Chapter 5

Thermodynamical Approach on FRW spacetime for various Dark energy conditions

5.1 Introduction

Recent studies suggest that it is a very challenging job for understanding the fate of the accelerating expansion of the universe in view of Type Ia Supernova by (Reiss, 2000; Permutter et al., 1999). Various studies have been done to explain this special discovery (Copeland, Sami and Tsujikawa, 2006; Ratra and Vogeley, 2008). It is an interesting component that considered as dark energy possessing a negative pressure and is recommended to understand the accelerating expansion of the universe. The current simplest candidate for a standard model of cosmology and a good understanding with most observations (Frieman, Turner and Huterer, 2008; Jassal, Bagla and Padmanabhan, 2010) is the ΛCDM model which is required due to two major problems as fine-tuning or why so small and coincidence (Weinberg, 1989; Zlatev, Wang and Steinhardt, 1999; Chen, Zhu, Alcaniz and Gong, 2010). There

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are mainly three different method to express the dark energy problem *i.e.,* dynamical dark energy (Copeland, Sami and Tsujikawa, 2006), the anthropic principle (Weinberg, 2000) and interacting dark energy (Amendola, 2000), (Caldera-Cabral, Maarteens and Urena-Lopez, 2009). Out of which, dynamical dark energy has an hypothetical form called as Quintessence which is reported as a scalar field minimally connected to gravity which can fall to the late time inflation accelerating cosmological expansion for some particular form of potential, Also, due to these particular potentials it lighten the cosmological coincidence model (Chen, Zhu, Alcaniz and Gong, 2010). In this chapter, we intend to focus on this quintessence and phantom phase models. By (Prigogine, Geheniau, Gunzig and Nardone, 1988) the role of thermodynamics in cosmology remain essential to study the transverse of irreversible energy flow from gravitational field to matter creation, which also helps to transform into matter from spacetime. As in (Prigogine, Geheniau, Gunzig and Nardone, 1988) the irreversible matter creation, the big-bang initial singularity remain unstable. This dissipative process of the Einstein field equations leads to the possibility of cosmological model from empty space to creation of matter and entropy. Gravitational entropy remain meaningful as associated with the entropy which is necessary to produce matter. This extend signifies the possibility of impact fullness of third law of thermodynamics.

As the source of dark energy of the current phase of universe can modify the horizon entropy, so its thermodynamics in both cosmological as well as gravitational set ups (Moradpur, Sadehnezhad, Ghaffari and Jahan, 2017; Samanta, Myrakulov and Shah, 2017; Shekh, Arora, Chirde and Sahoo, 2020; Jamil, Momeni, Raza and Myrzakulov, 2012, Patra, Sethi, Nayak and Swain, 2019; Hawking and Ellis, 1973; Schoen and Yau, 1981) have more impactful, Also, it seems that properties of such modifications to the thermodynamics are in line with non-extensive thermodynamics of spacetime and the current universe.

Here, using Einstein theory of gravitation, we investigate the mutual relationship between the thermodynamic laws with the Einstein field equations. In order to study the model here we apply the thermodynamical laws of Apparent Horizon of FRW universe. As FRW metric is an exact solution EFE of general relativity, which describes an isotropic, homogeneous and expanding universe that is path connected. The solution of this model are proved its generic properties that are different from dynamical FL Model which are specific solutions for $a(t)$ that assumes the only contribution to stress energy and are cold matter, radiation and cosmological constant.

The present chapter is organizes as in Section 5.2 we have presented thermodynamical behaviour and entropy of the model where we have expressed the entropy production rate, Apparent Horizon and Cui-Kim temperature of the apparent horizon. Section 5.3 we derived some basis of Einstein gravity with detail solutions of FRW metric in two different cases with entropy density, energy conditions and thermodynamical temperature. In Section 5.4, we summarize our findings in details.

