Chapter 2

Higher Dimensional Robertson-Walker universe Interacting With Brans-Dicke Field

2.1 Introduction

The study of Brans-Dicke scalar field has attained significant attention in the current years as it describes most of the important features of the progress of the universe during the later time dynamical epoch. Though Einstein's General Relativity is considered to be the most useful theories in investigating various cosmological models, it lacks in explaining certain physical observations. Some examples include present accelerated expansion of our universe, inconsistency with Mach's principle, existence of big bang singularity etc. To deal with those problems, in recent years, several alternative theories of gravity are studied. The scalar-tensor theories are considered to be the simplest and best understood modified theory of gravitation. Brans-Dicke theory is in fact a deformation of the Einstein's General Relativity

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allowing variable gravity coupling that the gravitational constant becomes time dependent, . It is somewhat classical in nature and for that reason it is expected to play a crucial role in the late-time evolution of the universe.

Many researchers investigated for new gravitational theories using extra dimensions beyond the existing four dimensional spacetime. Mention may be made of Nordstrom (1914) who investigated unified theory based on extra dimension and Kaluza (1921) and Klein (1926) theory in which five-dimensional relativity theory was established whereas an extra fifth dimension give rise to electrodynamics. The matter source of four dimensional spacetime can be taken as a manifestation of extra dimension. Thus five-dimensional field equations without matter sources can be reverted back to four dimensional field equations with matter source. In this way the extra dimension itself can manifest as some kind of matter source. The concept of Kaluza Klein theories, extra dimensions, higher dimensional unified theories, supergravity are studied by many researchers (Freund, 1982; Overduin, 1997; Cho, 1992; Weinberg, 1972). Manihar and Priyokumar (2012) investigated the string cosmological models in the Brans-Dicke theory for five-dimensional space time. They discussed the role of scalar field in evolving through different phases of universe and found a "bounce" at a particular instant of its evolution. Due to inconsistency of Mach's principle with general relativity, Brans and Dicke (1961) have given the idea of alternate relativistic theory of gravitation. The Brans-Dicke cosmology in four dimension from scalar-vacuum in five dimension was studied by Leon (2010a, 2010b). According to his study, the observed accelerated expansion of the universe can be explained by Brans-Dicke theory in five dimension without recurring to matter fields in five dimension or dark energy in four dimension. Aguilar (2008) showed that five-dimensional Brans-Dicke vacuum field equations turns out to be new Brans-Dicke theory when brought back to four dimension. The scalar field ϕ in Brans-Dicke theory, plays a great role in expansion and contraction of the universe. Priyokumar and Dewri (2015) found that ϕ behave as something reflecting the contraction of universe when increases with time and also behave as something reflecting the expansion of the universe when decreases with time.

Many authors (Naidu, 2013, 2015; Reddy, 2007, 2012) investigate cosmological models in five-dimensional spacetime in the context of BD theory of gravitation . The possible different candidates of dark energy are studied by many authors, Mishra and Sahoo (2014) , Rao et al. (2012) , Rao and Nilima (2013), Saha and Yadav (2012), Samanta (2013a), Samanta et al. (2013b), Katore et al. (2011), Bali and Singh (2012), Mahanta et al. (2014). Based on the concept of induced matter theory, Bahrehbakhsh (2011) gives geometrical interpretation of dark energy by considering non-vacuum five-dimensional version of general relativity. Khadekar (2015) assume linear combination of bulk viscosity and time dependent parameter Λ with inhomogeneous equation of state in FRW spacetime to demonstrate and to explain the dark energy dominated universe. Manihar and Priyokumar (2016) investigated the interaction of gravitational field and Brans-Dicke field in Robertson Walker universe containing Dark Energy like fluid. Priyokumar (2013) investigated on physical distributions in Brans-Dicke cosmology under flat Robertson-Walker universe.

Motivated from the above literatures, in this chapter, we consider a five-dimensional Brans-Dicke theory, to investigate about the role of the extra fifth dimension with some acceptable physical assumptions of scale factor for open, closed and flat models in three different cases. The chapter is presented as follows: Section 2.2 consist of metric and field equations, Section 2.3 deals with solution of the field equations, physical discussion is presented in Section 2.4 and conclusion is given in Section 2.5.

