Chapter 1

General Introduction

1.1 Introduction

Introduction is the part where the whole thesis begins. All the important definitions and terminology are mentioned in the introduction part.

We live in this universe and living on it is a great opportunity for the cosmologists to study about the universe. By forming different models using Einstein field equation and its modified theories, the cosmologists can easily known about our universe. By this inspiration, my thesis entitled "SOME PROBLEMS OF PHYSICAL DISTRIBUTION IN RELATIVISTIC COSMOLOGY" is taken up. This thesis comprises of eight chapters where first chapter and last chapter deals with general introduction and conclusion including future aspects. The other chapters *viz.,* Chapter 2 to Chapter 7 deals with some spatially homogeneous isotropic cosmological model of the universe in some modified theories of gravitation.

1.2 Einstein's theory of gravitation

Einstein theory of gravitation is the most important theory in the context of modern cosmology. First he developed special theory of relativity which relate to inertial frame of reference. Due to the limitation that it cannot be application to all kinds of motion, he developed general relativity, which is more popular and widely applicable to all types of motion. Basically Einstein was guided by the following three principles:

(i) Principle of covariance

(ii) Principle of equivalence

(iii) Mach's Principle.

1.2.1 Principle of covariance

This principle allows us to write the physical laws in covariant form so that their structure stays the same in all coordinate systems.In tensor form all the physical law should be expressed and so the spacetime metric can be expressed as

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2, \tag{1.2.1}$$

which is not invariant under general coordinate transformation, is replaced as

$$ds^2 = g_{ij}dx^idx^j; \;\; i,j = 1,2,3,4, \tag{1.2.2}$$

which is reasonable in any coordinate system. The transformation law satisfying by g_{ij} which is a symmetric tensor of rank two is given by

$$g^{'ij} = \frac{\partial x^\mu}{\partial x^{'i}}\frac{\partial x^\nu}{\partial x^{'j}}g_{\mu\nu} \tag{1.2.3}$$

where dashed ($'$) quantities belong to the new coordinate system $x^{'i}$.

1.2.2 Principle of equivalence

This principle brings in gravitational considerations into the development general relativity. It says that no physical experiment can distinguish whether the acceleration of a free particle is due to a gravitational field or due to the acceleration of a frame of reference. Thus, this leads to an intimate relationship between metric and gravitation.

1.2.3 Mach's Principle

This principle can be used to determine the geometry of the spacetime and there by the inertial properties of a test particle from the information of the density and mass energy distribution in its neighborhood. According to this principle:

(i) When masses are piled up in its neighbourhood, the inertia of a body must increase.

(ii) When neighbouring masses are accelerated, the body must also experience an accelerated force.

(iii) A "Coriolis field" deflects the moving bodies in the sense of rotation and a radial centrifugal.

The gravitational phenomena has been successfully describe by Einstein's general theory of relativity. Also the spacetime is given as

$$ds^2 = g_{ij}dx^i dx^j, ; \quad i,j = 1,2,3 \text{ and } 4 \tag{1.2.4}$$

and the components of the symmetric tensor g_{ij} act as gravitational potentials. The gravitational field manifests through the curvature of the spacetime and the general field equation's which govern the gravitational field are given by

$$G_{ij} \equiv R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = -8\pi T_{ij}, \tag{1.2.5}$$

where G_{ij} is the Einstein tensor, R_{ij} is the Ricci tensor, R is the scalar curvature and T_{ij} is the energy-momentum tensor due to matter and Λ is the cosmological constant . This cosmological constant was introduced by Einstein, while studying static cosmological model and was later discarded by him saying *"It is the greatest* *blunder of my life"*. In this connection, it may be mentioned that, in recent years, the cosmological constant is coming into lime light and attracting many researchers in general relativity but comes as a variable and not as a constant.

1.3 Relativistic cosmology

Relativistic cosmology is the description of the expanding universe based on general relativity. Cosmology has been transformed by dramatic progress in high-precision observations and theoretical modeling. The physical universe is the maximal set of physical objects which are locally casually connected to each other and to the region of space time that is accessible to us by astronomical observation. The scientific theory of cosmology is concerned with the study of the large-scale structure of the observable region of the universe and its relation to local physics on the one hand and to the rest of the universe on the other.

The recent astronomical data indicate that observable universe is currently accelerating. This observation, in turn, indicates that the universe has a positive cosmological constant. As a result it is likely that universe evolves into the future (asymptotically) de Sitter phase.

1.4 Einstein's field equations

The fact that the gravitational fields equations must locally reduce to those of Newtonian gravity considered by Einstein's, where the metric tensor components are related to the gravitational potential in the weak field limit and the field equations must reduce to Poisson equations.From the later, he imposed that the curvature side of the equations must contain only up to second order derivatives of the metric and must also be of the same tensor rank as the energy momentum tensor.Therefore, Einstein was considered the Ricci tensor, derived from contracting twice the Riemann curvature tensor, but there was a little bit more into it. He knew that the equations must satisfy conservation laws and thus be divergence free. The vanishing of the divergence of matter energy source side of the equations, on the curvature side, the Ricci tensor is not divergence free. For that, Einstein built precisely the tensor that holds his name which is divergence free and hence complies with conservation laws. Einstein (1915a) has proposed his gravitational field equations by taking into account the requirements coordinate invariance, conservation laws, and limits that must be consistent with Newtonian gravity. This field equations describe how space time reacts to the presence of mass energy. Hence , the Einstein's field equations becomes

$$G_{ij} \equiv R_{ij} - \frac{1}{2} R g_{ij} = -K T_{ij}, \tag{1.4.1}$$

where G_{ij} is the Einstein tensor, R_{ij} is the Ricci Tensor, $R = g^{ij} R_{ij}$ is the Scalar curvature and T_{ij} is the energy momentum tensor due to matter and and $K = \frac{8\pi G}{c^4}$, c is the velocity of light in vacuum and G is the Newtonian Gravitational constant.

Einstein's explained his general theory of relativity in a series of lectures (Einstein, 1915b, 1916). This theory connects space and time with matter, energy and gravitation. The theory itself is couched in the language of differential geometry and was a pioneer for the use of applied mathematics in physical theories, leading the way for the gauge theories that have followed.Einstein's field equations reduce to ordinary differential equations, because the inhomogenous degrees of freedom have been frozen out. They are thus different special terms in geometrical; nevertheless, they form a rich set of models where one can study the exact dynamics of the full nonlinear field equations. The solutions to Einstein's field equation will depend on the matter in the space time. Einstein (1917) resulted in a non statical universe when it was supposed a normal content of matter for the universe.

The idea of a non statical universe seemed senseless to Einstein and inconvenient for him, as addressed to the astronomer Willem de Sitter when he deduced the equations of an empty Universe which could be expanding. This fact persuaded Einstein to modify his field equations by introducing a new term proportional to a constant Lambda, the so called cosmological constant, which was interpreted as the energy density of the vacuum. The new Einstein's field equations took the following form which is obtained by adding a constant term to the Einstein's equations

$$G_{ij} + \Lambda g_{ij} = -KT_{ij}, \tag{1.4.2}$$

here $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$ is an Einstein tensor which is divergence free and Λ is the cosmological constant. This was indeed done by Einstein in order to permit a static solution for the cosmology. As a modification on the left hand side of the equations, the cosmological constant Λ is presaging a class of modification of gravity.

1.5 Cosmology and cosmological models

The study of large scale structure is deal within the Cosmology. Stars, star clusters and galaxies or the nebulae, pulsars, quasars as well as cosmic rays and background radiation are within the universe. The basic problem in cosmology is the dynamics of the system. The fundamental force keeping solar systems, stars and galaxies together is the force of gravity. The other long range interactions such as electromagnetic forces may be disregarded because the galaxies, which are major constituents of the universe as well as the intergalactic medium, are known to be electrically neutral.

It is well known that Einstein's general theory of relativity is a satisfactory theory of gravitation, correctly predicting the motion of test particles and photons in curved spacetime; but in order to apply to the universe one has to introduce simplifying assumptions and approximations. The first approximation that is usually made is that of continuous matter distribution. We assume that the universe is filled with a simple macroscopic perfect fluid (devoid of shear-viscous, bulk-viscous and heat conductive properties). Its energy momentum tensor T_{ij} is, then given by

$$T_{ij} = (\rho + p)u_i u_j - pg_{ij}, \tag{1.5.1}$$

where ρ is its proper energy density p is the isotropic pressure and u_i is four-velocity of the fluid particles (stars etc.)

The study of the large scale structure of the physical universe is the main aim of cosmology. Cosmologists construct mathematical model of the universe and they compare these model with the present day universe as observed by astronomers. The theory of cosmological model began with Einstein's development of the static universe in 1917. In 1922, Hubble published his famous law relating to apparent luminosities of distant galaxies to their red shifts.

$$i.e., \;\; V = HD, \tag{1.5.2}$$

where V is the speed of recession of galaxy at a distance D from us and H is Hubble's constant. Because of this observed red shift of spectral lines from distant galaxies and static model of the universe were ruled out and non-static model gained importance.

Friedmann (1922) was the first to investigate the most general non-static, homogeneous and isotropic spacetime described by the Robertson-Walker metric

$$ds^2 = dt^2 - R^2(t)\left\{\frac{dr^2}{1-kr^2} - r^2 d\theta^2 - r^2 \sin^2\theta d\theta^2\right\}, \tag{1.5.3}$$

where $R(t)$ is the scale factor, k is a constant which is by a suitable choice of r can be chosen to have values +1, 0 or -1 according as the universe is closed, flat or open respectively. The evolution of the function $R(t)$ using Einstein field equations for all three curvatures is also discussed by him with great interest. The present day universe is both spatially homogeneous and isotropic and therefore can be well described by a Friedmann-Robertson-Walker (FRW) model (Patricge and Wilkinson, 1967; Ehlers et al., 1968) which was proved both experimentally and theoritically. However, there is evidence for a small amount of anisotropy (Boughn et al., 1981) and a small magnetic field ever cosmic distant scales (Sofue et al., 1979).

1.6 Cosmological principle

The cosmological principle states that on a sufficiently large scale structure of the distribution of galaxies, the universe is both homogeneous and isotropic.

(i) *The homogeneity of space:* It is the property of matter and energy which are distributed uniformly over the large distance scales.

(ii) *The isotropy of space:* It is the property of looking the same in very direction *i.e.,* there is no preferred direction in space.

There is quite good observational evidence that the universe does have these properties, although this evidence is not completely watertight. Isotropy does not necessarily imply homogeneity without the additional assumption that the observer is not in a special place the so-called Copernican Principle. One would observe isotropy in any spherically symmetric distribution of matter, but only if one were in the middle of the pattern observed isotropy, together with the Copernican Principle, implies the cosmological principle. The cosmological principle was introduced by Einstein. He was particularly motivated by ideas associated with Ernst Mach in advocating the principle. There are many approaches one can take to this principle. Bondi, Gold and Hoyle in the 1940s together with the work of Milne in the 1930s developed the philosophical approach regarding this principle. The problem then arises as to how one explains the observation that the universe appears homogeneous on scales much larger than the scale one expect to have been in causal contact up to the present time.

