CHAPTER 5

COVERING L-LOCALLY UNIFORM SPACES

5.1 Introduction

In the previous chapter 3 the notion of CLS-uniform space is developed in the category of **C-TOP**.Various important results of uniform spaces concerning interior space, topological interior space, metrizablity and uniformly continuous functions have been developed and studied proximity relation in chapter 4 in the same context. It is, however observed that generating spaces are interior spaces but not $L-$topological spaces. So, it is become pertinent to investigate whether there are other generalised uniform space in the same category weaker than covering $L-$uniform space could be developed therein. While looking for an answer to the above question, the notion of covering $L-$locally uniform space is introduced. The compatibility of covering $L-$locally uniform spaces and $L-$topology is examined. Weakly uniform continuous functions are introduced in the same context as a generalisation uniformly continuous and for continuity with respect to the induced $L-$topological spaces. In other to show that the notion of covering $L-$locally uniform spaces lies between CLS-uniform space and covering $L-$uniform spaces, suitable examples are provided. Further, the problem of metrization of the introduced notion of covering $L-$locally uniform space is considered and a satisfactory answer has been provided.

5.2 Covering L-locally uniform spaces

In this section, the study of covering $L-$locally uniform spaces, by generalising covering $L-$uniform spaces. Covering L-uniform spaces shows that every covering $L-$uniform space is covering $L-$locally uniform space, and every covering $L-$locally uniform space is a CLS-uniform space. But the converse of the statement is not valid. Hence, it founded that CLS-Uniform spaces are weaker than covering $L-$locally uniform spaces, and covering $L-$locally uniform spaces is weaker than covering $L-$locally uniform spaces. Interior operator, closure operator, were studied in the context of covering $L-$locally uniform spaces, and some significant results were obtained.

Definition 5.2.1. A CLS-Uniform spaces $(L^X, \mathfrak{U})$ is said to be a covering $L-$ locally uniformity on L^X, if it satisfies **SC1** and **SC2** and the following axiom:

LC. For each $\mathscr{A} \in \mathfrak{U}$ and for all $x_\alpha \in L^X$, there exists $\mathscr{B} \in \mathfrak{U}$ such that $st(x_\alpha, st(\mathscr{B})) \subseteq st(x_\alpha, \mathscr{A})$.

In that case we called the pair $(L^X, \mathfrak{U})$ as covering $L-$ locally uniform space.
Let $\mathfrak{U}_1$ and $\mathfrak{U}_2$ be covering $L-$uniform spaces on L^X. If $\mathfrak{U}_1 \subset \mathfrak{U}_2$, then $\mathfrak{U}_2$ is called finer than $\mathfrak{U}_1$.

Theorem 5.2.1. *Every covering L-locally uniform space is a covering CLS-uniform space.*

Proof. It follows from the Definition 5.2.1. □

Converse of above Theorem is not true, for this an example is given below.

Example 5.2.1. Let $X = \{a, b, c\}$ with $L = [0, 1]$. Consider $\mathscr{A} = \{\{a\}, \{b\}, \{a, b\}\{c\}\}$ and $\mathscr{B} = \{\{a\}, \{a, b\}, \{b, c\}, \{c\}\}$ are $L-$covers , then $\mathfrak{B} = \{\mathscr{A}, \mathscr{B}\}$ is a base for CLS-uniform space. We have $st(\mathscr{A}) = st(\mathscr{B}) = \{a, b, c\}$; $st(a, \mathscr{A}) = st(a, \mathscr{B}) = st(b, \mathscr{A}) = \{a, b\}$; st(b, $\mathscr{B}$) $= \{a, b, c\}$; $st(c, \mathscr{A}) = \{c\}$; $st(c, \mathscr{B}) = \{b, c\}$; $st(a, st(\mathscr{A})) = st(a, st(\mathscr{B})) = st(a, st(\mathscr{B})) = st(b, st(\mathscr{A})) = st(b, st(\mathscr{B})) = st(c, st(\mathscr{A})) = st(c, st(\mathscr{B}))$ $= \{a, b, c\}$; But for a and $\mathscr{A}$, there is no $\mathscr{B}$ such that $st(a, st(\mathscr{B})) \subseteq st(a, \mathscr{A})$.