2.2 Metric and field equations

The vacuum Brans-Dicke field equations in the general form are given by

$$R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = -\frac{8\pi}{\phi}T_{ij} - \frac{\omega}{\phi^2}\left[\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi^{,s}\phi_{,s}\right] - \frac{1}{\phi}\left(\phi_{,ij} - g_{ij}\phi^{,s}_{;s}\right), \quad (2.2.1)$$

$$(4+3\omega)\phi^{,s}_{;s} = 8\pi T, \quad (2.2.2)$$

where ϕ is the scalar field and Λ is the cosmological constant. For a perfect fluid matter distribution, the energy-momentum tensor is given by,

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij}, \tag{2.2.3}$$

where ρ is the energy density and p is the isotropic pressure. Let us consider the Robertson-Walker space time metric

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] - Q^2(t)d\psi^2, \tag{2.2.4}$$

where $R(t)$ is the scale factor and k is the curvature index which can take up the values $(-1, 0, +1)$ for open, flat, closed model of the universe respectively.

The Brans-Dicke field equations (2.2.1) and (2.2.2) together with (2.2.3) for the metric (2.2.4), becomes

$$\dot{Q}Q + \frac{\dot{R}^2}{R^2}\frac{\dot{Q}}{Q} + 2\frac{\ddot{R}}{R} + \frac{k}{R^2} - \Lambda = -\frac{8\pi\rho}{\phi} + \frac{\omega}{2}\frac{\ddot{\phi}^2}{\phi^2} - \frac{1}{\phi}\left[-2\frac{R\dot{\phi}}{R} - \phi + \frac{\ddot{Q}}{Q}\dot{\phi}\right], \tag{2.2.5}$$

$$\frac{\ddot{Q}}{Q} + \frac{\dot{R}^2}{R^2}\frac{\dot{Q}}{Q} + 2\frac{\ddot{R}}{R} + \frac{k}{R^2} - \Lambda = -\frac{8\pi\rho}{\phi} - \frac{\omega}{2}\frac{\ddot{\phi}^2}{\phi^2} + \frac{1}{\phi}\left[-2\frac{R\dot{\phi}}{R} + \ddot{\phi} - \frac{\ddot{Q}}{Q}\dot{\phi} + \ddot{\phi}\right], \tag{2.2.6}$$

$$3\left(\frac{k}{R^2} + \frac{\dot{R}^2}{R^2} + \frac{\ddot{R}}{R}\right) - \Lambda = \frac{8\pi p}{\phi} + \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} - \frac{1}{\phi}\left[3\frac{\dot{R}\dot{\phi}}{R} + \ddot{\phi}\right], \tag{2.2.7}$$

$$3\left(\frac{k}{R^2} + \frac{\dot{R}^2}{R^2} + \frac{\dot{R}}{R}\frac{\dot{Q}}{Q}\right) - \Lambda = \frac{8\pi p}{\phi} - \frac{\omega}{2}\frac{\dot{\phi}^2}{\phi^2} - \frac{1}{\phi}\left[-3\frac{\dot{R}\dot{\phi}}{R} - \frac{\dot{Q}}{Q}\dot{\phi}\right], \tag{2.2.8}$$

and

$$(4 + 3\omega)\left[\ddot{\phi} + 3\frac{\dot{R}}{R}\dot{\phi} + \frac{\dot{Q}}{Q}\dot{\phi}\right] = 8\pi(\rho - 4p), \tag{2.2.9}$$

Here a overhead dot denotes differentiation with respect to time t. From (2.2.5) and (2.2.6), we get

$$Q = M\phi^{\frac{\omega}{2}}, \tag{2.2.10}$$

Q being the interacting term between matter (including dark matter) and dark energy. To get the determinate solution, we assume the physical condition for five-dimensional radiating model (Reddy, 2015)

$$\rho = 4p. \tag{2.2.11}$$

Thus, from equation (2.2.9) and (2.2.11), we obtain

$$\frac{\phi^{\frac{\omega}{2}+1}}{\frac{\omega}{2}+1} = N\int \frac{1}{R^3}dt, \tag{2.2.12}$$

where N is an arbitrary constant.