1.7 Cosmological constant

In physical cosmology especially Einstein general relativity, Albert Einstein introduces a term called the cosmological constant (usually denoted by the Greek capital letter Lambda: Λ) for achieving a stationary universe .But due to the result obtained by Hubble red shift that the universe might not be stationary, he left the concept of introducing his new term.

However, the discovery of cosmic acceleration in the 1990s has renewed interest in a cosmological constant. So it is an extra term in Einstein's equations of general relativity. The cosmological constant is thought to represent the energy density of empty space and is taken as a constant density even as space expands. It has the same effect as an intrinsic energy density of the vacuum. A positive vacuum energy density resulting from a cosmological constant implies a negative pressure, and vice versa. If the energy density is positive, the associated negative pressure will drive an accelerated expansion of empty space. In lieu of the cosmological constant, cosmologists often quote the ratio between the energy density due to the cosmological constant and the current critical density of the universe. This ratio is usually called Ω_Λ. In a flat universe Ω_Λ corresponds to the fraction of the energy density of the universe which is associated with the cosmological constant. For Ω_Λ less than 1, the universe has negatively curved or hyperbolic geometry i.e. the universe is open. For $\Omega_\Lambda = 1$, the universe has Euclidean or flat geometry. For Ω_Λ greater than 1, the universe has positively curved or spherical geometry *i.e.*, the universe is closed. The critical density changes with cosmological time, but the energy density due to the cosmological constant remains unchanged throughout the history of the universe. Deceleration parameter (observable) now depends on both matter content and Λ. By incorparating this cosmological constant to Einstein's field equation in order to get a static universe, Einstein felt that he commited a Himalayan Blunder by introducing this term as he predicted a non-static universe.Also he felt a bad conscience because of the contradictory observation made by Edwin Hubble thet the universe is eapnading which is consistent with Friedmann. . Since it no longer seemed to be needed, Einstein abandoned the cosmological constant and called it the "biggest blunder" of his life.

1.8 Hubble's law and Hubble's constant

Hubble's law states that the galaxies recede with a velocity proportional to their distance from the earth. Thus, Recessional velocity = Hubble's constant times distance.

$$i.e., \quad V = HD \tag{1.8.1}$$

where V is the observed velocity of the galaxy away from us, usually in km/sec. H is Hubble's constant in km/sec/MPc. D is the distance of the galaxies in MPc. We know that the Hubble parameter or Hubble constant H defines the rate of cosmic expansion. The recession velocity of V of an object situated at a distance D given by $H = V/D$. Also, it is the logarithmic derivative of the scale factor $R(t)$

$$H = \frac{\dot{R}(t)}{R(t)}. \tag{1.8.2}$$

Bret from the latest source the Hubble space telescope key project team came up with the answer.

$$\mathrm{H = 75 + / - 8kmS - 1MPc - 1.}$$

And finally, WMAP came up with

$$\mathrm{H = 71 + / - 3.5kmS - 1MPc - 1.}$$

where 1MPc = 3.26 million light years.

1.9 Equation of state parameter

In cosmology, the equation of state (EoS) parameter ω of a perfect fluid is equal to the ratio of its pressure p and its energy density ρ. Thus

$$\omega = \frac{p}{\rho}, \tag{1.9.1}$$

were ω may be a constant or a function of the cosmic time t. It plays a significant role to describe the evolution and the ultimate fate of the universe.

- If $\omega = 1$, then the universe is stiff fluid dominated
- If $\omega = \frac{1}{3}$, then the universe is radiation dominated
- If $\omega = 0$, then the universe is dust dominated
- If $\omega = -1$, then the universe is vacuum energy dominated (negative pressure)
- If $\omega = -\frac{1}{3}$, then the universe is dark energy dominated (accelerated expansion)
- If $-1 < \omega < -\frac{1}{3}$, then the universe is quintessence dominated
- If $\omega < -1$, then the universe is phantom energy dominated and it expands exponentially to reach Big rip.

1.10 Deceleration parameter

The deceleration parameter is a dimensionless measure of the acceleration of the expansion of the universe, which is denoted by q and is defined by

$$q = -\frac{R\ddot{R}}{\dot{R}^2}. \tag{1.10.1}$$

Here R denote the scale factor of the universe and the dots indicating the derivatives with respect to cosmic time t. The expansion of the universe is accelerating if $\ddot{R} > 0$ or $q < 0$ and decelerating if $\ddot{R} < 0$ or $q < 0$.
Again from the Hubble parameter $H = \frac{\dot{R}}{R}$, we have

$$\frac{d}{dt}\left(\frac{1}{H}\right) = -\frac{\dot{H}}{H^2} = \frac{\dot{R}^2 - R\ddot{R}}{\dot{R}^2} = 1 - \frac{R\ddot{R}}{\dot{R}^2}. \tag{1.10.2}$$

Therefore q can be expressed in terms of H as

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 = -\frac{\dot{H}}{H^2} - 1. \tag{1.10.3}$$

1.11 Expansion scalar

The expansion scalar is denoted by θ and is defined by

$$\theta = 3H = 3\frac{\dot{R}}{R}. \tag{1.11.1}$$

This measures the relative rate of expansion or contraction of the universe.

1.12 Anisotropy parameter

The anisotropy parameter is denoted by A_m and is defined by

$$A_m = \frac{1}{3}\sum_{i=1}^{3}\frac{(H_i - H)^2}{H}, \tag{1.12.1}$$

where $H_i \quad (i = 1, 2, 3)$ are directional Hubble parameters. If A_m, then the universe becomes isotropic.

1.13 Shear scalar

The shear scalar is denoted by σ and is defined by

$$\sigma^2 = \frac{1}{2}\left[\sum_{i=i}^{3} H_i^2 - \frac{1}{3}\theta^2\right] = \frac{3}{2}A_m H^2 = \frac{1}{6}A_m\theta^2. \tag{1.13.1}$$

If $\sigma = 0$, then the universe becomes isotropic.

1.14 Perfect fluid and energy-momentum tensor

A perfect fluid is a friction-less homogeneous and in-compressible fluid which is incapable of sustaining any tangential stress or action in the form of a shear but the normal force acts between the adjoining layers of fluid. The energy-momentum tensor describing matter is given by

$$T_{ij} = \rho u_i u_j + S_{ij}, \tag{1.14.1}$$

where ρ is the mass density, u^i is the four-velocity $u^i = \frac{dx^i}{ds}$ of the individual particles, and S_{ij} is the stress tensor where the speed of the light $c = 1$. If the matter consists of perfect fluid, namely, one whose pressure is isotropic the stress tensor can be expressed as

$$S^{ij} = p(u^i u^j - g^{ij}), \tag{1.14.2}$$

where p is the pressure. Thus the energy-momentum tensor becomes

$$T^{ij} = (\rho + p)u^i u^j - pg^{ij}, \tag{1.14.3}$$

Thus for pressure, its equation of state is

$$p = \frac{1}{3}\rho. \tag{1.14.4}$$

1.15 Bulk viscosity

Since bulk viscosity leads to the accelerated expansion phase in the early Universe, there has been a considerable interest in cosmological models with bulk viscosity.

The possibility of bulk viscosity leading to inflationary like solution in general relativistic FRW models has been discussed by several researchers like Barrow (1986); Lima et al. (1993). It contributes a negative pressure term giving rise to an effective total negative pressure term giving rise to an effective total negative leading to an repulsive gravity. This overcomes the attractive gravity and gives an impetus for rapid expansion of the Universe. Barrow (1986), Padmanabhan and Chitre (1987), Pavon et al. (1991), Lima et al. (1993) are some of the authors who have investigated the possibility of bulk viscosity leading to inflationary like solution in general relativistic FRW models. Shri Ram and Singh (1997) have discussed early cosmological models with bulk viscosity in Brans-Dicke theory. Mahanthy and Pattanaik (1991); Bali and Pradhan (2007); Tripathy et al. (2009); Katore et al. (2010) and Saadat and Pourhassan (2013) are some of the authors who have studied cosmological models with bulk viscosity in different context. Rao et al. (2011) have obtained anisotropic universe with cosmic strings and bulk viscosity in a scalar tensor theory of gravitation. Samatha et al. (2013) have discussed five-dimensional bulk viscous cosmological model with wet dark fluid in general relativity. Katore and Shaik (2014) have studied various string cosmological models in the presence of bulk viscosity. Khadekhar et al. (2015) have discussed FRW viscous cosmology with inhomogenous equation of state and future singularity. Rao et al. (2015a, 2015b) have analysed FRW bulk viscous cosmological model in some scalar tensor theories of gravitation. Santhi et al. (2017) have investigated bulk viscous string cosmological models in modified theory of gravity. Santhi et al. (2018) have discussed bulk viscous string cosmological models in the framework of modified theory of gravity. Mishra et al. (2019) have studied bulk viscous embedded hybrid dark energy models. Yadav et al. (2020) have investigated existence of bulk viscous Universe in modified theory of gravity and confrontation with observational data. They have estimated thepresent values of Hubble and deceleration parameters with observational Hubble data and SN la data sets. Sanjay et al. (2021) have discussed stability analysis of viscous fluid models in modified theory of gravity.

1.16 Strings and string cosmology

The concept of string theory was developed to describe events of the early stages of the evolution of the universe. It can give rise to various forms of topological defects as the symmetry of the universe is broken during phase transition. A defect is a discontinuity in the vacuum (Kibble, 1976; Pando et al., 1998) have proposed that the topological defects are responsible for structure formation of the universe. Among the above topological defects strings have important astrophysical consequences, namely, double quasar problem and galaxy formation can well, be explained by strings (Vilenkin and Shellard, 1994). Vilenkin (1985) has shown that the strings can act as a gravitational lense and hence astronomical observations may detect these objects. String theory is also considered the promising candidate for unification of all forces. Schwarz (2001) presented a brief chronology of some of the major developments that has taken place in string theory. The general relativistic treatment of strings was initiated by Stachel (1980) and Letelier (1983). According to Letelier (1983), the massive strings are nothing but geometric strings (massless) with particles attached along its extension. So, the total energy-momentum tensor for a cloud of massive strings can be written as a detailed derivation of energy-momentum tensor for a cloud of strings one can refer (Letelier, 1979,1983; Stachel, 1980)

$$T_{ij} = \rho u_i u_j, \tag{1.16.1}$$

where ρ is the rest energy density for a cloud of strings with particles attached to them. So, we can write

$$\rho = \rho_p + \lambda, \tag{1.16.2}$$

ρ_p being the particle energy density and λ being the tension density of the string. The four-velocity u^i for the cloud of particles and the four-vector x^i the direction of string, satisfy

$$u_i u_j = 1 = -x_i x^j \quad \text{and} \quad u_i x^i = 0. \tag{1.16.3}$$

Recently, there has been a lot of interest in cosmic strings and string cosmological models. The gravitational effects of cosmic strings have been extensively discussed by Vilenkin (1981), Gott (1985), Letelier (1983), Stachel (1980) in general relativity. Relativistic string models in the context Bianchi spacetime have been obtained by Krori et al. (1990), Banerjee et al. (1990), and Tikekar and Patel (1990) while Tikekar et al. (1994) have presented a class of cylindrically symmetric models in string cosmology. The string cosmological models with magnetic field are investigated by Chakraborty (1991) and Tikekar and Patel (1992). Yavus and Tarhan (1996), Baysal

(2001) have also investigated Bianchi type string cosmological models which are exact solutions of Einstein's field equations, the curvature source being a cloud of strings which are one dimensional objects. They have also discussed the effect of cosmic strings on the cosmic microwave background (CMB) anisotropies. Recently, Rajbali et al. (2005) has investigated Bianchi type-I string dust magnetized cosmological model in general relativity. There has been a considerable progress in our understanding of string theory over the past few decades. The recent attempts to bring string theory closer to cosmology which have been reviewed here, are only a preview of things to come. As string theory develops further, a meaningful dialogue with cosmology will ensure leading to much more excitement in this field.