From Theorem 5.2.1, and Example 5.2.1, it may conclude that every CLS-unform spaces is generalisation of covering $L-$locally uniform spaces. and also clearly, by Definition 2.4.4, it follows that every covering $L-$ uniform space is covering $L-$locally uniform spaces. But the converse is not true for this an example is given below,

Example 5.2.2. Let $X = \{a, b, c, d\}$ and $L = [0,\ 1]$.
Let $\mathscr{A} = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$,
$\mathscr{B} = \{\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$.
Then $\mathscr{A} \bigcap \mathscr{B} = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$.
Let $\mathfrak{B} = \{\mathscr{A}, \mathscr{B}, \mathscr{A} \bigcap \mathscr{B}\}$. Then clearly, $\mathfrak{B}$ satisfies the axiom (SC2). Now we have $st(\mathscr{A}) = \{a, b, c, d\}$; $st(\mathscr{B}) = \{a, b, c, d\}$; $st(\mathscr{A} \bigcap \mathscr{B}) = \{a, b, c, d\}$; Also, $st(a, st(\mathscr{A}))$ $= st(b, st(\mathscr{A})) = st(c, st(\mathscr{A})) = st(d, st(\mathscr{A})) = st(a, st(\mathscr{B})) = st(b, st(\mathscr{B})) = st(c, st(\mathscr{B}))$ $= st(d, st(\mathscr{B})) = \{a, b, c, d\}$; $st(a, st(\mathscr{A} \bigcap \mathscr{B}))= st(b, st(\mathscr{A} \bigcap \mathscr{B})) = st(c, st(\mathscr{A} \bigcap \mathscr{B}))$ $= st(d, st(\mathscr{A} \bigcap \mathscr{B})) = \{a, b, c, d\}$ and $st(a, \mathscr{A}) = st(a, \mathscr{B}) = st(a, \mathscr{A} \bigcap \mathscr{B}) = st(b, \mathscr{A})$ $= st(b, \mathscr{B}) = st(b, \mathscr{A} \bigcap \mathscr{B}) = st(c, \mathscr{A}) = st(c, \mathscr{B}) = st(c, \mathscr{A} \bigcap \mathscr{B}) = st(d, \mathscr{A}) = st(d, \mathscr{B}) = st(d, \mathscr{A} \bigcap \mathscr{B}) = \{a, b, c, d\}$. Thus $\mathfrak{B}$ satisfies the axioms (LC) and consequently, $\mathfrak{B}$ is a base for some covering $L-$local uniformity on L^X. But for $\mathscr{A}$, there is

no $\mathscr{B}$ such that $st(\mathscr{B}) \preceq \mathscr{A}$. This implies $\mathfrak{B}$ is not a base for covering $L-$uniformity on L^X.

Thus this can be conclude that covering $L-$locally uniform spaces is a generalisation of covering $L-$uniform spaces in the sense of Garcı́a et at.[59].

In order to show that every covering $L-$local uniformity generates an $L-$topology the following lemma is considered.

Lemma 5.2.2. *Let $(L^X, \mathfrak{U})$ be covering $L-$locally uniform space. Then the mapping, $int : L^X \rightarrow L^X$ defined by*

$$int(A) = \bigcup\{x_\alpha \in L^X : st(x_\alpha, \mathscr{A}) \subseteq A \text{ for some } \mathscr{A} \in \mathfrak{U}\},$$

for all $A \in L^X$ *is an interior operator on* L^X.

Proof. Clearly, (IO1) $int(\underline{1}) = \underline{1}$ and (IO2) For all $A \in L^X$, $int(A) \subseteq A$ satisfied trivially.

(IO3) Since L is completely distributive complete lattice, therefore by Proposition 2.2.2(5), L^X is also so. Hence, $int(A \bigcap B) = int(A) \bigcap int(B)$ follows immediately form (LC2).