5.2 Thermodynamical Behaviour and entropy

Thermodynamical study has been an important tool to incept a gravitational theory. As pivotal event, black hole thermodynamics and recent Conformal field theory correspondence shows a strong corelation between gravity and thermodynamics also it has great significance on recent observations. From the thermodynamics, the interaction between first and second law of thermodynamics with volume V (Prigogine, Geheniau, Gunzig and Nardone, 1988) can be expressed as

$$\tau ds = d(\rho V) + \rho dV, \tag{5.2.1}$$

where τ and s represents the temperature and entropy respectively. Above equation can be written as

$$\tau ds = d(p + \rho)V - Vdp, \tag{5.2.2}$$

to define a perfect fluid as a thermodynamic system there is an integrability condition is required which can be written as

$$dp = \left(\frac{p+\rho}{\tau}\right) d\tau. \tag{5.2.3}$$

Using equations (5.2.2) and (5.2.3) we have the differential equation

$$ds = \frac{1}{\tau} d(p+\rho)V - (p+\rho)V \, \frac{d\tau}{\tau^2}. \tag{5.2.4}$$

Rewriting above equation

$$ds = d\left[\frac{(p+\rho)V}{\tau}\right]. \tag{5.2.5}$$

Therefore, the entropy is defined as

$$s = \left[\frac{(p+\rho)V}{\tau}\right]. \tag{5.2.6}$$

Let the entropy density is

$$s^* = \frac{s}{V} = \frac{p+\rho}{\tau} = \frac{(1+\omega)\rho}{\tau}. \tag{5.2.7}$$

Consider the apparent horizon of the universe for the assumed model is appeared at r_A where

$$r_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}}. \tag{5.2.8}$$

The entropy density in terms of temperature with the help of first law of thermodynamics can be expressed as

$$d(\rho V) + \omega \rho dV = (1+\omega)\tau d\left(\frac{\rho V}{\tau}\right), \tag{5.2.9}$$

which on integration yields

$$\tau = \rho^{\frac{\omega}{1+\omega}}. \tag{5.2.10}$$

From equation(5.2.3), we obtain

$$s^{*} = (1+\omega)\rho^{\frac{\omega}{1+\omega}}. \tag{5.2.11}$$

Now the Cui-Kim temperature of the apparent horizon can be obtained as

$$\tilde{T} = \frac{1}{2\pi r_A}. \tag{5.2.12}$$

Equation (5.2.6) represents the entropy which does not depends on any individual fluids and is only depends on the isotropic pressure and total matter density of the fluid. Many authors have investigated on thermodynamical aspects of cosmological model using different theories with different fluid contents such as Samant et al. (2017) have investigated on the validity of second law of thermodynamics using Kaluza-Klein metric using bulk viscosity in the context of $f(R,T)$ theory and found that the second law of thermodynamics doesn't hold for the assumed model. The remark on the actions of thermodynamic parameters is directly related to the energy density of the universe. Recently, Shekh et.al., (2020) investigated thermodynamical aspects of relativistic hydrodynamics in $f(R,G)$ gravity for accelerated spatially homogeneous and isotropic FRW cosmological model with a non-perfect (um-magnetized) fluid in the framework of $f(R,G)$ gravity model by defining entropy density by using the condition stated above. Jamil et al. (2012) have investigated on Horava-Lifshitz cosmology for thermodynamical validity in different types of universe and found that the model remain valid for closed and flat universe but conditionally valid for open universe which matches to the result obtained during our study.

5.3 Einstein field equation and their solutions

Einstein field equation can be written as follows

$$R_{ij} - \frac{1}{2}Rg_{ij} = -KT_{ij}, \tag{5.3.1}$$

where G_{ij} the Einstein tensor, Λ is cosmological constant which can be regarded as dark energy of the model introduced by Einstein and T_{ij} the energy-momentum tensor.

To study the universe it is quiet necessary to consider a metric by which the Einstein field equations can be evaluated and further solutions can be evaluated. Let's consider FRW metric with a maximally symmetric spatial section as

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{5.3.2}$$

where $R(t)$ the cosmic scale factor and the spatial curvature index $k = -1, 0, +1$ corresponds to spatially open, flat and closed universe respectively.
Now, consider the fluid representation for the energy-momentum tensor which can be written as follows

$$T_{ij} = [\rho, p, p, p]. \tag{5.3.3}$$

In a co-moving coordinate system the Einstein field equation (5.3.1) with the help of (5.3.2) and (5.3.3) yields

$$3H^2 + \frac{3k}{R^2} = \rho + \Lambda, \tag{5.3.4}$$

$$3H^2 + \frac{k}{R^2} + 2\dot{H} = -p + \Lambda, \tag{5.3.5}$$

where H stands for well-known Hubble parameter.