1.17 Higher dimensional cosmology

In the general relativistic physics, our present universe seems to be four dimensional of which three are used to denote usual spatial dimensions and the fourth dimension represents time. But many researchers established their theories about the universe in higher dimensional spacetime mainly due to the significant achievement in solving long-standing problems relating to the stability of the results in general relativity and quantum mechanics. Before Einstein, two mathematicians namely, Herman Weyl (1918) and Theodor Kaluza (1921) attempted to unify gravity with the electromagnetic force. In the standard four dimensional spacetimes, the first unified theory was suggested by Herman Weyl on the basis of generalizing the Riemannian geometry. But in the five-dimensional spacetimes, a unified theory of gravitation and electromagnetic force was established the first time by the mathematician Kaluza. Also in the year 1926, Oskar Klein, Swedish physicist, suggested the unification law of the gravitational force and the electromagnetic force by using the fifth dimension. This theory is known as Kaluza-Klein theory. Later on, it was established that their approaches where to some extent erroneous, but this theory provides a basis to the researchers for further investigation over the last few decades. Einstein (1927), later on, showed that in general relativity, the

Kaluza's idea gives a rational foundation for Maxwell's electromagnetic equations and combines them with gravitational equations to a formal whole.

1.18 Robertson-Walker metric

Around our galaxy, the extra-galactic nebular clusters in space is isotropic and hence we assume the matter is distributed homogeneously. Einstein took the time coordinate t as in (Narlikar, An Introduction to Cosmology) as

$$ds^2 = c^2dt^2 - g^{ij}dx^i dx^j, \tag{1.18.1}$$

where g_{ij} are functions of space coordinates $x^i(i, j = 1, 2, 3)$. Also, in cartesian coordinates , we have x_1, x_2, x_3, x_4 by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2.$$

Therefore the spatial line element on the surface is given by

$$\begin{aligned} d\sigma^2 &= (dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2 \\ &= R^2[d\chi^2 + \sin^2\chi(d\sin^2\theta d\phi^2)], \end{aligned}$$

where $x_1 = R\sin\chi\cos\theta, x_2 = R\sin\chi\cos\theta\cos\phi, x_3 = R\sin\chi\cos\theta\sin\theta, x_4 = Ra\cos\chi$ and the ranges of θ, ϕ and χ are given by $0 \le \chi \le \pi, 0 \le \theta \le \pi, 0 \le \phi \le 2\pi$.
Another way to express $d\sigma^2$ through coordinates r, θ, ϕ with $r = \sin\chi, (0 \le r \le 1)$ is

$$d\sigma^2 = R^2\left[\frac{dr^2}{1 - r^2} + r^2(d\theta + \sin^2\theta d\phi^2)\right]. \tag{1.18.2}$$

The line element for the Einstein universe is therefore given by

$$ds^2 = c^2dt^2 - d\sigma^2 = c^2dt^2 - R^2\left[\frac{dr^2}{1 - r^2} + r^2(d\theta + \sin^2\theta d\phi^2)\right]. \tag{1.18.3}$$

This line element is for positive curvature only.

In general we have,

$$ds^2 = c^2dt^2 - R^2\left[\frac{dr^2}{1-kr^2} + r^2(d\theta + \sin^2\theta d\phi^2)\right], \tag{1.18.4}$$

where, $k = 0, +1, -1$ for zero, positive, negative curvatures respectively and are also known as flat, closed, open models and $R(t)$ is known as the scale factor or expansion factor. Thus for $c = 1$, FRW line element reduces to

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta + \sin^2\theta d\phi^2)\right]. \tag{1.18.5}$$

Thus FLRW five-dimensional line element is

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] - S^2(t)d\psi^2, \tag{1.18.6}$$

where $R(t)$ is the scale factor of the universe, $k = 1, 0, -1$ for space of positive, vanishing and negative curvature representing closed, flat and open models of the universe respectively. The fifth co-ordinate ψ is also assumed to be space like coordinate.

1.19 Literature review of relevant works on Friedmann-Lemaitre-Robertson-Walker metric

Lemaitre (1931) has independently derived similar results as Friedmann.

Robertson (1935) and Walker (1937) independently have shown the non-static cosmological metric known as the Robertson-Walker metric.

Bondi and Gold (1948) studied the steady state model of the universe. They have shown that the separation of the matter increases exponentially and the continuous matter creation is necessary to clarify the density of matter.

Buchdahl (1972) considered the time-like geodesics of the Robertson-Walker spaces from the Lagrangian point of view. He obtained the characteristic function V of an arbitrary Robertson-Walker space from the Hamiltonian point of view by integrating the differential equations that govern V.

Heller and Suszycki (1974) have studied dust model of FRW universe with bulk viscosity and shown that initial singularity may be avoided for suitable conditions.

Barrow and Matzner (1977) have studied the homogeneity and isotropy of the universe and found that the chaotic cosmology is essentially ruled out.

Starobinsky (1980) recognized a new kind of isotropic cosmological models with no singularity.

Berman (1983) provide a constant value for deceleration parameter by considering a variation law for Hubble's parameter in an evolutionary model. Beesham (1986) have considered a FRW universe with a time-varying gravitational constant G as variable and has shown that there is no creation even though the rest mass of matter particles is constant in a universe with variable G and Λ.

Ozer and Taha (1987) have introduced a cosmological model with an additional term $\Lambda(x)$, $g_{\mu\nu}$ in the energy-momentum tensor and found that the whole universe is causally connected slightly after Planck time. There exists a period of phase transition during part of which pressure is negative. This model indicated an energy flow from the curvature to the matter such that the entropy of the matter is not conserved.

Johri and Sudharsan (1988) conclude that the presence of little time independent bulk viscosity play an important role steady state universe through the effect of bulk viscosity on the evolution of the Friedmann models.

Tarachand and Ibotombi (1989) carried out a study of imperfect fluid interacting with the gravitational field for spherically symmetric Robertson-Walker metric and found that the Big Bang does not take place when the viscous fluid interact with the gravitational field at the early stages.

Abdel-Rahman (1990) considered a cosmological model wherein the universe has its critical density and the gravitational G and cosmological constant Λ are time-dependent. The horizon and monopole problems may be solved by this model.

Moreover, it predicted an expanding universe wherein G increases and Λ decreases with time in a way dependable with the conservation of the energy-momentum tensor.

Sistero (1991) studied about the cosmology with the gravitational and Gravitational constants generalized as coupling scalars in Einstein's theory. He found exact solutions for zero pressure models satisfying certain condition.

Beesham (1991) confirmed that the scale covariant theory of gravity admits the possibility of a time varying gravitational constant with a gauge function wherein there is no independent equation. He also investigated the situation of obtaining the explicit forms of the gauge function in the Friedmann-Robertson-Walker cosmological models.

Ibotombi and Biren (1992) found an exact solution of the Einstein's field equations for a conformally invariant scalar field with the trace free energy-momentum tensor for the Robertson-Walker models with $k = +1, -1$ and discussed the physical properties of the solution.

Beesham (1993) derived exponenetially expanding solutions by discussing the stability of the models by considering non-Flat variable- Lambda cosmological models with bulk viscosity.

Johri and Desikan (1994) investigated cosmological models by considering the constant deceleration parameter in Brans-Dicke theory and studied each singular and non-singular models of the universe. They explained that the growth of singular models supported big-bang impulse and also the growth of non-singular models because of creation of matter particles.

Abdussattar and Vishwakarma (1997) studied some Robertson-Walker models considering a contracted Ricci collineation with the fluid flow vector and having time-varying G and Λ. They obtained the character of the growth of the models within the cases $k = \pm 1$ and located to be interchanged from the corresponding standard FRW models.

Banerjee and Sen (1998) investigated the character of the potential function V (ϕ) relevant to power law inflation in an exceedingly minimally coupled scalar field cosmology together with a perfect or causal viscous fluid and discovered that if the coefficient of viscosity is proportional to the square root of the density of the fluid, the desired potential is an exponential function of the scalar field ϕ.

Friedmann (1999) has deduced some necessary results of the Friedmann- Lemaitre-Robertson-Walker (FLRW) model.

Vishwakarma (2001) considered four variable Λ-models to investigate the magnitude-redshift relation and angular size-redshift relation for the Type Ia supernovae and updated compact radio sources data respectively.

Kremer and Devecchi (2003) shown that a present acceleration with a past deceleration may be a possible solution to the Friedmann equation by considering the universe as a combination of a scalar with a matter field and by together with a non equilibrium pressure term within the energy-momentum tensor. They additionally concluded that the dark energy density decays a lot of slowly with reference to the time than the matter energy density does.

Debnath and Paul (2006) thought-about the evolution of a flat Friedmann-Roberstson-Walker universe in higher derivative theories, together with αR^2 terms to the Einstein-Hilbert action within the presence of variable gravitational and cosmological constants. They additionally studied the evolution of the gravitational and cosmological constants within the presence of radiation and matter domination era of the universe.

Akbar and Cai (2007) shown that the differential form of a Friedmann equations may be derived by applying the first law of thermodynamics at the apparent horizon of an FRW universe with entropy.

Singh et al. (2007) thought-about Einstein field equations with variable gravitational and cosmological constants within the presence of bulk viscosity for a spatially flat homogenized and isotropic universe. They also studied the cosmological model with the constant and time-dependent bulk viscosity.

Akbar (2008) shown that the differential type of Friedmann equations of Friedman-Robertson-Walker (FRW) universe may be recast as the same type of the first law of thermodynamics at the apparent horizon of FRW universe full of the viscous fluid.

Arbab (2008) studied a cosmological model of phantom energy using a variable cosmological constant Λ that depend on the energy density (ρ). He thought-about the cosmological constant in such the way that it varies reciprocally proportional the energy density of the universe.

Copeland et al. (2009) investigated the dynamics of a particular scalar field within the Friedmann-Robertson-Walker universe through the spatial curvature and obtained the fixed point solutions that are indicated to be late time at tractors. They additionally determined the corresponding scalar field potentials that correspond to those stable solutions.

Ibotombi et al. (2009) investigated FRW models of universe in presence of viscous fluid within the cosmological theory supported Lyra's Manifold. They obtained exacts solutions by considering the deceleration parameter to be a variable and the viscosity coefficient of bulk viscous fluid to be a constant and investigated the physical properties of the models.

Leon and Saridakis (2010) studied various varying-mass models of dark matter particle within the framework of phantom cosmology and investigated whether or not there exist late-time cosmological solutions, equivalent to an accelerating universe and having the dark energy and dark matter densities of a similar order. They ended that the coincidence problem cannot be solved or may be relieved.

El-Nabulsi (2010a) studied a new cosmological model of the universe supported a spatially flat FRW metric. He created a new kind of extended modified gravity theory to explain a dark energy dominated accelerating universe employing a Gauss–Bonnet invariant term and a new Einstein-Hilbert term.