(IO4) For any $A \in L^X$, let $x_\alpha \subseteq int(A)$, then there exists $\mathscr{A} \in \mathfrak{U}$ such that $st(x_\alpha, \mathscr{A}) \subseteq A$. Since $\mathfrak{U}$ is a covering L-local uniformity, therefore for $x_\alpha \in L^X$ and $\mathscr{A} \in \mathfrak{U}$, there exists $\mathscr{B} \in \mathfrak{U}$ such that $st(x_\alpha, st(\mathscr{B}) \subseteq st(x_\alpha, \mathscr{A})$. Now by Proposition

2.4.2(5), we have

$$\begin{aligned} st(st(x_\alpha, \mathscr{B}), \mathscr{B}) &\subseteq st(x_\alpha, st(\mathscr{B})) \\ &\subseteq st(x_\alpha, \mathscr{A}) \subseteq A \\ \therefore\ st(x_\alpha, \mathscr{B}) &\subseteq int(A) \\ \text{So} \quad x_\alpha &\subseteq int(int(A)) \end{aligned}$$

Thus by (IO2), $int(A) = int(int(A))$.

□

Now by Theorem 2.2.21 in [48], it we can conclude the following theorem

Theorem 5.2.3. *Every covering $L-$locally uniformity $\mathfrak{U}$ on L^X, generates an $L-$topology on L^X.*

In that case, the symbol $\mathbb{F}(\mathfrak{U})$ to denote the respective generated $L-$topology on L^X.

Subsequently, $int(A)$ is the interior of A in $(L^X, \mathbb{F}(\mathfrak{U}))$.

Theorem 5.2.4. *Let $\mathfrak{U}_1$ and $\mathfrak{U}_2$ be two covering $L-$ local uniformities on L^X such that $\mathfrak{U}_2$ is finer than $\mathfrak{U}_1$. Then $\mathbb{F}(\mathfrak{U}_2)$ will finer than $\mathbb{F}(\mathfrak{U}_1)$.*

Proof. Straight forward. □

Lemma 5.2.5. *Let $(L^X, \mathfrak{U})$ be covering $L-$locally uniform space. Then the mapping, $cl : L^X \to L^X$ defined by*

$$cl(A) = \bigcap \{st(A, \mathscr{A}) \mid \mathscr{A} \in \mathfrak{U}\}$$

for all $A \in L^X$ is a closure operator on L^X.

Proof. Clearly, (CO1) $cl(\underline{0}) = \underline{0}$ and (CO2) For all $A \in L^X$, $A \subseteq cl(A)$ satisfied trivially.

(CO3) For any A, $B \in L^X$, we have

$$cl(A \bigcup B) = \bigcap_{\mathscr{A} \in \mathfrak{U}} st(A \bigcup B, \mathscr{A})$$

$$\Rightarrow cl(A \bigcup B) = \bigcap_{\mathscr{A} \in \mathfrak{U}} [st(A, \mathscr{A}) \bigcup st(B, \mathscr{A})] \text{ [By Proposition 2.4.2 (4)]}$$

Also since L is completely distributive complete lattice, therefore by Proposition 2.2.2(5)

$$\Rightarrow cl(A \bigcup B) = [\bigcap_{\mathscr{A} \in \mathfrak{U}} st(A, \mathscr{A})] \bigcup [\bigcap_{\mathscr{A} \in \mathfrak{U}} st(B, \mathscr{A})]$$

$$= cl(A) \bigcup cl(B)$$

(CO4) For any $A \in L^X$, we have

$$cl(cl(A)) = \bigcap_{\mathscr{A} \in \mathfrak{U}} st(cl(A), \mathscr{A})$$

$$= \bigcap_{\mathscr{A} \in \mathfrak{U}} st(\bigcap_{\mathscr{A} \in \mathfrak{U}} st(A, \mathscr{A}), \mathscr{A})$$