The above system of equation consists of two equation and four unknowns. To make the system consistent two additional constraints required. The well-known relation between pressure and energy density on considering the shear scalar and expansion scalar are proportional to each other as

$$p = \gamma\rho. \tag{5.3.6}$$

Here "γ" stands for equation of state parameter. This parameter takes a vital role to model the universe for different values of γ. For $\gamma = 0, \frac{1}{3}$ and -1 the model represents a dust, radiating and vacuum energy of the fluid. Similarly, for $\gamma < 0$ is considered as an accelerating expansion of the universe in the context of dark energy. Moreover, for different range of γ such as quintessence ($-1 < \gamma < 0$), phantom ($\gamma < -1$) and for quantum ($\gamma = -1$). Still, there is no clear understanding on EoS of Dark Energy yet.

To study the model in a simpler way here we have considered a linearly varying deceleration parameter (Patra, Sethi, Nayak, and Swain, 2019) as

$$q = -\alpha t + m - 1 \tag{5.3.7}$$

where α and m are scalar and $q = -\frac{R\ddot{R}}{\dot{R}^2}$ the deceleration parameter which helps to predict whether the model is accelerating or decelerating in nature.
The proposed form of deceleration parameter yields

$$q = a_1\left(\frac{\alpha t}{2m - \alpha t}\right)^{\frac{1}{m}}, \quad \alpha > 0, m > 0. \tag{5.3.8}$$

The Hubble parameter can be immediate calculated on using (5.3.8) as

$$H = -\frac{2}{t(\alpha t - 2m)}. \tag{5.3.9}$$

Hence the apparent horizon for the model is appeared at

$$r_A = \left(\frac{4}{t^2(\alpha t - 2m)^2} + \frac{k}{a_1^2}\left(\frac{\alpha t}{2m - \alpha t}\right)^{-\frac{2}{m}}\right). \tag{5.3.10}$$

The energy density and pressure can be calculated as follows

$$\rho = \frac{12}{t^2(\alpha t - 2m)^2} + \frac{3k}{a_1^2}\left(\frac{\alpha t}{2m - \alpha t}\right)^{-\frac{2}{m}} - \Lambda, \tag{5.3.11}$$

$$p = \frac{8(m - \alpha t) - 12}{t^2(\alpha t - 2m)^2} - \frac{k}{a_1^2}\left(\frac{\alpha t}{2m - \alpha t}\right) + \Lambda. \tag{5.3.12}$$

Now, the equation of state parameter can be calculated on using (5.3.6)

$$\gamma = \frac{8(m - \alpha t) + \Lambda t^2(\alpha t - 2m)^2 - \frac{k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2}{m}} t^2(\alpha t - 2m)^2 - 12}{12 - \Lambda t^2(\alpha t - 2m)^2 + \frac{k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2}{m}} t^2(\alpha t - 2m)^2}. \tag{5.3.13}$$

5.3.1 Case : I

For $\Lambda \propto t^{-2}$, i.e., $\Lambda = \Lambda_1 t^{-2}$.

This case gives the results of the model parameter as

$$\rho = \frac{12}{t^2(\alpha t - 2m)^2} + \frac{3k}{a_1^2}\left(\frac{\alpha t}{2m - \alpha t}\right)^{-\frac{2}{m}} - \Lambda_1 t^{-2}, \tag{5.3.14}$$

$$p = \frac{8(m - \alpha t) - 12}{t^2(\alpha t - 2m)^2} - \frac{k}{a_1^2}\left(\frac{\alpha t}{2m - \alpha t}\right) + \Lambda_1 t^{-2}, \tag{5.3.15}$$

$$\gamma = \frac{8(m - \alpha t) + \Lambda_1 t^2(\alpha t - 2m)^2 - \frac{k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2}{m}} t^2(\alpha t - 2m)^2 - 12}{12 - \Lambda_1 t^2(\alpha t - 2m)^2 + \frac{k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2}{m}} t^2(\alpha t - 2m)^2}. \tag{5.3.16}$$