El-Nabulsi (2010b) presented a four-dimensional dilaton-Brans-Dicke cosmological situation related to the multiverse occupied by dark energy or phantom energy with a positive cosmological constant containing countless without end big rip singularities.

Jamil and Debnath (2011) thought-about a cosmological model of variable G and Λ for the FRW universe and obtained the solutions in the form of cosmological constant for the flat model. They additionally found the cosmological parameters for dust, radiation and stiff matter. The state finder parameters analyzed and shown that this depends only on ω and ϵ.

Mostafapoor and Gron (2011) studied the flat Λ cold dark matter models through the bulk viscosity and investigated the role of the bulk viscosity in case of evolution of the universe. They obtained the dynamical equations for these models and resolved for a few cases of bulk viscosity. They additionally obtained the differential equations for the Hubble parameter and also the energy density of dark matter.

Akarsu and Dereli (2012) planned a cosmological model for the deceleration parameter that varies linearly with time and covers Berman's law wherever it's constant and it provides an improved work with information (from SNIa, BAO and CMB), particularly regarding the late time behavior of the universe. They thought-about the models for the spatially closed and flat universe and located that the cosmological fluid behaves like quintom and also the universe ends with a big-rip in each the cases.

Mohajan (2013) established Friedmann, Robertson-Walker (FRW) models on the premise of the idea that the universe is homogeneous and isotropic altogether epochs to explain the FRW models with easier mathematical calculations, physical interpretations and diagrams wherever necessary.

Amirhashchi et al. (2014) examined the evolution of the equation of state parameter of the dark energy for the spatially homogeneous and isotropic Friedmann-Robertson-Walker (FRW) model containing the barotropic fluid. They additionally discovered that in each interacting and non-interacting cases, the equation of state parameter for dark energy is decreasing function of time and forever variable in quintessence region for all open, closed and flat models.

Singh and Bishi (2015) studied the FRW metric for the universal gravitational constant G and cosmological constant Λ within the $f(R,T)$ gravity using the modified Chaplygin gas equation of state. By using hybrid exponential law (HEL) for the scale factor they obtained the field equation's solution and also talk over about some physical behaviour of the model.

Chand et al. (2016) established the exact solution for spatially homegenous and isotrpic FRW spacetime within the Brans-Dicke scalar tensor theory of gravity. Tiwari et al. (2017) concluded for the spatially homogeneous FRW model together with barotropic fluid by studying the equation of state parameter of dark energy. Zhang and Kuang (2018) studied the quantum effect of the modified Friedmann equation within the Friedmann-Robertson-Walker universe. They also investigated the bounce cosmological solution for the spatially flat geometry in the modified Friedmann equation.

Archana et al. (2019) investigated Tsallis holographic dark energy in flat FRW universe considering IR cutoff as Hubble horizon with time-varying deceleration parameter for the evolution of the universe. From the study of statefinder, They observed that THDE model is in good agreement with a flat ΛCDM model.

Taser (2020) investigated conformal symmetric Freidmann–Robertson–Walker (FRW) universe with perfect fluid in the framework of $f(R,T)$ gravitational theory. The exact solutions of conformal FRW universe with perfect fluid are attained for matter part of the $f(R,T)$ model in the case of $h(T) = \lambda T$. Energy conditions are investigated.

Bochniak and Andrzej (2021) motivated by the models of geometry with discrete spaces as additional dimensions we investigate the stability of cosmological solutions in models with two metrics of the Friedmann-Lemaître-Robertson-Walker type. They proposed an effective gravity action that couples the two metrics in a similar manner as in bimetric theory of gravity and analyze whether standard solutions with identical metrics are stable under small perturbations.

1.20 Modified theories of gravitation

Since Einstein published his first theory of gravitation there has been many criticisms. Brans and Dicke (1961), Saez and Ballester (1986), Nordtvedt (1970), Ross (1972), Dunn (1974) and scale covariant theory of gravitation proposed by Canuto et al. (1977), Schmidt et al. (1981) are the most important modified theories for the substitute of the Einstein's theory. In the scale covariant theory Einstein field equations are valid in gravitational units whereas physical quantities are measured in atomic units. The metric tensor in the two systems of units is related by a conformal transformation

$$\bar{g}_{ij} = \phi^2(x^k) g_{ij}, \tag{1.20.1}$$

where bar denotes gravitational units and unbar denotes atomic units.

Another recent modification of Einstein theory is the generalized $f(R,T)$ gravity proposed by Harko et al. (2011) to explain dark energy and accelerated expansion of the universe. In this thesis we concentrate on the investigation of some cosmological models in scalar-tensor theories of gravitation proposed by Saez and Ballester (1986), scale covariant theory of gravitation formulated by Canute et al. (1977) and $f(R,T)$ gravity proposed by Harko et al. (2011). We now present a detailed discussion of the above modified theories of gravitation.

1.21 Brans-Dicke scalar-tensor theory of gravitation

A theory of gravitation in which the gravitational field is described by the tensor field of general relativity and by a new scalar field, which is determined by the distribution of mass-energy in the universe and replaces the gravitational constant. scalar-tensor theories of gravitation have great appreciation for the community academic in the 1960s, when the first version was established. In this theory, a scalar field has a dynamic that affects the structure of space time. The Brans-Dicke theory was first proposed as an alternative to general relativity, however, require testing in the solar system a lower limit of the Brans-Dicke parameter is around 4000, as a result ended up putting the theory into sensitive positions. On the other hand, an analysis of the dynamics of the scalar field of Brans-Dicke in a cosmological context presents a clear-cut result: the usual general relativity is an attractor of a natural gravitation of Brans-Dicke cosmology evolved, *i.e.*, the dynamics of the scalar field is gradually suppressed. Now we propose the study of gravitation of Brans-Dicke. We believe that the study of such a generalization of general relativity, and provide a better perspective on the area in question, allows a better understanding of gravitation, as well as their own general relativity itself. In theoretical physics, the Brans–Dicke theory of gravitation (sometimes called the Jordan-Brans-Dicke theory) is a theoretical framework to explain gravitation. It is a well-known competitor of Einstein's more popular theory of general relativity. It is an example of a scalar-tensor theory, a gravitational theory in which the gravitational interaction is mediated by a scalar field as well as the tensor field of general relativity. The gravitational constant G is not presumed to be constant but instead G is replaced by a scalar field which can vary from place to place and with time. The theory was developed in 1961 by Robert H. Dicke and Carl H. Brans building upon, among others, the earlier 1959 work of Pascual Jordan. At present, both Brans–Dicke theory and general relativity are generally held to be in agreement with observations.

As usually formulated, Mach's principle requires that the geometry of spacetime and hence the inertial properties of every infinitesimal test particle be determined by the distribution of mass-energy throughout the universe (Wheeler, 1964). Although being one of the foundation stones of Einstein's philosophy, this principle is contained only to a limited extent in general relativity (Dicke, 1964). Some examples of 'non-machian' solutions are (Heckmann and Schucking, 1962), Minkowski space which has inertial properties but no matter, the Godel (1949) universe which contains such unphysical as closed time-like curves and the closed but empty Taub (1951) model. Wheeler (1964) has suggested that these unsatisfactory solutions might be excluded by means of boundary conditions. Brans and Dicke (1961) have argued against this possibility by considering a static massive shell. The inertial properties of test particles inside shell are according to general relativity, unchanged even if the mass of the shell is increased.

In the hope of extending general relativity in such a way as to incorporate Mach's principle, Brans and Dicke (1961) have proposed a theory which includes a long range scalar field interacting equally with all forms of matter (with the exception of electromagnetism). They noted, following Dirac (1938) and Sciama (1959), that the Newtonian gravitational constant G is related to the mass M and radius R of the visible universe by

$$G \sim \frac{Rc^2}{M}. \tag{1.21.1}$$

(The numbers are approximate). This suggests that G is a (scalar) function determined by the matter distribution. Their theory is formally equivalent to the one previously considered by Jordan (1955).

In order to generalize the equations of general relativity, Brans and Dicke (1961) formulated their variational principle, which differs from that of general relativity, namely,

$$\delta \int \{R + (16G)\} \sqrt{-g} d^4x = 0. \tag{1.21.2}$$

In that G is replaced by ϕ^{-1} which now comes inside the action integral. There are also additional terms to take account of the scalar nature of ϕ

$$\delta \int \left\{ \phi R + 16RL - \frac{\omega \phi_{,i} \phi^{,i}}{\phi} \right\} \sqrt{-g} d^4x = 0. \tag{1.21.3}$$

where ϕ is the scalar field, R is the usual scalar curvature, L is a function of matter variables and metric tensor components (not of scalar field ϕ) and ω is a dimensionless constant.

The field equations obtained by the variation of g_{ij} and ϕ take the form

$$R_{ij} - \frac{1}{2} R g_{ij} = -8\pi \phi^{-1} T_{ij} - \omega \phi^{-2} \left(\phi_{,i} \phi_{,j} - \frac{1}{2} g_{ij} \phi_{,k} \phi^{,k} \right) - \phi^{-1} \left(\phi_{i;j} - \frac{1}{2} g_{ij} \phi^{,k}_{;k} \right) \tag{1.21.4}$$

and

$$\phi^{,k}_{;k} = 8\pi (3 + 2\omega)^{-1} T, \tag{1.21.5}$$

where $T = g^{ij} T_{ij}$. Here, the metric has signature +2, a comma denotes partial differentiation, a semi colon (;) denotes covariant differentiation and the velocity of light 'c' is taken to be unity. The main difference between the Brans-Dicke theory and Einstein theory lies in the gravitational field equations, which determine the metric field g_{ij}, rather than in the equations of motion. The energy momentum tensor of matter T_{ij} satisfies the local matter-energy conservation law

$$T^{ij}_{\ ;j} = 0, \qquad (1.21.6)$$

which also represents equation of motion and is a consequence of the field equations (1.2.5).

A comparison of the above equations with Einstein's equations shows that the Brans-Dicke theory goes over to general relativity in the limit , constant $G = 1$. The above modification of Einstein's theory involves violation of 'strong principle of equivalence' on which Einstein's theory is based. But this does not violate *'weak principle of equivalence'* for example, the paths of test particles in a gravitational field are still independent of their masses. Thus, Brans-Dicke theory now can be described as a theory for which the gravitational force on an object is partially due to the interaction with a scalar field, and partially due to a tensor interaction. Further discussions, by Brans and Dicke (1961), of the field equations (1.2.5) have included and analysis of the weak field equations, study of the three standard tests, comparison with the work of Jordan (1955), discussions of boundary conditions for , investigations of cosmology and the general relationship to Mach's principle. At present there is no evidence to preclude the validity of the Brans-Dicke scalar-tensor theory. While this theory doesn't predict an anomalous gravitational red shift, it gives values for the gravitational deflection of light rays and the perihelion advance of planetary orbits different from those of Einstein's theory (Dicke, 1964; Brans and Dicke, 1961). But in view of the relatively large discrepancies in the measurements of the deflection of star light near the sun's limb during a total eclipse (Dicke, 1967) and the measurements of the oblate ness of the sun (Dicke and Goldenberg, 1967), it is concluded that Brans-Dicke theory is not a conflict with observations.

