr{A} \in \mathfrak{U}$ be an open cover, then by totally boundedness there exists finite $L-$ fuzzy set F such that $st(F, \mathscr{A}) = \underline{1}$. For each $x_{\alpha_i} \in F$ we consider one A_i for some $x_\alpha \in A_i \in \mathscr{A}$. Then it is clear that $\bigcup A_i = \underline{1}$ implies $\{A_i\}$ is a finite subcover of $\mathscr{A}$ as F is finite $L-$fuzzy set. Hence $(L^X, \mathfrak{U})$ is compact covering $L-$locally uniform space. □

Thus in a totally bounded covering $L-$locally uniform spaces, compactness and strong completeness are equivalent.

Lemma 6.3.8. *Let $(X, \mathbb{F})$ be a regular $L-$ topological space. For each open cover $\mathscr{U}$ such that there exists an open cover $\mathscr{V}$ such that $cl(\mathscr{V}) \preccurlyeq \mathscr{U}$*

Proof. Straight forward. □

Theorem 6.3.9. *Let $(L^X, \mathbb{F})$ compact regular $L-$topolgy. Then the $L-$topology generates a unique covering $L-$uniform space.*

Proof. Let $\mathfrak{U}$ and $\mathfrak{U}^*$ two covering $L-$locally uniform spaces on $(L^X, \mathbb{F})$ for the compact regular $L-$topological spaces. Let $\mathscr{A} \in \mathfrak{U}$, then there exists finite subcover say $\{A_i : 1 \leq i \leq n\}$ also by Lemma 6.3.8 there exists a covering $\mathscr{B} \in \mathfrak{U}$ such that $cl(\mathscr{B}) \preccurlyeq \{A_i : 1 \leq i \leq n\}$ i.e, for each i there exists some $cl(B) \in cl(\mathscr{B})$ such that $cl(B) \subseteq A_i$. Let k be a positive integer such that $k \leq n$. For each $x_\alpha \in cl(B) \subseteq A_k$ there exists $\mathscr{A}^* \in \mathfrak{U}^*$ with $st(x_\alpha, st(\mathscr{B}^*)) \subseteq st(x_\alpha, \mathscr{A}^*)$ for some $\mathscr{B}^* \in \mathfrak{U}^*$. Put $\mathscr{A}_k^* = \{st(x_\alpha, \mathscr{A}^*) : x_\alpha \in cl(B) \subseteq A_k\}$, since $cl(B)$ is compact so, $\mathscr{A}_k^*$ has finite sub cover $\mathscr{C}_k^*$. For each $\mathscr{A}^* \in \mathfrak{U}^*$ there is a $\mathscr{A}_k^*$ such that $st(x_\alpha, \mathscr{A}) \in \mathscr{D}_k^*$ for each $x_\alpha \in cl(B) \subseteq A_k$. which implies $st(x_\alpha, \mathscr{C}_k^*) \subseteq A_k$ for each $x_\alpha \in cl(B) \subseteq A_k$.

Next we choose $\mathscr{A}^* \in \mathfrak{U}^*$ such that $\mathscr{A}^* \preccurlyeq \mathscr{A}_k^*$ for each $k = 1, 2, 3 \ldots n$. Let $x_\alpha \in L^X$, then $x_\alpha \in cl(B) \subseteq A_j$ for some $j \leq n$. Therefore $x_\alpha \in st(x_\alpha, \mathscr{A}_k^*) \subseteq A_k$. Consequently $\mathscr{A}^* \preccurlyeq \{A_i : 1 \leq i \leq n\}$. But then $\mathscr{A} \in \mathfrak{U}$, we conclude that $\mathfrak{U} \subset \mathfrak{U}^*$. Similarly $\mathfrak{U}^* \subset \mathfrak{U}$. Hence the theorem. □

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