Temperature

$$\tau = \left(\frac{12}{t^2(\alpha t - 2m)^2} + \frac{3k}{a_1^2}\left(\frac{\alpha t}{2m - \alpha t}\right)^{-\frac{2}{m}} - \Lambda_1 t^{-2}\right)^{\beta}, \tag{5.3.17}$$

where

$$\beta = \frac{\frac{8(m-\alpha t)+\Lambda_1 t^2(\alpha t-2m)^2-\frac{k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2}{m}} t^2(\alpha t-2m)^2-12}{12-\Lambda_1 t^2(\alpha t-2m)^2+\frac{k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2}{m}} t^2(\alpha t-2m)^2}}{1+\frac{8(m-\alpha t)+\Lambda_1 t^2(\alpha t-2m)^2-\frac{k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2}{m}} t^2(\alpha t-2m)^2-12}{12-\Lambda_1 t^2(\alpha t-2m)^2+\frac{k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2}{m}} t^2(\alpha t-2m)^2}}. \tag{5.3.18}$$

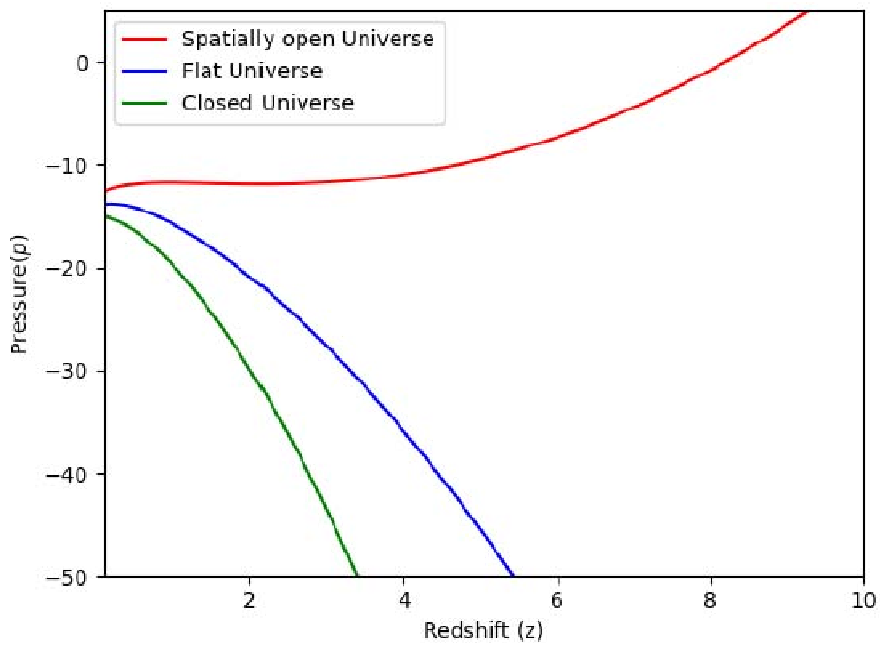

Figure 5.1: Pressure vs. Redshift for case-I

The entropy density

$$S_1 = (1+\gamma)\left(\frac{12}{t^2(\alpha t - 2m)^2} + \frac{3k}{a_1^2}\left(\frac{\alpha t}{2m - \alpha t}\right)^{-\frac{2}{m}} - \Lambda_1 t^{-2}\right)^{1-\beta}. \tag{5.3.19}$$

Fig. 5.1 and Fig. 5.2 depicts behavior of the universe for spatially open universe, flat and closed universe for pressure and energy density versus red shift respectively. From the above geometrical representation we conclude that the pressure is initially negative constant and later on it diverges. Whereas, the energy density remain positive constant initially and later on it increases for flat and closed universe but for open universe it diverges towards negative side which predicts that the model can't sustain for open universe at late time.

5.3.2 Energy conditions

In general relativity, the energy conditions have significant role, like the Hawking's Penrose singularity strong energy condition (Hawking and Ellis 1973) whereas to