2.3 Solution of the field equations

The field equations (2.2.5) – (2.2.9) are a system of four independent equations connecting five unknowns Q, R, ϕ, ρ and p. To get the determinate solution, let us assume that: The scale factor $R(t)$ is

$$R(t) = \sqrt{a + bt - kt^2}. \tag{2.3.1}$$

The scalar field ϕ is

$$\phi = \left[-\frac{N(w+2)(b-2kt)}{(4ak+b^2)\sqrt{(a+bt-kt^2)}}\right]^{\frac{2}{\omega+2}}. \tag{2.3.2}$$

Now,

$$Q = M\phi^{\frac{\omega}{2}} = M\left[-\frac{N(w+2)(b-2kt)}{(4ak+b^2)\sqrt{(a+bt-kt^2)}}\right]^{\frac{\omega}{\omega+2}}. \tag{2.3.3}$$

The gravitational constant G is

$$G = \frac{1}{\phi}\left(\frac{4+2\omega}{3+2\omega}\right),$$

$$G = \left[-\frac{N(w+2)(b-2kt)}{(4ak+b^2)\sqrt{(a+bt-kt^2)}}\right]^{-\frac{2}{\omega+2}}\left(\frac{4+2\omega}{3+2\omega}\right). \tag{2.3.4}$$

The Hubble's parameter H is

$$H = \frac{1}{2}\left[\frac{b-2kt}{a+bt-kt^2}\right]. \tag{2.3.5}$$

The Expansion factor θ is

$$\theta = \frac{3}{2}\left[\frac{b-2kt}{a+bt-kt^2}\right]. \tag{2.3.6}$$

For Λ (cosmological constant) = 0 :

The Energy density ρ is

$$\rho = \frac{1}{8\pi}\left[-\frac{N(w+2)(b-2kt)}{(4ak+b^2)\sqrt{(a+bt-kt^2)}}\right]^{-\frac{2}{\omega+2}}\left[\frac{3}{4}(\frac{b-2kt}{a+bt-kt^2})^2(\frac{\omega+6}{\omega+2})+\frac{3k}{a+bt-kt^2}\right]. \tag{2.3.7}$$

The Pressure p is

$$p = \frac{1}{32\pi}\left[-\frac{N(w+2)(b-2kt)}{(4ak+b^2)\sqrt{(a+bt-kt^2)}}\right]^{-\frac{2}{\omega+2}}\left[\frac{3}{4}(\frac{b-2kt}{a+bt-kt^2})^2(\frac{\omega+6}{\omega+2})+\frac{3k}{a+bt-kt^2}\right]. \tag{2.3.8}$$

2.3.1 Case I : $k = 0$, flat universe

The scale factor $R(t)$ is

$$R(t) = \sqrt{a+bt},\ L = -\frac{N}{b}. \tag{2.3.9}$$

The scalar field is

$$\phi = \left[\frac{L(\omega+2)}{\sqrt{(a+bt)}}\right]^{\frac{2}{\omega+2}}. \tag{2.3.10}$$

Now,

$$Q = M\phi^{\frac{\omega}{2}} = M\left[\frac{L(\omega+2)}{\sqrt{(a+bt)}}\right]^{\frac{\omega}{\omega+2}}. \tag{2.3.11}$$

The gravitational constant G is

$$G = \left[\frac{L(\omega+2)}{\sqrt{(a+bt)}}\right]^{\frac{-2}{\omega+2}}\left(\frac{4+2\omega}{3+2\omega}\right). \tag{2.3.12}$$

The Hubble's paramater H is

$$H = \frac{1}{2}\left[\frac{b}{a+bt}\right]. \tag{2.3.13}$$

The expansion scalar θ is

$$\theta = \frac{3}{2}\left[\frac{b}{a+bt}\right]. \tag{2.3.14}$$

The energy density ρ is

$$\rho = \frac{1}{8\pi}\left[\frac{L(\omega+2)}{\sqrt{(a+bt)}}\right]^{\frac{2}{\omega+2}}\left[\frac{3}{4}\left(\frac{b}{a+bt}\right)^2\left(\frac{\omega+6}{\omega+2}\right)\right]. \tag{2.3.15}$$