Very recently, this theory has been applied to more interesting problems in astrophysics in order to appreciate fully the implications of the addition of along range scalar interaction. By comparing the predictions of this theory with those of Einstein's theory, one may hope to obtain important differences which might be used to decide between the two theories. For example, while Solmona (1967) has shown that certain gross features of a cold neutron star remain unchanged by the presence of the strength of the scalar field, Morganstern and Chiu (1967) have shown that if a neutron star is observed to exhibit symmetric radial pulsation, then the existence of the scalar field may be ruled out.

As a consequence of the recent lunar ranging experiments (Williams et al., 1976, Shapiro et al., 1976) one can conclude that Brans-Dicke parameter 500. It has been pointed out that there is no theoretical reason to restrict to positive values (Smalley and Eby, 1976). In view of this one might well conclude that Brans-Dicke theory with some large values of is the correct theory.

1.21.1 Review on the work related to Brans-Dicke theory

In this thesis, some of our works is related to the Brans-Dicke scalar-tensor theory of gravitation and Quintessence. We are highlighting here the work carried out by various authors.

Chauvet (1982) found general solutions of the Friedmann vacuum universe by rescaling the scalar ϕ of the Jordan-Brans and Dicke cosmology. Each solution is characterised by the sign of the second order derivative of the rescaled field Robertson-Walker line element. $\phi(= \phi R^3)$, R being the scale factor.

Banerjee (1995) studied an isotropic homogeneous cosmological models with Robertson-Walker line element in general scalar tensor where the coupling parameter is a function of the scalar field. Exact solutions are obtained in Dicke's conformally transformed units for stiff fluid and radiation universe.

Liddle and Scherrer (1998) have obtained scaling attractor solutions with power law potentials in non-minimally coupled theories.

Bertolo and Pietroni (1999) have investigated an approach to find tracking solutions in general scalar tensor theories with inverse power law potentials.

Amendola (1999) have obtained an accelerated expanding solutions at present time, the so-called scaling attractor in some theories of gravity with non-minimal coupling of the form $[1 + \epsilon f(\phi)]$ and $V(\phi) = Af(\phi)^M$, where $f(\phi)$ is a power-law or exponential function of the scalar field and A and M are constants.

Uzan, Holden and Wands (1999) have studied the self-interacting Brans-Dicke theory by considering a scalar field coupled non-minimally with gravity. Although accelerated solutions can be obtained in non-minimally coupled gravity theories the constraint on the variability of the gravitational coupling is quite strong and that it implies a universe that is considerably older than H^{-1}. Uzan (1999) have found scaling attractor solutions in the literature with exponential potentials in non-minimally coupled theories.

Banergee and Pavon (2000) investigated the possibility of obtaining a non-decelerating expansion for the universe for open, flat and closed Friedmann Robertson-Walker models in Brans-Dicke theory with the help of scalar field which is minimally coupled to gravity and serves as the Bertolami and Martins (2000) analysed the conditions under which the dynamics of a self-interacting Brans-Dicke field can account for the accelerated expansion of the universe. They have shown that accelerated expanding solutions can be achieved with a quadratic self-coupling of the Brans-Dicke field and a negative coupling constant ω.

Faraoni (2000) have studied different potentials with a non-minimal coupling term $\psi R \frac{\phi^2}{2}$.

Saini, Raychaudhury and Starobinsky (2000) have reconstructed the potential from the luminosity-redshift relation available from the observations in context of scalar tensor theory.

Chimento, Jakubi and Pavon (2000) have shown that a combination of dissipative effects such as a bulk viscous stress and a quintessence scalar field gives an accelerated expansion for an open universe ($k = -1$) as well.

Lopez and Matos (2000) proposed a quintessence model with the potential $V(\phi) = V_0[\sin h(\alpha \sqrt{k_0})\Delta\phi]$ which asymptotic behaviour corresponds to an inverse power law potential at early times and to an exponential one at late time. They demonstrate that this is an tracker solution and that it could have driven the universe into its current inflationary stage.

Banerjee and Pavon (2001a) have shown that with $\omega(\phi)$ one can have a decelerating radiation dominated era in the early time and accelerated matter dominated era in the late time. They have also shown that in BD theory the nucleosynthesis problem can be avoided by considering ω to be a function of ϕ.

Banerjee and Pavon (2001b) investigates the possibility of obtaining a non-decelerating expansion for the universe in Brans-Dicke theory with the help of another scalar field which is minimally coupled to gravity and serves as the quintessence matter.

Sen and Sen (2001) have found an accelerating solutions in Brans-Dicke cosmology with a potential which has a time dependent mass squared term which has recently become negative.

Sen and Seshadri (2003) investigated the nature of the potential relevant to the power law expansion of the universe in a self interacting Brans-Dicke cosmology with a perfect fluid distribution. The density perturbation is also studied to check the consistency of the structure formation scenario.

Chakraborty et al. (2008) have shown that minimally coupled scalar field in Brans-Dicke theory with varying speed of light can solve the quintessence problem and it is possible to have a non-decelerated expansion of the present universe with Brans-Dicke theory for anisotropic models without any matter.

Sotiriou and Faraoni (2010) reviewed $f(R)$ theories of gravity in an attempt to comprehensively present their most important aspects and cover the largest possible portion of the relevant literature. All known formalisms are presented—metric, Palatini, and metric affine—and the following topics are discussed: motivation; actions, field equations, and theoretical aspects; equivalence with other theories; cosmological aspects and constraints; viability criteria; and astrophysical applications.

Shobhan Babu et al., (2013) investigated a five-dimensional Kaluza-Klein space-time in the frame work of Brans-Dicke scalar-tensor theory of gravitation when the source of energy momentum tensor is a bulk viscous fluid containing one dimensional cosmic strings. They have obtained a determinate solution of the field equations using the special law of variation for Hubble's parameter proposed by Bermann. They have also used a barotropic equation of state for the pressure and density. They further discussed some physical properties of the model.

Das and Mamon (2014) show that in non-minimally coupled Brans-Dicke theory containing a self-interacting potential, a suitable conformal transformation can automatically give rise to an interaction between the normal matter and the Brans-Dicke scalar field. Considering the scalar field in the Einstein frame as the quintessence matter, they had shown that such a non-minimal coupling between the matter and the scalar field can give rise to a late time accelerated expansion for the universe preceded by a decelerated expansion for very high values of the Brans-Dicke parameter ω. They have also studied the observational constraints on the model parameters considering the Hubble and Supernova data.

Kiran, Reddy and Rao (2015) studied the anisotropic and homogeneous Bianchi type-V universe filled with two minimally interacting fields, matter and holographic dark energy components in the frame work of Brans and Dicke theory of gravitation. To obtain a determinate solution of the field equations they have used (i) the scalar expansion is proportional to the shear scalar and (ii) special law of variation for Hubble's parameter proposed by Berman. They also discussed some physical and kinematical properties of the model.

Chand, Mishra and Anirudh Pradhan (2016) considered exact solution of modified Einstein's field equations within the scope of spatially homogeneous and isotropic Fraidmann-Robertson-Walker (FRW) space-time filled with perfect fluid in the frame work of Brans-Dicke scalar-tensor theory of gravity. They have investigated the flat, open and closed FRW models and the effect of dynamic cosmological term on the evolution of the universe. Two types of FRW cosmological models are obtained by setting the power law between the scalar field ϕ and the scale factor a and deceleration parameter (DP) q as a time dependent.

Goswami (2017) investigated late time acceleration for a spatially flat dust filled universe in Brans-Dicke theory in the presence of a positive cosmological constant Λ. They have obtained the expressions for Hubble's constant, luminosity distance and apparent magnitude of the model.

Aditya and Reddy (2018) investigate non-Ricci, non-compact Friedmann-Robertson-Walker type Kaluza-Klein cosmology in the presence of pressureless matter and modified holographic Ricci dark energy in the frame work of Brans and Dicke scalar-tensor theory of gravitation. They solve the field equations of this theory using a hybrid expansion law for the five-dimensional scale factor. They have also used a power law and a form of logarithmic function of the scale factor for the Brans-Dicke scalar field. Consequently, they obtain two interesting cosmological models of the Kaluza-Klein universe. They have evaluated the cosmological parameters, namely, the equation of state parameter, the deceleration parameter, and the density parameters. To check the stability of the models they use the squared speed of sound. Some well-known cosmological ($\omega_{\rm de}$-$\omega'_{\rm de}$ and statefinder) planes are constructed for our models. They have also analyzed the physical behavior of these parameters through graphical representation. They observed that the FRW type Kaluza-Klein dark energy models presented are compatible with the present day cosmological observations.

Jawad, Aslam and Rani (2019) Tsallis entropy has been widely applied to analyze the gravitational and cosmological setups. They discuss the dark energy (DE) model by its cosmological consequences using Tsallis holographic entropy in the framework of modified Brans–Dicke (BD) gravity. They consider the Hubble horizon as infrared cutoff to study the nature of DE that is responsible for current cosmic acceleration. They focus on flat FRW universe in interacting and non interacting scenarios between DE and dark matter (DM). In this framework, they discuss the cosmological parameters like equation of state parameter, deceleration parameter, Om-diagnostic, squared speed of sound and planes like evolving equation of state parameter and statefinders. They discuss graphical presentation of these parameters and planes. They compare the results with observation data to check the consistency of results.

Tripathy et al. (2020) constructed some dark energy cosmological models in the framework of a generalized Brans–Dicke theory which contains a self interacting potential and a dynamical coupling parameter. The models are constructed in the background of an anisotropic metric. The dark sector of the universe is considered through a unified linear equation of state. Also, on the basis of the generalised Brans–Dicke theory, they have estimated the time variation of the Newtonian gravitational constant.

Mukhopadhyay, Saha and Chaudhury (2021) study the generalized Brans-Dicke theory of gravity, time dependence of various cosmological parameters for a spatially flat, homogeneous and isotropic universe filled with pressure-less matter. Mathematical formulations have been carried out with the help of two models, based upon two different expressions for the scale factor. In the first model, the entire matter content (dark matter + baryonic matter) of the universe has been assumed to be conserved. A smooth transition from a state of decelerated expansion to a state of accelerated expansion of the universe has been obtained from an exact solution of the field equations, without incorporating any parameter in the theoretical formulation that represents the dark energy. Time dependence of the scalar field has been determined from this solution with the help of astrophysical characteristics of the expanding universe. They found the nature of dependence of the Brans-Dicke parameter upon time and also upon the scalar field. The Brans-Dicke parameter has been found to have a small negative value and it becomes more negative as the scalar field decreases with time. It has been found in the present study that the gravitational constant, which is reciprocal of the scalar field parameter, increases with time. In the second of the two models discussed here, an ansatz has been assumed regarding the mode of change of the dark energy content of the universe with time. Using this model, they determined the time dependence of the densities of matter and dark energy and also the density parameters corresponding to these two constituents of the universe.