$$= \bigcap_{\mathscr{A} \in \mathfrak{U}} st(st(A, \mathscr{A}), \mathscr{A})$$

$$\subseteq \bigcap_{\mathscr{A} \in \mathfrak{U}} st(A, st(\mathscr{A})) \text{ (By Proposition 2.2.2(5))}$$

$$= \bigcap_{\mathscr{A} \in \mathfrak{U}} st(\bigcup_{x_\alpha \in A} x_\alpha, st(\mathscr{A})) \text{ Since for any } (A \in L^X, \ A = \bigcup_{x_\alpha \in A} x_\alpha)$$

$$= \bigcap_{\mathscr{A} \in \mathfrak{U}} \bigcup_{x_\alpha \in A} st(x_\alpha, st(\mathscr{A})) \text{ (By Proposition 2.2.2(4))}$$

$$= \bigcup_{x_\alpha \in A} \bigcap_{\mathscr{A} \in \mathfrak{U}} st(x_\alpha,\, st(\mathscr{A}))$$

$$\subseteq \bigcup_{x_\alpha \in A} \bigcap_{\mathscr{A} \in \mathfrak{U}} st(x_\alpha,\, \mathscr{A}) \; (By\ LC)$$

$$= \bigcap_{\mathscr{A} \in \mathfrak{U}} \bigcup_{x_\alpha \in A} st(x_\alpha,\, \mathscr{A})$$

$$= \bigcap_{\mathscr{A} \in \mathfrak{U}} st(\bigcup_{x_\alpha \in A} x_\alpha,\, \mathscr{A})$$

$$= \bigcap_{\mathscr{A} \in \mathfrak{U}} st(A,\, \mathscr{A})$$

$$= cl(A)$$

Hence, by (CO2), we have $cl(cl(A)) = cl(A), \quad$ for all $A \in L^X$.

□

Lemma 5.2.6. *For every $L-$covers $\mathscr{A}$ and for each $A \in L^X$, we have*

$$st(A,\, \mathscr{A}) = \bigcap \{B \mid st(B', \mathscr{A}) \subseteq A'\}$$

Proof. It follows from the fact that for any $B \in L^X$, $B \subseteq st(A,\, \mathscr{A})$ if and only if $A \subseteq st(B,\, \mathscr{A})$ as $A \bigcap B \neq \underline{0}$ if and only if $B \bigcap A \neq \underline{0}$. □

Lemma 5.2.7. *Let $(L^X, \mathfrak{U})$ is covering $L-$locally uniform space, then $(cl(A))' = int(A')$.*

Proof. For any $A \in L^X$, we have

$$
\begin{aligned}
int(A') &= \bigcup\{x_\alpha \in L^X \mid st(x_\alpha, \mathscr{A}) \subseteq A' \text{ for some } \mathscr{A} \in \mathfrak{U}\}.\\
&= \bigcup\{\bigcup\{x_\alpha \in L^X \mid st(x_\alpha, \mathscr{A}) \subseteq A'\}, \mathscr{A} \in \mathfrak{U}\}.\\
&= \bigcup\{st(A,\, \mathscr{A})' \mid \mathscr{A} \in \mathfrak{U}\} \quad \text{[By Lemma 5.2.6]}.\\
\text{Hence, } int(A')' &= \bigcap\{st(A,\, \mathscr{A}) \mid \mathscr{A} \in \mathfrak{U}\}.\\
&= cl(A).
\end{aligned}
$$

□

Now by Lemma 5.2.5 and Lemma 5.2.7, it has the following theorem:

Theorem 5.2.8. *Let $(L^X, \mathfrak{U})$ be covering $L-$locally uniform space. Then for any $A \in L^X$, $cl(A) = \bigcap\{st(A, \mathscr{A}) \mid \mathscr{A} \in \mathfrak{U}\}$, is the closure of A in $(L^X,\, \mathbb{F}(\mathfrak{U}))$.*

Theorem 5.2.9. *Let $(L^X, \mathfrak{U})$ be covering $L-$locally uniform space. Then the topology $(X, \mathbb{F}(\mathfrak{U}))$ generated by covering $L-$locally uniform space is regular.*