The pressure p is

$$p = \frac{1}{32\pi}\left[\frac{L(\omega+2)}{\sqrt{(a+bt)}}\right]^{\frac{2}{\omega+2}}\left[\frac{3}{4}\left(\frac{b}{a+bt}\right)^2\left(\frac{\omega+6}{\omega+2}\right)\right]. \tag{2.3.16}$$

2.3.2 Case II : $k = 1$, closed universe

The scalar factor $R(t)$ is

$$R(t) = \sqrt{a+bt-t^2},\ L = -\frac{N}{(4a+b^2)}. \tag{2.3.17}$$

The scalar field ϕ is

$$\phi = \left[\frac{L(\omega+2)(b-2t)}{\sqrt{(a+bt-t^2)}}\right]^{\frac{2}{\omega+2}}. \tag{2.3.18}$$

Now,

$$Q = M\phi^{\frac{\omega}{2}} = M\left[\frac{L(\omega+2)(b-2t)}{\sqrt{(a+bt-t^2)}}\right]^{\frac{\omega}{\omega+2}}. \tag{2.3.19}$$

The graviational constant G is

$$G = \left[\frac{L(\omega+2)(b-2t)}{\sqrt{(a+bt-t^2)}}\right]^{\frac{-2}{\omega+2}}\left(\frac{4+2\omega}{3+2\omega}\right). \tag{2.3.20}$$

The Hubble's paramater H is

$$H = \frac{1}{2}\left[\frac{b-2t}{a+bt-t^2}\right]. \tag{2.3.21}$$

The expansion scalar θ is

$$\theta = \frac{3}{2}\left[\frac{b-2t}{a+bt-t^2}\right]. \tag{2.3.22}$$

The energy density ρ is

$$\rho = \frac{1}{8\pi}\left[\frac{L(\omega+2)(b-2t)}{\sqrt{(a+bt-t^2)}}\right]^{\frac{2}{\omega+2}}\left[\frac{3}{4}\left(\frac{b-2t}{a+bt-t^2}\right)^2\left(\frac{\omega+6}{\omega+2}\right)+\frac{3}{a+bt-t^2}\right]. \tag{2.3.23}$$

The pressure p is

$$p = \frac{1}{32\pi}\left[\frac{L(\omega+2)(b-2t)}{\sqrt{(a+bt-t^2)}}\right]^{\frac{2}{\omega+2}}\left[\frac{3}{4}\left(\frac{b-2t}{a+bt-t^2}\right)^2\left(\frac{\omega+6}{\omega+2}\right)+\frac{3}{a+bt-t^2}\right]. \tag{2.3.24}$$

2.3.3 Case III : $k = -1$, open universe

The scale factor $R(t)$ is

$$R(t) = \sqrt{(a+bt+t^2)}, \quad L = -\frac{N}{(-4a+b^2)}. \tag{2.3.25}$$

The scalar field ϕ is

$$\phi = \left[\frac{L(\omega+2)(b+2t)}{\sqrt{(a+bt+t^2)}}\right]^{\frac{2}{\omega+2}}. \tag{2.3.26}$$

Now,

$$Q = M\phi^{\frac{\omega}{2}} = M\left[\frac{L(\omega+2)(b+2t)}{\sqrt{(a+bt+t^2)}}\right]^{\frac{\omega}{\omega+2}}. \tag{2.3.27}$$

The gravitational constant G is

$$G = \left[\frac{L(\omega+2)(b+2t)}{\sqrt{(a+bt+t^2)}}\right]^{\frac{-2}{\omega+2}}\left(\frac{4+2\omega}{3+2\omega}\right). \tag{2.3.28}$$