1.22 Lyra geometry

In recent years, there has been a lot of interest in alternative theories of gravitation. Noteworthy among them is the theory of gravitation proposed by Sen (1957) based on Lyra (1951) geometry. This geometry is a modified Riemannian geometry in which a gauge function has been introduced into the structure less manifold as a result of which the cosmological constant arises naturally from the geometry. In general relativity, Einstein succeeded in geometrizing gravitation by identifying the metric tensor with gravitational potentials. In scalar tensor theory of Brans-Dicke on the other hand, the scalar field remains alien to the geometry. Lyra's geometry is more in keeping with spirit of Einstein's principle of geometrisation since both the scalar and tensor fields have more or less intrinsic geometrical significance. Lyra (1951) defined the displacement vector PP' between two neighboring points $P(x^i)$ and $P'(x^i + dx^i)$ which has the components $\xi^i = x^0 dx^i$ where $x^0(x^i)$ is a non-zero gauge function. The coordinate system x^i together with x^0 form reference system (x^0,x^i). Tensors are characterized by the way in which components transform under a general transformation of coordinates.

The metric

$$ds^2 = g_{ij} x^0 dx^i x^0 dx^j, \tag{1.22.1}$$

is an absolute invariant (*i.e.,* invariant under change of reference system). The components of the affine connection are no longer symmetric in the lower indices and cannot be identified with the Christoffel symbols as is the case in Riemannian geometry. In Lyra geometry the components of the affine connection are not only the functions of Christoffel symbols but also of ϕ_i . The Lyra curvature tensor K^h_{ij} , the contracted curvature tensor K_{ij} and the scalar curvature K can be obtained similar to the Riemannian one. The scalar constant K is given by

$$K = R\left(x^0\right)^{-2} + 3\left(x^0\right)^{-1}, \phi^i_{;i} + \frac{3}{2}\phi^i\phi_i, \tag{1.22.2}$$

where R is the Riemannian curvature scalar and ϕ_i is defined by

$$\phi_i = \left(x^0\right)^{-1} \frac{\partial}{\partial x^i} \left\{\log(x^0)^2\right\}, \tag{1.22.3}$$

choosing the so-called normal gauge *i.e.,* $x^0 = 1$ the volume integral in space becomes

$$I = \int k \sqrt{-g} d^4x, \tag{1.22.4}$$

where

$$K = R + 3\phi^i_{;i} + \frac{3}{2}\phi_i\phi^i. \tag{1.22.5}$$

The field equations in this manifold may be obtained from the variational principle

$$\delta \left(I + J\right) = 0, \tag{1.22.6}$$

where I is given by (1.22.4) and

$$J = \int L \sqrt{-g} d^4x. \tag{1.22.7}$$

Here L is the Lagrangian density of matter.

The field equations given by Sen (1951) are

$$R_{ij} - \frac{1}{2}Rg_{ij} + \frac{3}{2}\phi_i\phi^i - \frac{3}{4}g_{ij}\phi_k\phi^k = -8\pi T_{ij}. \tag{1.22.8}$$

Lyra Geometry is use in my study because it is a modification of Riemannian geometry, which is relatively easy and helpful to introducing a gauge function into the structureless manifold, which terminates the non-integrability state of the length of a vector under parallel transport. Lyra geometry, along with constant gauge vector ϕ_i will either play the role of cosmological constant or creation field.

1.22.1 Literature review on Lyra Geometry

Some works are highlighted as a literature review by different cosmologists and authors regarding Lyra geometry as follows:

Lyra (1951) suggested a modification of Riemannian geometry, which is also regarded as a modification of Weyl's geometry, by introducing a Gauge function (or scale function) into the structure less manifold which removes the non-integrability condition of the length of a vector under parallel transport (*i.e.,* the metricity condition is restored) and a cosmological constant is naturally introduced from the geometry. Lyra (1951) and Scheibe (1952) completed the study of this geometry, which is known as Lyra's Geometry. In Lyra's geometry, the connection is metric preserving as Riemannian geometry, and length transfers as integrable in contrast to Weyl's geometry. This alternating theory of Lyra's geometry is of interest since it produces effects similar to Einstein's theory.

On the basis of Lyra's geometry, Sen (1957) studied a static cosmological model universe similar to the Einstein's static model, which had a finite density and showing a red shift. He also showed that the red shift of spectral lines from extragalactic nebulae was nothing but an outcome of an intrinsic geometrical property of the model independent of expansion. Also, he obtained the field equations in normal gauge as

$$R_{ij} - \frac{1}{2}Rg_{ij} + \frac{3}{2}g_{ij}\phi_i\phi_j - \frac{3}{4}g_{ij}\phi_k\phi^k = -kT_{ij}, \tag{1.22.9}$$

where R_{ij} is the Ricci tensor; R is the Ricci scalar, g_{ij} is a metric tensor, ϕ^i is a displacement field and T_{ij} is the energy-momentum tensor.

Sen (1960) and Sen and Dunn (1971) showed that, unlike Riemannian geometry, the auto parallels associated with the affine connection in Lyra geometry did not coincident with the geodesics arises from the metric. In the Lyra's geometry, they also constructed a new scalar-tensor theory where both the scalar and tensor field had natural geometrical significance.

Sen and Vanstone (1972), in their paper "On Weyl and Lyra Manifolds", showed that the Lyra's geometry and Weyl's geometry are special cases of manifolds with more general connections. Also, they showed the relationship between Lyra's geometry and Weyl's geometry and obtained the relationship of them with Riemannian geometry by giving a global formulation of Lyra's geometry.

Halford (1970) designed a cosmological theory within the framework of Lyra's geometry and showed that the constant displacement vector field in Lyra's geometry plays the role of the cosmological constant in the normal general relativistic study. Also, Halford (1972) obtained a closed-form exact solution of the field equations corresponding to a scalar-tensor theory similar to the Brans-Dicke theory and showed that the scalar-tensor treatment based on Lyra's geometry predicts the same effect, within observational limits, as far as the classical solar system test are concerned (as in the Einstein's theory of relativity.

Bhamra (1974) obtained a spherically symmetric cosmological model of class-one in the framework of Lyra's geometry and showed that the static universe is physically unrealistic whereas the non-static universe is similar to Lemaitre's model in Riemannian geometry in which the mass-energy conservation law did not hold.

Jeavons et al. (1975), in their study of "A Correction to the Sen and Dunn Gravitational Field Equations", showed that the field equations formulated by Sen and Dunn (1971) cannot be derived from the normal variational principle and they suggested the modified field equations as

$$R_{ij} - \frac{1}{2}Rg_{ij} + \phi^{-1}\left(\phi_{i;j} - g_{ij}\phi\right) - \omega\phi^{-2}\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{ik}\phi^{,k}\right) = -\phi T_{ij}, \tag{1.22.10}$$

where R_{ij} is the Ricci tensor, R is the Ricci scalar (Riemann curvature scalar), ω = constant =1 , and T_{ij} is the material energy-momentum tensor (in our units $c = 8\pi G = 1$).

Reddy (1973, 1977) investigated the Birkhoff's theorem of general relativity both in the Brans-Dicke theory and in the scalar-tensor theory suggested by Sen and Dunn (1971). Reddy (1973) showed that the Birkhoff's theorem of general relativity is hold good in the scalar-tensor theory suggested by Sen and Dunn (1971) for all scalar field irrespective of nature of the scalar field. But in the Brans-Dicke theory, Birkhoff's Theorem is valid only for the scalar field which is independent of time. Considering time-independent scalar field in the scalar-tensor theory suggested by Sen and Dunn, Reddy (1977) showed that the Birkhoff's Theorem of general relativity is also valid in presence of electromagnetic field. So, he suggested that the scalar-tensor theory of Sen and Dunn (1971) may be considered as a superior version of the Brans-Dicke theory.

Karade and Borikar (1978) studied the effects of the thermodynamic equilibrium of a gravitating fluid sphere in Lyra's Geometry and obtained a static model universe with a zero red shift in it. Singh and Rai (1979) investigated the Birkhoff's theorem of general relativity in the scalar-tensor theory suggested by Jeavons et al. (1975) and showed that when the scalar field is independent time then in presence of electromagnetic fields in the scalar-tensor theory suggested by Jeavons et al. (1975), the spherically symmetric gravitational and electromagnetic fields turn out to be static.

Kalyanshetti and Waghmode (1982) obtained a static cosmological model in Einstein-Cartan theory in the framework of Lyra's geometry. Assuming the spin of each fluid particle along the radial direction, he observed that only constant spin has existed in his Einstein's static model universe that can be expressed in terms of central density. Considering a metric described by a scale constant associated with the size of the universe, Rosen (1983) modified the Weyl-Dirac theory of gravitation and electromagnetism.

Reddy and Innaiah (1985) formulated an anisotropic and spatially homogeneous Bianchi type-I cosmological model in Lyra's manifold with perfect fluid as a source of gravitational field by considering energy density equal to pressure.

Reddy and Innaiah (1986) constructed a plane-symmetric cosmological model in Lyra manifold with perfect fluid as a source of gravitational field by taking energy density equal to pressure.

Beesham (1986) obtained vacuum FRW cosmological models in the framework of Lyra's geometry and a number of new solutions are discussed in the de Sitter universe.

In the study of "Cosmologies Based on Lyra's Geometry", Soleng (1987) discussed that the Lyra Geometry together with gauge vector ϕ_i will play either the role of cosmological constant or the creation field (equal to the Hoyle's creation field (Hoyle, 1948; Hoyle and Narlikar, 1963, 1964). He also showed that the solutions in the first case are equal to the solutions in general relativistic cosmologies with a cosmological term.

Considering Friedmann-Lemaitre-Robertson-Walker (FLRW), Beesham (1988) formulated cosmological models in Lyra's manifold with time-dependent displacement field. In this model, not only he solved the existing problems like singularity, horizon and entropy in the standard cosmological models based on Riemannian geometry but also studied the asymptotic behavior of the models. Singh and his co-authors (1991a,bb,bc,bd; 1992a, b; 1993b; 1997) studied Bianchi types $I, II, III, V, VI_0, VIII, IX$, Kantowski-Sachs, and a new class of cosmological model universe with and without time-dependent displacement field in the framework of Lyra geometry. Comparative study of the Cosmological theory based on Lyra's geometry and the Friedmann-Robertson-Walker (FRW) model universe with a constant deceleration parameter in the Einstein's theory of relativity were also made by them.

Khadekar and Nagpure (2001) studied a Higher Dimensional Static conformally flat spherically symmetric Cosmological Model in Lyra Geometry in presence of perfect fluid and observed that in Lyra's manifold, the displacement vector plays the role of the spin density.

Rahaman et al. (2002) investigated an Inhomogeneous cosmological model in Lyra Geometry and obtained the exact solutions of the field equations. He has got an anisotropic model universe where the displacement vector is always non-zero, so the concept of Lyra geometry exists even after infinite time.

Rahaman (2003) discussed a five-dimensional spherically symmetric metric in presence of a homogeneous perfect fluid in the framework of Lyra geometry and obtained a cosmological model for vacuum energy type universe together with matter filled-universe for dust case, Zeldovich fluid and stiff fluid.

Considering a time-dependent displacement field, Pradhan and Vishwakarma (2004) investigated a locally rotationally symmetric Bianchi type-I metric and a new class of exact solutions of the field equations in the framework of Lyra geometry is obtained for constant deceleration parameter. Also, they studied the characteristics of the energy density and displacement field in the power law expansion and exponential expansion of both flat and non-flat universe.

Rahaman et al. (2005) obtained two model universe namely axially symmetric Bianchi type-I and Kantowski-Sach cosmological models with negative constant deceleration parameter based on Lyra geometry.