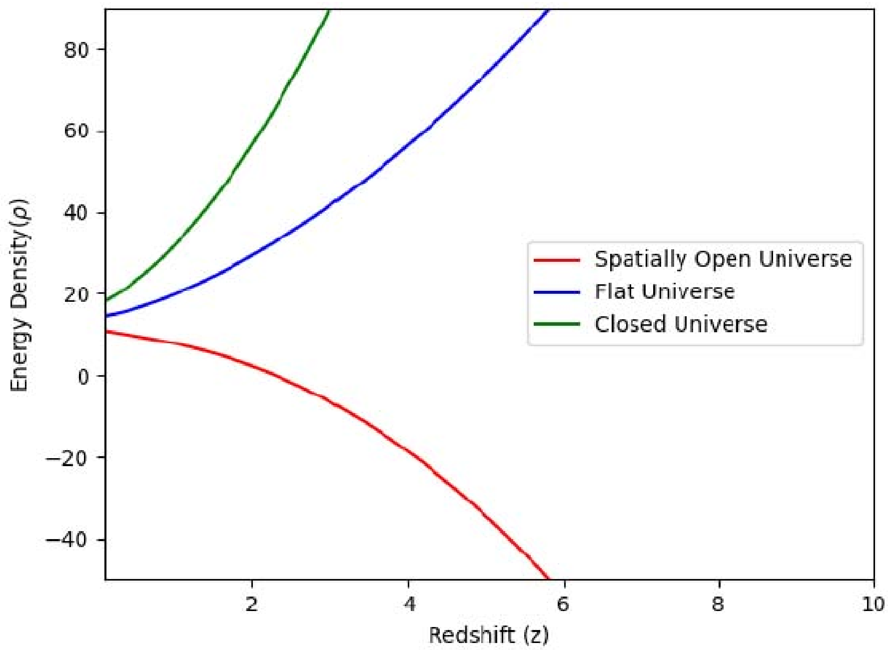

Figure 5.2: Energy Density vs. Redshift for case-I

verify the positive mass theorem, the dominant energy condition (Schoen and Yau, 1981) is required. It consists of a couple of constraints which characterise the nature of the obscurity of light-like, time-like or space-like curves. Furthermore, to identify the second law of black hole thermodynamics, null energy condition plays a major role (Caroll, 2004). The four different types of energy conditions are null, weak, strong and dominant energy conditions are respectively (i) $\rho + p \geq 0$ (ii) $\rho + p \geq 0$, $\rho \geq 0$ (iii) $\rho + 3p \geq 0$ (iv) $\rho > |p|$ (Visser, Woodbury,1995). The geometrical representation of the energy conditions are presented in Fig. 5.3, Fig. 5.4 and Fig. 5.5.

The above figures depicts that the Null energy condition and weak energy conditions are satisfied for flat and closed universe whereas for open universe it violates, as expected. also, for the case of dominant energy condition the closed, flat universe satisfied and the open universe violated. Moreover, the strong energy condition is violated for all the universe which indicates the universe is accelerating in nature, this matches to the results of Shekh et. al, (2020).