Proof. Let $(X, \mathfrak{U})$ be a covering $L-$locally uniform space. Now for any $x_\alpha \in L^X$ and $\mathscr{A} \in \mathfrak{U}$ there exists $\mathscr{B}$ such that $st(x_\alpha, st(\mathscr{B})) \subseteq st(x_\alpha,\, \mathscr{A})$. Again for $x_\alpha \in L^X$ and $\mathscr{B} \in \mathfrak{U}$ there exists $\mathscr{C} \in \mathfrak{U}$ such that $st(x_\alpha, st(\mathscr{C})) \subseteq st(x_\alpha,\, \mathscr{B})$. Then by (CO3), we have $cl(st(x_\alpha,\, st(\mathscr{C}))) \subseteq cl(st(x_\alpha,\, \mathscr{B}))$. But by definition of cl, we have $cl(st(x_\alpha,\, \mathscr{B})) \subseteq st(st(x_\alpha,\, \mathscr{B}),\, \mathscr{B})$. We have $cl(st(x_\alpha,\, st(\mathscr{C}))) \subseteq st(st(x_\alpha,\, st(\mathscr{B}))$ [So, by the Proposition 2.2.2(5)]. This implies $cl(st(x_\alpha,\, st(\mathscr{C}))) \subseteq st(x_\alpha,\, \mathscr{A})$, as $st(x_\alpha, st(\mathscr{B})) \subseteq st(x_\alpha,\, \mathscr{A})$. Hence for each $x_\alpha \in L^X$ there exists a neighbourhood base at x_α consisting of closed sets and consequently the space is regular. □

Theorem 5.2.10. *Any regular $L-$topology is generated by a covering $L-$locally uniform space.*

Proof. Let $(L^X, \mathbb{F})$ be a regular $L-$topology and $\mathfrak{U}$ be the collection of all open covers in L^X, then it follows easily (LC1) and (LC2). The only thing left is for every $\mathscr{A} \in \mathfrak{U}$ there exists $\mathscr{B} \in \mathfrak{U}$ such that $st(x_\alpha, st(\mathscr{B})) \subseteq st(x_\alpha, \mathscr{A})$. Now by regularity of L^X, we have for $x_\alpha \in L^X$ and $\mathscr{A} \in \mathfrak{U}$ there exists an $L-$fuzzy open set G such that $x_\alpha \subseteq G \subseteq cl(G) \subseteq st(x_\alpha, \mathscr{A})$. Again since G is open and $x_\alpha \subseteq G$, therefore there exists an open cover $\mathscr{B}$ such that, $st(x_\alpha, \mathscr{B}) \subseteq G$. But then $st(x_\alpha, st(\mathscr{B})) \subseteq st(G, \mathscr{B})$ (as $x_\alpha \subseteq G$). Thus $st(x_\alpha, st(\mathscr{B})) \subseteq st(G, \mathscr{B}) = G$ (as G is open) $\subseteq cl(G) \subseteq st(x_\alpha, \mathscr{A})$. Also, by the construction of $\mathfrak{U}$, it follows from Lemma 5.2.2 that the $L-$topologies $\mathbb{F}(\mathfrak{U})$ and $\mathbb{F}$ are identical. Hence the result. □

By Theorem 2.2.13, It has the following Corollary.

Corollary 5.2.11. *Let $\{(L^{X_t}, \mathbb{F}_t) \mid t \in \Lambda\}$ be a family of L-topological spaces. Then the product topology of L-topologies $\{\mathbb{F}_t \mid t \in \Lambda\}$ on L^X is generated by a covering L-local uniformity if and only if for each $t \in \Lambda$, $(L^{X_t}, \mathbb{F}_t)$ is generated by a covering L-local uniformity.*

5.3 Weakly Uniformly Continuous Functions

This section establishes that every weakly uniform continuous function on covering $L-$locally uniform spaces is continuous for the induced $L-$topologies. Towards the end of this section, we have shown that the products of $L-$ regular topologies are generated by the product covering $L-$locally uniform spaces.