Casana et al. (2005) studied the coupling of the curved and torsioned Lyra manifold with the electromagnetic field and showed that the coupling between torsion and the massless electromagnetic field was related to scale transformations in Lyra setting. Also, they showed that the suitable choice of the connection of gauge transformations with scale invariance in Lyra manifold would remove the problem of breaking the local gauge invariance connected with this coupling.

Casana et al. (2006) discussed the Dirac field in Lyra geometry and obtained the equation of motions and conservation laws for spin and energy-momentum. They, also, obtained the scale relation, which is a fundamental property of matter fields in Lyra geometry, connecting the spin tensor and energy-momentum tensor.

Studying five-dimensional LRS Bianchi type-I spacetime in presence of bulk viscous fluid, Mohanty et al. (2007) constructed a higher dimensional string cosmological model in Lyra Manifold for time-dependent displacement field and constant coefficient of bulk viscosity. This model had no initial singularity.

In a scalar-tensor theory of Sen (1957) based on Lyra manifold, Rao and Vijaya Santhi (2008a) formulated a Bianchi type-V cosmological model in presence of perfect fluid for a constant displacement vector. Also, when displacement vector is a function of cosmic time then by using negative constant deceleration parameter they had shown that this model exists only for radiation universe. Kumar and Singh (2008) investigated a spatially homogeneous and anisotropic Bianchi type-I in presence of perfect fluid and obtained a cosmological model universe based on

Lyra geometry. Using the special law of Hubble's parameter that gives a constant deceleration parameter, they had obtained the exact solutions of the field equations which are consistent with the recent observational data from supernovae Type Ia.

Considering five-dimensional plane symmetric metric, Mohanty et al. (2009b) attempted to obtain a string cosmological model universe both in Riemannian geometry and in Lyra geometry. But they had observed that, in both the theories, the string cosmological models were not survived. Accordingly, they had formulated the vacuum cosmological models and discussed their properties.

Investigating plane-symmetric metric under the influence of perfect fluid, Yadav (2010) obtained an inhomogeneous cosmological model universe with electromagnetic field based on Lyra geometry and the exact solutions of the field equations for this model are consistent with the recent observational data from supernovae type Ia.

In the framework of Lyra geometry, Gad (2011) obtained a new class of axially symmetric cosmological model universe in presence of the mesonic stiff fluid with time-dependent displacement field which are expanding, shearing and non-rotating.

Adhav (2011) obtained an anisotropic dark energy model based on Lyra geometry by ex- amining a LRS Bianchi type-I metric under the influence of anisotropic fluid. Considering exponential volumetric expansion, exact solutions of the field equations were determined for constant and time-dependent displacement field and isotropic properties of the space and fluid were examined.

In the framework of Lyra geometry, Mahanta and Biswal (2012) obtained cosmological model universe for both string cloud and domain walls with quark matter by solving the Einstein's field equations using anisotropy property of the universe, time-dependent displacement field and special law for Hubble's parameter that gives the constant value of deceleration parameter.

Shchigolev (2013) obtained a cosmological model within the framework of Lyra's geometry with an effective Λ-term in the field equations that appeared due to the interaction of the displacement vector field with an auxiliary Λ-term.

In a cosmological model in the framework of Lyra's geometry, Hova (2013) established a relationship between the displacement vector field, the energy density of matter and Hubble's parameter through an arbitrary function $\alpha(t)$ and obtained an effective equation of state parameter ω_{eff} in terms of $\alpha(t)$ and constant equation of state ω_m . The effective equation of state parameter ω_{eff} was completely determined for pressure-less matter by $\alpha(t)$. Consequently, he had obtained exact solutions for the models in Lyra's geometry that yield the ΛCDM and Power-Law Expansion.

Studying an inhomogeneous Bianchi Type-I metric in presence of an electromagnetic field, Megied et al. (2014) obtained a cosmological model in the framework of Lyra geometry. Assuming the metric potentials and displacement field as functions of coordinates x ant t, they had obtained a class of exact solutions of the Einstein's field equations.

In the framework of Lyra's geometry, Darabi et al. (2015) studied about the existence of the Einstein's static universe for homogeneous scalar perturbations together with the stability condition and obtained the stability condition in terms of the equation of state parameter ω as $\omega = p$. Also, they had studied the stability conditions for tensor and vector perturbations. They showed that, in the framework of Lyra's geometry, Einstein's static universe can be obtained for appropriate values of physical parameters.

In order to describe the evolution of the universe, Saadat (2016) formulated a new cosmological model based on extended Chaplygin gas with varying Λ-term in the context of Lyra geometry where extended Chaplygin gas is taken as dark matter and quintessence scalar field is considered as dark energy.

Sahoo, Nath and Sahu, (2017) studied Bianchi type-III cosmological model for a cloud of string with bulk viscosity in Lyra geometry. To get deterministic models of universe, they have assumed two conditions (i) $\xi = \xi_0$ = constant and (ii) shear scalar (σ) proportional to the scalar expansion (Θ). This condition leads to$B = C^n$ where ξ is the coefficient of bulk viscosity, B and C are metric potentials and n a constant. They also discussed some physical and geometrical aspects of the model.

Mollah, Singh and Singh (2018) studied with the investigation of a homogeneous and anisotropic spacetime described by Bianchi type-III metric with perfect fluid in Lyra geometry setting. Exact solutions of Einstein's field equations have been obtained under the assumption of quadratic equation of state (EoS) of the form $p = a\rho^2 - \rho$, where a is a constant and strictly $a > 0$. The physical and geometrical aspects are also examined in detail.

Maurya and Zia (2019) developed a new cosmological model in Einstein's modified gravity theory using two types of modification: (i) Geometrical modification, in which they have used Lyra's geometry in the left-hand side of the Einstein field equations (EFE), and (ii) modification in gravity (energy momentum tensor) on the right-hand side of EFE, as per the Brans-Dicke (BD) model. With these two modifications, they have investigated spatially homogeneous and anisotropic Bianchi type-I cosmological models of Einstein's Brans-Dicke theory of gravitation in Lyra geometry.

Mollah and Singh (2021) studied the aspects of bulk viscous fluid cosmological model with quadratic equation of state in the presence of strings loaded with particles in a higher dimensional (5- dimensional) Bianchi type-III geometry in Lyra's Manifold (Lyra, 1951). Interestingly they saw that both bulk viscosity and quadratic equation of state are acting crucial jobs throughout the evolution of the model which is expanding with acceleration so it represents dark energy model universe.

1.23 Saez-Ballester scalar-tensor theory

Saez and Ballester (1986) developed a scalar-tensor theory in which the metric is coupled with a dimensionless scalar field. Saez and Ballester (1986) assumed the Lagrangian

$$L = R - \omega\phi^n(\phi^{\mu}_{,k}\phi^{,k}), \tag{1.23.1}$$

where R is the curvature, ϕ is the dimensionless scalar field, ω and n are arbitrary dimensionless constants and $\phi^{,i} = g^{ij}\phi_{,j}$. For scalar field having the dimension $\phi =$

G^{-1}, the Lagrangian given by equation (1.23.1) has different dimensions. However, it is a suitable Lagrangian in the case of a dimensionless scalar field. From the Lagrangian one can build the action

$$I = \int_{\Sigma} (L + GL_m) \sqrt{-g}\, dx\, dy\, dz\, dt, \tag{1.23.2}$$

here L_m is the matter Lagrangian, $g =\mid g_{ij} \mid$, Σ is an arbitrary region of integration and $G = 8\pi$. Also the variational principle

$$\delta I = 0 \tag{1.23.3}$$

leads to the Saez and Ballester (1986) field equations for combined scalar and tensor fields given by

$$R_{ij} - \frac{1}{2}Rg_{ij} - \omega\phi^n \left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}\right) = -8\pi\phi^{-1}T_{ij} \tag{1.23.4}$$

and the scalar field ϕ satisfies the equation

$$2\phi^n\phi^{,i}_{;i} + n\phi^{n-1}\phi_{,k}\phi^{,k} = 0 \tag{1.23.5}$$

also

$$T^{ij}_{\ ;j} = 0. \tag{1.23.6}$$

The study of cosmological models in the framework of scalar tensor theories has been the active area of research for the last few decades. In particular, Singh and Agrawal (1991); Shari Ram and Tiwari (1998) are some of the authors who have investigated several aspects of the cosmological modles in Saez-Ballester scalar tensortheory. Bisabr (2009) has studied the holographic DE model in a generalized scalar tensor theory.He has shown that various types of potentials,the equation of state parameter is negative and transition from deceleration to acceleration expansion of the universe is possible. In recent years, Rao et al. (2011) have discussed anisotropic universe with cosmic strings and bulk viscosity in this scalar tensor theory of gravitation.

Zeyauddin and Shri Ram (2011) have studied Bianchi type-V viscous fluid cosmological models in saez-Ballester theory of gravitation. Jamil et al. (2012) have obtained Bianchi type-I cosmology in generalized Saez-Ballester theory via noether gauge symmetry. Pradhan et al. (2013) have studied accelerating Bianchi type-V cosmology with perfect fluid and heat flow in Saez-Ballester theory. Reddy et al. (2014) have investigated Bianchi type-V bulk viscous string cosmological model in Saez-Ballester scalar–tensor theory of gravitation. Chand and Shri Ram (2015) have discussed anisotropic cosmological models with bulk viscosity and particle creation in Saez-Ballester theory of gravitation.

Rao and prasanthi (2017) have analysed some Bianchi type modified holographic RDE modles in SB scalar-tensor theory of gravity with a variable deceleration parameter. Aditya and Reddy (2018a) have investigated anisotropic new holographic dark energy model in the framework of Saez-Ballester theory of gravitation. Mishra and Dua (2019) have studied bulk viscous string modles in Saez-Ballester theory of gravitation. Sharma et al. (2019a) have discussed transit cosmological models with perfect fluid and heat flow in Saez-Ballester theory of gravitation.

Archana et al. (2020) investigate a new class of LRS Bianchi type-II cosmological by considering a new deceleration parameter (DP) depending on the time in string cosmology for the modified gravity theory suggested by Sáez–Ballester. They have considered the energy–momentum tensor proposed by Letelier for bulk viscous and perfect fluid under some assumptions. They have substantiated a new class of cosmological transit models for which the expansion takes place from the early decelerated phase to the current accelerated phase. Also, they have studied some physical, kinematic and geometric behaviour of the models, and have found them consistent with observations and well-established theoretical results. They observed that the results are better, stable under perturbation and in good agreement with cosmological reflections.

Naidu et al. (2021) investigate the dynamical behaviour of Kaluza-Klein (KK)

FRW type dark energy cosmological models. Three cosmological models are presented by solving the field equations using (i) hybrid expansion law given by Pradhan et al. (ii) varying deceleration parameter proposed by Mishra et al. and (iii) linearly varying deceleration parameter defined by Akarsu and Dereli. They have evaluated the dynamical parameters for each of the models, namely, the equation of state (EoS) parameter, the deceleration parameter, statefinder parameter and total energy density parameter of dark energy. They have also found the scalar field in the models and discussed the dynamical behavior of the parameters through graphical representation with special reference to Planck Collaboration data. It is observed that the models describe accelerated expansion of the universe and our theoretical results are, reasonably, in good agreement with the observational data.