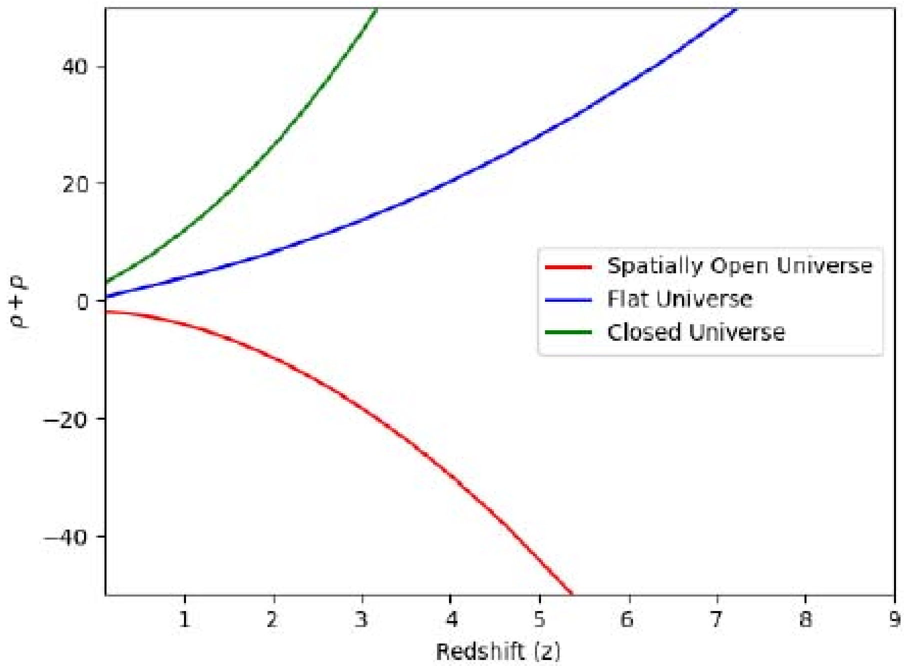

Figure 5.3: $\rho + p$ vs. Redshift for case-I

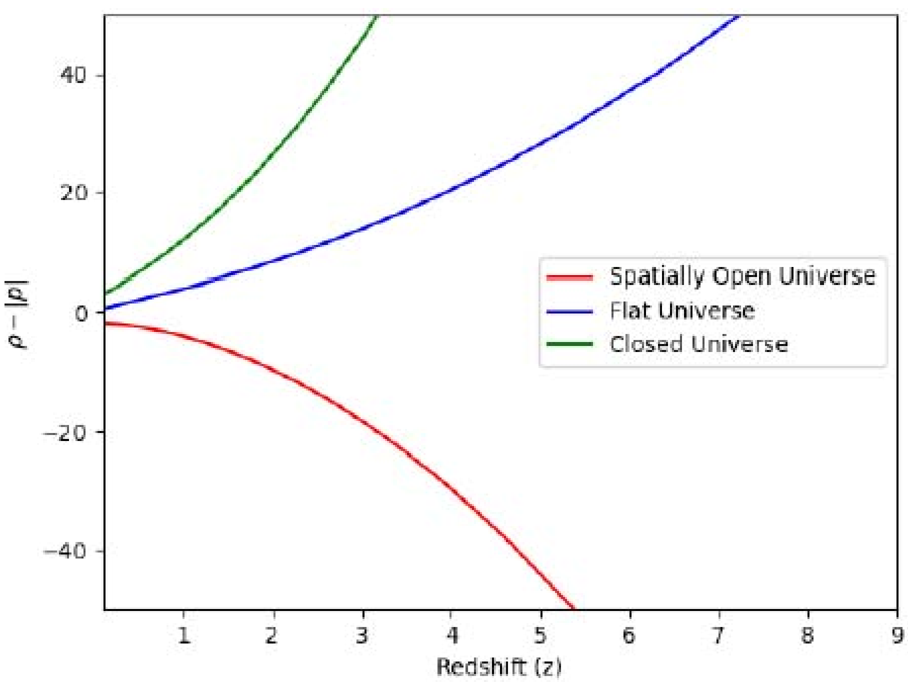

Figure 5.4: $\rho-\mid\rho\mid$ vs. Redshift for case-I

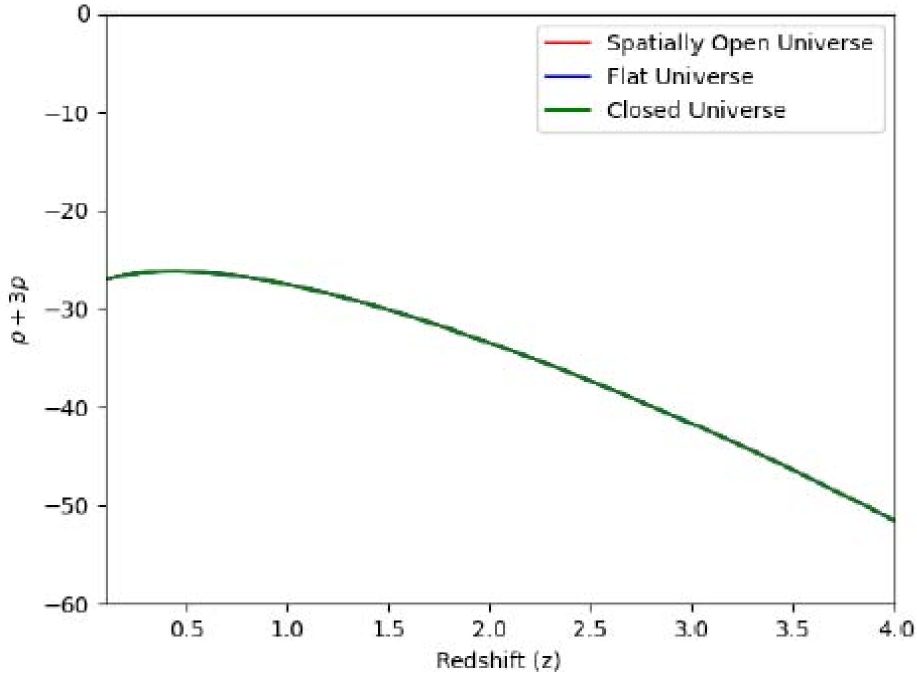

Figure 5.5: $\rho + 3p$ vs. Redshift for case-I

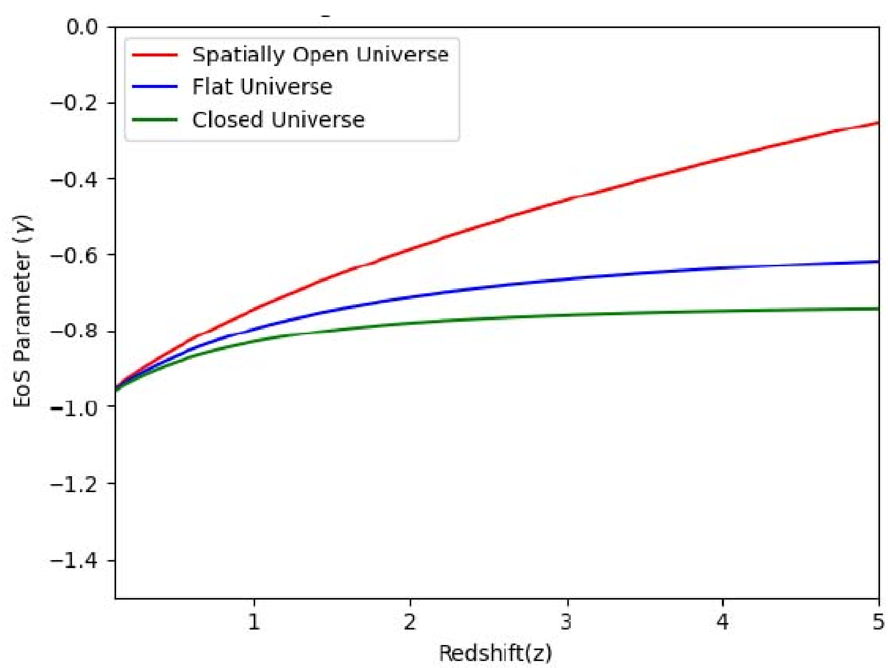

Figure 5.6: EoS vs. Redshift for case-I