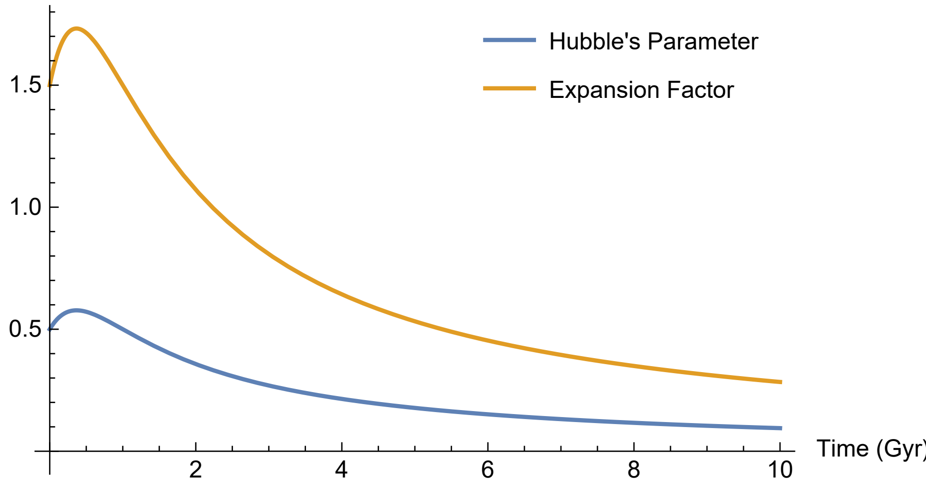

Figure 2.1: The variation of H and θ vs. Time t in Gyr for $k = 0$.

The Hubble's paramater is

$$H = \frac{1}{2}\left[\frac{b+2t}{a+bt+t^2}\right]. \tag{2.3.29}$$

The expansion scalar θ is

$$\theta = \frac{3}{2}\left[\frac{b+2t}{a+bt+t^2}\right]. \tag{2.3.30}$$

The energy density ρ is

$$\rho = \frac{1}{8\pi}\left[\frac{L(\omega+2)(b-2t)}{\sqrt{(a+bt+t^2)}}\right]^{\frac{2}{\omega+2}}\left[\frac{3}{4}\left(\frac{b+2t}{a+bt+t^2}\right)^2\left(\frac{\omega+6}{\omega+2}\right)-\frac{3}{a+bt+t^2}\right]. \tag{2.3.31}$$

The pressure p is

$$p = \frac{1}{32\pi}\left[\frac{L(\omega+2)(b+2t)}{\sqrt{(a+bt+t^2)}}\right]^{\frac{2}{\omega+2}}\left[\frac{3}{4}\left(\frac{b+2t}{a+bt+t^2}\right)^2\left(\frac{\omega+6}{\omega+2}\right)-\frac{3}{a+bt+t^2}\right]. \tag{2.3.32}$$

Taking $a = 1$, $b = 1$, $L = 1$, and $\omega = -1.5$, some of the parameters are plotted against time for $k = 0$ and $k = -1$.

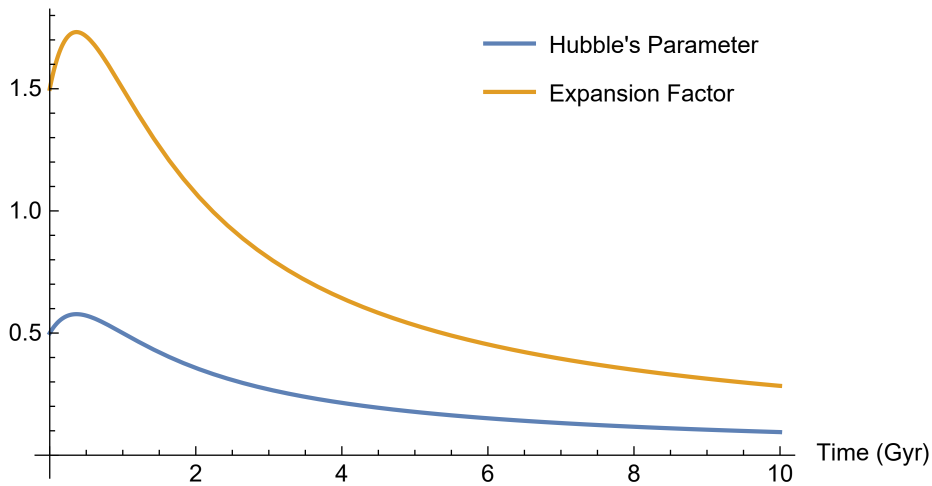

Figure 2.2: The variation of H and θ vs. Time t in Gyr for $k = -1$.