1.24 Entropy

Entropy is a property of the equilibrium states of a system. It is the measure of a system's thermal energy per unit temperature. The entropy of our entire universe is enormous dominated by supermassive black holes, the entropy density is remarkably small. Because of our universe having singularity that ballooned out and continues expanding on the time entropy is constantly growing in our universe.

The role of entropy in cosmology remain essential to study the transverse of irreversible energy flow from gravitational field to matter creation. It also influence of the specific models of dark energy on the thermodynamical properties of the FLRW universe and examines the energy conditions in these scenarios.

1.25 Dark energy

The universe is expanding which is believed to be driven by some exotic dark energy (Perlmutter et al., 1999; Reiss et al., 1998; Spergel et al., 2003, 2007). The nature and composition of dark energy is still an open problem. Also, it is commonly believed by the cosmological community that this hitherto unknown exotic physical entity known as dark energy is a kind of repulsive force which acts as antigravity responsible for gearing of the universe.

It has been conjectured that the simplest dark energy candidate is the cosmological constant, but it needs to be extremely fine tuned to satisfy the current value of the dark energy. (Srivatsva, 2005; Bertolami et al., 2004; Bento et al., 2002) considered Chaplygin gas as a source of dark energy dark energy because of negative pressure. Some authors have also suggested that interacting and non-interacting two fluids scenario are possible dark energy candidates (Setare, 2007; Setare et al., 2009; Pradhan et al., 2011). Dark energy causing late time acceleration of the universe by adding a function (Nojiri and Odinstov, 2003; Caroll et al., 2004; Abdalla et al., 2005; Mena et al., 2006). A review on modified gravity as an alternative to dark energy is made available by Nojiri and Odinstov (2007) and Copeland et al. (2006). In spite of these attempts cosmic acceleration is, still, a challenge for modern cosmology.

Cosmological models based on dark energy have been widely investigated by Sami et al. (2005), Li et al. (2011), Wang et al. (2007), Jamil and Rashid (2008), Zimdahl and Pavon (2007). These models yield stable solutions of FRW equations at late times of evolving universe. Farooq et al. (2011) have investigated dynamics of interacting phantom and quintessence dark energy. Yadav and Yadav (2011) and Adhav et al. (2011) are some of the authors

Caozziello and Luongo (2018) have analysed the information entropy and dark energy evolution. The adiabatic evolution can be investigated in such an approach by defining a dark temperature that matches information entropy with standards thermodynamics. Capozziello and Sen (2019) have investigated model independent constraints on dark energy evolution from low-redshift observations.They have presented, the constraint on sound speed for the total fluid of the universe, and for the dark energy fluid, rules out the possibility of a barotropic fluid model for unified dark sector and barotropic fluid model for dark energy.Very Recently, Mishra et al. (2021) have studied stability analysis of two fluid dark energy models.

1.26 Objectives of the thesis

(i) To study whether the fifth dimension in Robertson Walker universe plays a vital role in early stage evolution of universe and in driving the present accelerated expansion of the universe.

(ii) To discuss the role of perfect fluid in investigating an inflationary model universe in modified theory of gravitation.

(iii) To study the role of bulk viscosity, which plays a significant role in the present scenario of the evolution of the universe which in turn contribute to a better understanding of spatially homogeneous and isotropic accelerating universe in five dimensions.

(iv) To test the stability of FRW model by examining the different energy condition.

(v) To investigate the FRW cosmological model in the context of Einstein theory of gravitation.

(vi) To study five-dimensional FRW model universe in scalar-tensor theory of Gravitation using quadratic equation of state.

(vii) To investigate the higher dimensional flat FRW model with cosmological Variable G and Λ.

1.27 Problems investigated

In this section, we mention, in brief, the problems investigated and the results achieved in this thesis.

In **Chapter 2**, we discussed about five-dimensional Robertson-Walker universe interacting with Brans-Dicke field. To obtain determinate solution of the field equations, we have used the relation for scale factor and curvature index which can take different values of $-1, 0, +1$ by considering three different cases. Physical properties of the model are also discussed in detail. Interestingly, it is found that the fifth dimension itself acts as a source of dark energy. In this chapter, Robertson-Walker universe have been studied in the context of Brans-Dicke theory of gravitation corresponding to perfect fluid distribution of matter source. After solving the field equations for this theory of gravitation, we have presented closed, open and flat Robertson-Walker radiating universe corresponding to perfect fluid in five-dimensional spacetime. In this Brans-Dicke scalar-tensor theory, the fifth dimension plays a vital role in early stage evolution of universe and in driving the present accelerated expansion of the universe. At a particular case, if $\omega \to -2$, we see that the scalar field $\phi \to 0$. In such case, the scale factor of the fifth dimension is considered to be the source of dark energy. Though it acts as a source of dark energy, the fifth dimension contracts and is therefore not visible to the present epoch. The solution obtained here represents a five-dimensional expanding universe and helps to discuss the role of perfect fluid in investigating an inflationary model universe in this modified theory of gravitation. So, our findings will be useful for better understanding of the present universe.

In **Chapter 3**, we investigate the role of bulk viscosity in present scenarios of the evolution in FRW model universe in the framework of Lyra's geometry. We derived the field equations when the source for energy-momentum tensor is composed of a bulk viscous fluid with cosmic strings. The Einstein's field equations are solved by assuming a constant deceleration parameter. In this work, the displacement vector is considered to be a function of time. The kinematic and physical properties of the model are also discussed by using some acceptable physical assumptions of scale factor for flat, open, and closed universe. We restricted our study to a constant deceleration parameter as predicted from observation. The solutions of the model have been obtained for flat, closed, and open bulk viscous string FRW universe in five dimensions. The physical parameters have been plotted for $b \neq -1$. However, in the case of $b = -1$, all the parameters vanish rapidly within a short period of time. This fact indicates that the solution represents an early era of the evolution of the universe. The incorporation of bulk viscosity in our investigation is to replace the condition of material distribution other than perfect fluid. The bulk viscosity plays a significant role in the present scenario of the evolution of the universe. So, our model will contribute to a better understanding of spatially homogeneous and isotropic accelerating universe (Bamba, 2012) in five dimensions.

In **Chapter 4**, we discussed the Einstein's field equations based on Lyra's manifold in normal gauge is studied in a FRW line element for a five-dimensional cosmological model universe. Considering the power law expansion as $a(t) = t^n$ where n is a parameter and the shear scalar to be proportional to expansion scalar so as to obtain $A = R^m$ where m is an arbitrary constant, we have examined some of the energy conditions such as null energy condition (NEC), weak energy condition (WEC), dominant energy condition (DEC) and strong energy condition (SEC) for the open and closed universe. Here in this chapter we have presented a five-dimensional FRW cosmological model by considering power law expansion as $a(t) = t^n$ with certain physical assumption of the scalar σ and expansion scalar θ interact with perfect fluid. We also made the assumption based on observed relation between velocity and red shift for an extra galactic source which predicted the Hubble expansion is isotropic. Our solution supports the finding of Thore (1967) and Kristran and Sachs (1966). We also presented for the different model of universe like open and closed universe. In order to test the stability of our proposed model we have examined the Energy Condition such as Null energy condition (NEC) ,Weak energy Condition (WEC), Dominated energy condition (DEC) and Strong energy Condition (SEC). In all the conditions we found that our proposed model supports the condition of present observational findings. Such a model will be benefited to the new researchers to investigate about the evolution of our present day universe other than the other Cosmological Models.

In **Chapter 5**, the modern astronomical research is more attractive with different fluid contents present in the universe which yields significant mysterious results that gives moral boost to study the contents of the universe with various alternate theories as well. Here we have analysed the Einstein theory as a source of discussion with thermodynamical effect within it. To study the model in a diversified way we have considered the dark energy of the universe in terms of time varying cosmological parameters of the universe. For a specific assumption the obtained model indicated a phantom phase during spatially open universe and quintessence phase for other different assumptions. We conclude from our observations that the obtained model is valid for flat and closed universe but remain conditionally valid for open universe which is acceptable one. Study of early stage of the universe with FRW cosmological models in the frame work of Einstein theory plays an important role. Also, it is well established that the mathematical formulation of different cosmological models through the laws of physics becomes an essential component in understanding the nature of the universe. Hence, in this chapter, we have investigated FRW cosmological model in the context of Einstein theory of gravitation. We have studied time varying dark energy states of two different assumptions, from which we found a phantom phase during spatially open universe for $\Lambda \propto [a(t)]^n$ and all remaining results indicates a quintessence phase. We observed that the Hubble parameter approaches to infinite when time approaches to zero, this indicates the universe describes a power law inflation. The temperature and entropy density of the model remain positive for both the cases. In view of energy conditions, the assumptions yields identical results. Our study suggests the Strong Energy Condition violates for our model, that indicates an accelerating expansion of the universe. From our discussion we conclude that during both the assumptions the second law of thermodynamics remain impactless. Moreover, the study suggests our universe is of finite life time. All the obtained results are consistent with respect to observational constraints.

In **Chapter 6**, a five-dimensional Friedmann-Robertson-Walker (FRW) cosmological spacetime is considered in the scalar-tensor of gravitation proposed by Saez and Ballester using Quadratic equation of state. The Einstein field equation is solved using Scale factor $R = e^{\alpha t}$ (de Sitter universe) where α is constant, which always give a deceleration parameter $q = -1$. The behavior of flat, open and closed models is presented and discussed under various scenarios. In this chapter, we attempt to explain the behaviour some of unknown phenomenon of the universe in five-dimensional FRW model universe in scalar tensor theory of gravitation using quadratic equation of state is studied with the use of certain physical assumptions, which are agreeing with the present observational findings. The field equations for five-dimensional FRW model universe in scalar tensor theory of gravitation have been obtained and exact solutions are obtained. The model represents to have anisotropic phase throughout the evolution of the universe which is in agreement with the present observational data made by COBE (Cosmic Background Explorer) and WMAP (The Wilkinson Microwave Anisotropy Probe). Also, the model represents an expanding universe that starts with small finite volume at cosmic time $t = 0$ and expands with acceleration. Our model satisfies the energy conditions $\rho \geq 0$. Also, the shear scalar become nonzero as $t \to \infty$. So, our model represents a shearing cosmological model universe for large values of cosmic time t.

In **Chapter 7**, this chapter deals by considering a five-dimensional homogeneous and isotropic FRW model with varying gravitational and cosmological constant with time t. Exact solution of the Einstein field equations are obtained by using the equation of state $p = (\gamma - 1)\rho$ (gamma law), where γ which is an adiabatic parameter varies continuously as the universe expands. We obtained the solutions for flat model using $R = e^{\beta t}$, where β is a constant as the scale factor. Physical parameters of the models are discussed. In this chapter we have investigated a higher dimensional flat FRW model with variable G and Λ. The cosmological parameters and state finder parameters have been obtained for dust, radiation and stiff matter. The different models are obtained for different stages of the universe. We have discussed the physical parameters of the models.

The constant G and Λ are allowed to depend on the cosmic time t. We hope that our results may throw some light in understanding of the real universe. This study will throw some light on the structure formation of the universe, which has astrophysical significance. The expanding universe has singular at $t = 0$. In this way the unified description of early evolution of the universe is possible with variables G and Λ in the framework of higher dimensional space time.