Definition 5.3.1. Let $(L^X, \mathfrak{U}_1)$ and $(L^Y, \mathfrak{U}_2)$ be two covering $L-$locally uniform spaces. Then a function $f^{\rightarrow} : (L^X, \mathfrak{U}_1) \rightarrow (L^Y, \mathfrak{U}_2)$, is called weakly uniformly continuous if and only if $f^{-1}(\mathscr{C}) \in \mathfrak{U}_1$, whenever $\mathscr{C} \in \mathfrak{U}_2$, where $f^{-1}(\mathscr{C}) = \{f^{\leftarrow}(C) : C \in \mathscr{C}\}$.

Theorem 5.3.1. *Every weakly uniform continuous function is continuous.*

Proof. Let $f^{\rightarrow} : (L^X, \mathfrak{U}_1) \rightarrow (L^Y, \mathfrak{U}_2)$ be a weakly uniformly continuous functions and $A \in L^Y$ be any member.

Then by definition of int, we have $int(A) = \bigcup\{x_\alpha : st(x_\alpha, \mathscr{A}) \subseteq A, \text{for some}\, \mathscr{A} \in \mathfrak{U}_2\}$.

This implies

$$f^{\leftarrow}(int(A)) = \bigcup\{f^{\leftarrow}(x_\alpha) : st(x_\alpha, \mathscr{A}) \subseteq A\, \text{for some}\, \mathscr{A} \in \mathfrak{U}_2\}.$$

$$\text{[Since by Theorem 2.1.17 (i) in [48]}, f^{\leftarrow}\text{is arbitrary join preserving].} \quad (5.3.1)$$

Since $f^{\leftarrow}$ is order preserving, therefore

$$st(x_\alpha, \mathscr{A}) \subseteq A \text{ implies } f^{\leftarrow}(st(x_\alpha, \mathscr{A})) \subseteq f^{\leftarrow}(A) \quad (5.3.2)$$

Then by Proposition 2.2.2(5) and Line (5.3.2) we have

$st(f^{\leftarrow}(x_\alpha, f^{-1}(\mathscr{A})) \subseteq f^{\leftarrow}(st(x_\alpha, \mathscr{A})) \subseteq f^{\leftarrow}(A)$.

Now from line (5.3.1), we have

$$f^{\leftarrow}(int(A)) \subseteq \bigcup\{f^{\leftarrow}(x_\alpha) : st(f^{\leftarrow}(x_\alpha), f^{-1}(\mathscr{A})) \subseteq f^{\leftarrow}(A)\, \text{for some}\, \mathscr{A} \in \mathfrak{U}_2\}. \quad (5.3.3)$$

But since $f^{\rightarrow}$ is weakly uniformly continuous, therefore $\mathscr{A} \in \mathfrak{U}_2$ implies $f^{-1}(\mathscr{A}) \in \mathfrak{U}_1$.

So by Line (5.3.3), we have $f^{\leftarrow}(int(A)) \subseteq int(f^{\leftarrow}(int(A)))$.

This implies $f^{\leftarrow}(int(A)) \in \mathbb{F}(\mathfrak{U}_1)$.

Hence $f^{\rightarrow} : (L^X, \mathbb{F}(\mathfrak{U}_1)) \rightarrow (L^Y, \mathbb{F}(\mathfrak{U}_2))$ is continuous. □

Theorem 5.3.2. *The composition of weakly uniformly continuous function is weakly uniformly continuous.*

Proof. Let $f^{\rightarrow} : (L^X, \mathfrak{U}_1) \rightarrow (L^Y, \mathfrak{U}_2)$ and $g^{\rightarrow} : (L^Y, \mathfrak{U}_2) \rightarrow (L^Z, \mathfrak{U}_3)$ be two weakly uniformly continuous functions. Let $\mathscr{C} \in \mathfrak{U}_3$ be any member. Then by Theorem