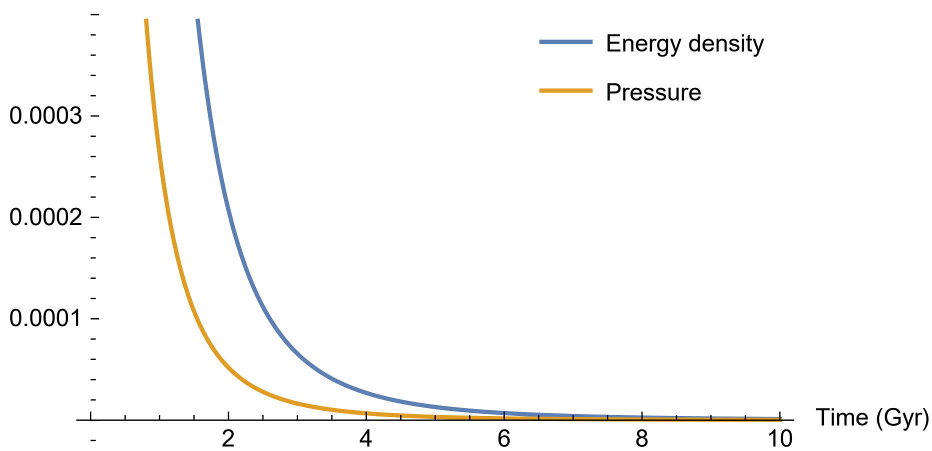

Figure 2.3: The variation of ρ and p vs. Time t in Gyr for $k = 0$.

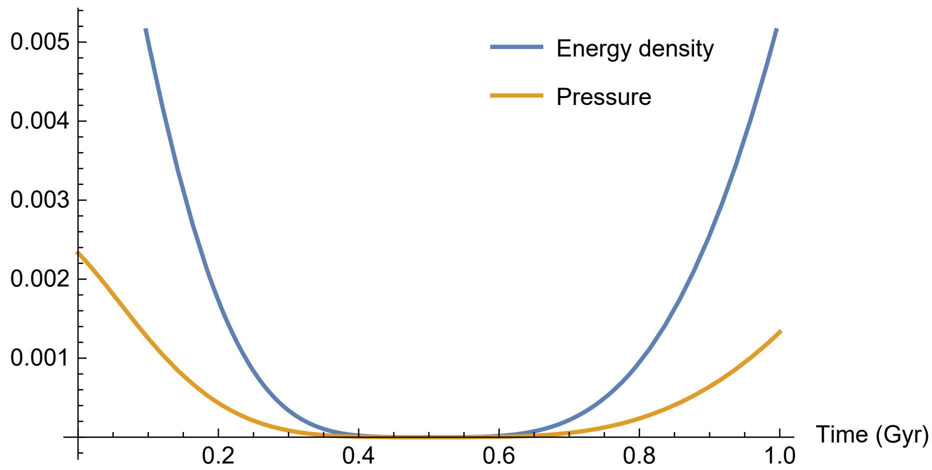

Figure 2.4: The variation of ρ and p vs. Time t in Gyr for $k = -1$.