2.1.23(ii) in [48], we have $(g \circ f)^{\leftarrow}(\mathscr{C}) = f^{\leftarrow}(g^{\leftarrow}(\mathscr{C}))$. Since $g^{\rightarrow}$ is weakly uniformly continuous, therefore $\mathscr{C} \in \mathfrak{U}_3$ implies $g^{\leftarrow}(\mathscr{C}) \in \mathfrak{U}_2$. This further implies $f^{\leftarrow}(g^{\leftarrow}(\mathscr{C})) \in \mathfrak{U}_1$ as $f^{\rightarrow}$ is weakly uniformly continuous. Hence $(g \circ f)^{\leftarrow}(\mathscr{C}) \in \mathfrak{U}_1$ for every $\mathscr{C} \in \mathfrak{U}_3$. Hence the result.

□

Definition 5.3.2. Let $\{(L^{X_t}, \mathscr{U}_t) \mid t \in \Lambda\}$ be a family of covering L-locally uniform spaces, where Λ is the index set.

Let $X = \Pi_{t\in\Lambda} X_t$.

The product covering L-local uniformity on L^X is defined as the coarsest covering L-local uniformity such that for every $t \in \Lambda$, projection $\pi_t^{\rightarrow} : L^X \rightarrow L^{X_t}$ is weakly uniformly continuous.

By Theorem 5.2.4, the following Theorem is now obvious.

Theorem 5.3.3. *The L-topology generated by the product covering L-local uniformity is the product topology and conversely product of regular $L-$topologies is generated by product covering L-local uniformity.*

5.4 Metrization for Covering L-locally Uniform space

In this section, the problem of metrization of covering $L-$locally uniform spaces were considered and successfully obtained significant result on metrization.

Lemma 5.4.1. *If $(L^X, \mathfrak{U})$ is covering L-locally uniform space with countable base, then $(X, \mathbb{F}(\mathfrak{U}))$ has countable base.*

Proof. Let $\mathfrak{U}^* = \{\mathscr{A}_n : n \in N\}$ be countable base for the covering $L-$locally uniform space.

For fixed n, let us define,

$$B_n = st(st(x_\alpha, \mathscr{A}_n), \mathscr{A}_m), \text{ for some } m \in N, x_\alpha \in L^X$$

By Lemma (5.2.2) it is clear that $int(B_n) = B_n$.

Let us denote the collection $\mathfrak{B} = \{B_n : B_n = st(st(x_\alpha, \mathscr{A}_n), \mathscr{A}_m), m \in N\}$. Let $x_\alpha \in B \subseteq L^X$ be any open set. Then $int(B) = B$, and since $\mathscr{U}^*$ is base for $\mathscr{U}$. By Covering $L-$locally uniform space, for $\mathscr{A}_j$ there exists $\mathscr{A}_k$ such that $st(x_\alpha, st(\mathscr{A}_k)) \subseteq st(x_\alpha, \mathscr{A}_j)$. Again by Proposition 2.4.2(5)

$$\begin{aligned}
& st(st(x_\alpha, \mathscr{A}_k), \mathscr{A}_k) \subseteq st(x_\alpha, st(\mathscr{A}_k)) \\
\Rightarrow\ & st(st(x_\alpha, \mathscr{A}_k), \mathscr{A}_k) \subseteq st(x_\alpha, \mathscr{A}_j) \\
\Rightarrow\ & st(st(x_\alpha, \mathscr{A}_k), \mathscr{A}_k) \subseteq B \\
\Rightarrow\ & x_\alpha \in st(st(x_\alpha, \mathscr{A}_k), \mathscr{A}_k) \subseteq B \\
\Rightarrow\ & x_\alpha \in B_k \subseteq B.
\end{aligned}$$

On the other hand, each B_n is assign to some member of $\mathfrak{U}^*$, $\mathfrak{U}^*$ is countable implies $\mathfrak{B}$ is countable. □

Now by Theorem 2.5.3, the following result holds.

Theorem 5.4.2. *Every covering $L-$locally uniform with countable base is pointwise pseudo-metrizable.*

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