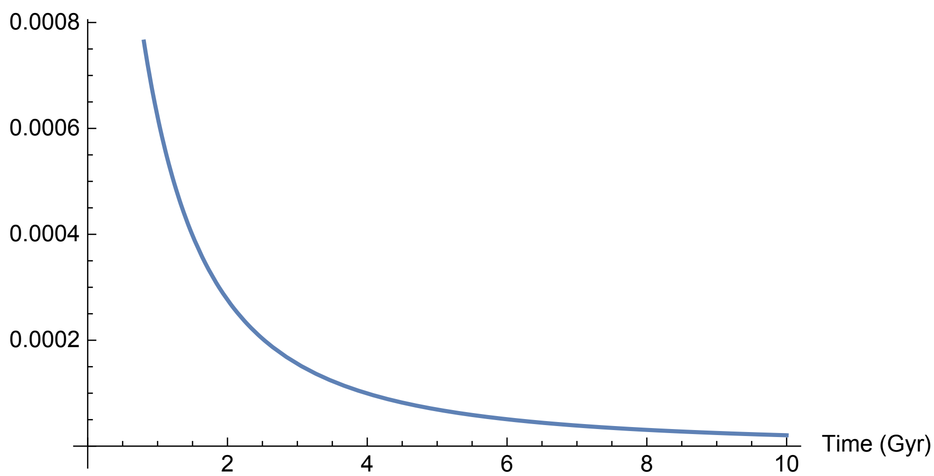

Figure 2.5: The variation of ϕ vs. Time t in Gyr for $k = 0$.

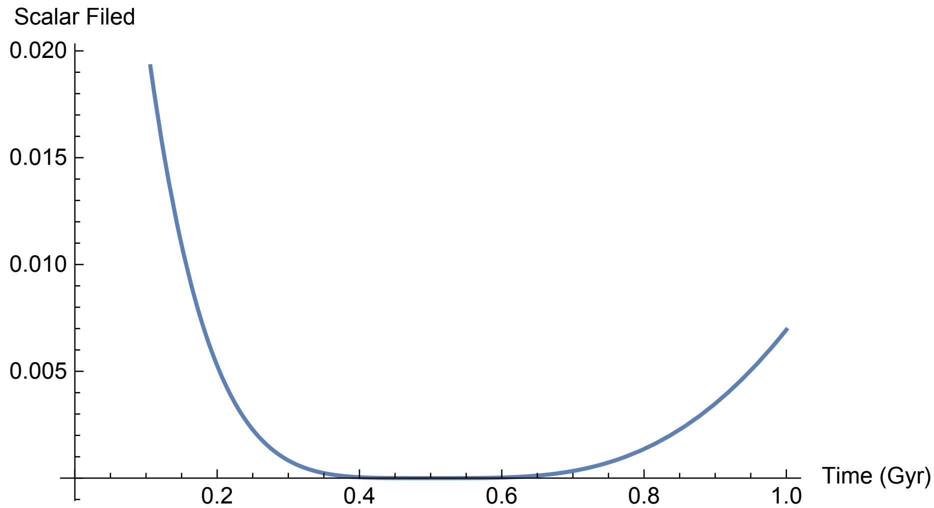

Figure 2.6: The variation of ϕ vs. Time t in Gyr for $k = -1$

2.4 Physical Interpretations

From the above analytical expressions and with the help of their graphical representation, their physical behaviours can be smoothly understood. In the cases, $k = 0$ and $k = -1$, both the Hubble's parameter and expansion factor decreases with time and vanishes at infinitely large t. The pressure and energy density are decreasing function of time t and their positivity condition are achieved in flat and open model universe. The behaviour of scalar field ϕ for flat model is quite familiar with that for open model. It decreases with time and tends to zero for infinitely large time for $k = 0$. Also, we see that Hubble's parameter decreases with time which represents an expanding universe with an accelerated rate and is consistent with the present observational findings.

2.5 Conclusion

In this chapter, Robertson-Walker universe have been studied in the context of Brans-Dicke theory of gravitation corresponding to perfect fluid distribution of matter source. After solving the field equations for this theory of gravitation, we have presented closed, open and flat Robertson-Walker radiating universe corresponding to perfect fluid in five-dimensional spacetime. In this Brans-Dicke scalar-tensor theory, the fifth dimension plays a vital role in early stage evolution of universe and in driving the present accelerated expansion of the universe. At a particular case, if $\omega \to -2$, we see that the scalar field $\phi \to 0$. In such case, the scale factor of the fifth dimension is considered to be the source of dark energy. Though it acts as a source of dark energy, the fifth dimension contracts and is therefore not visible to the present epoch. The solution obtained here represents a five-dimensional expanding universe and helps to discuss the role of perfect fluid in investigating an inflationary model universe in this modified theory of gravitation. So, our findings will be useful for better understanding of the present universe.