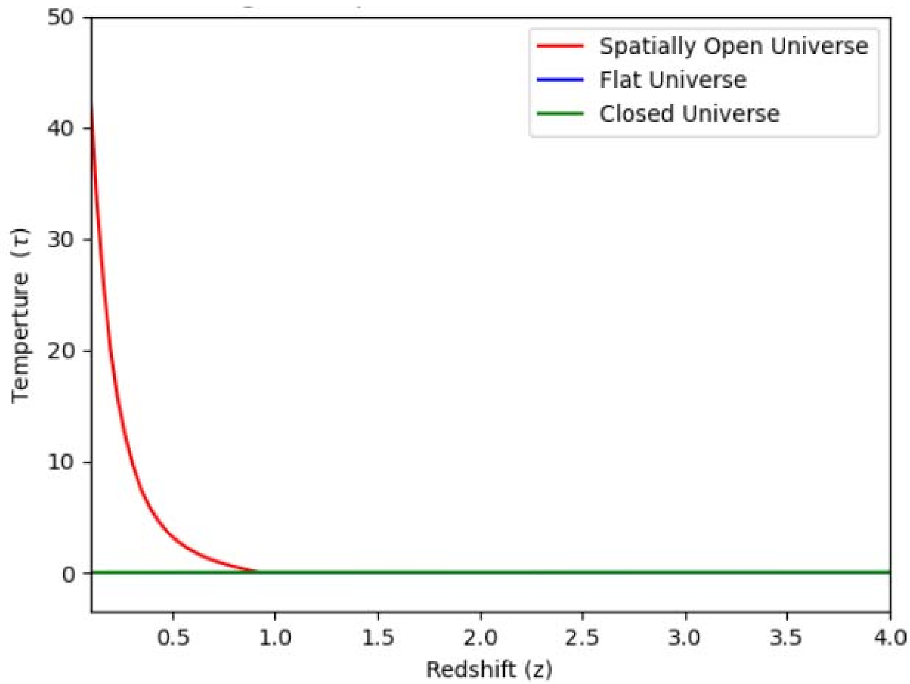

Figure 5.7: Temperature vs. Redshift for case-I

The Fig. 5.7 represents the temperature of the model decreases when time passes on for spatially open universe But for flat and closed universe the temperature vanishes during entire life span. In this case we observed that the entropy density of our model is remain positive for open, flat and closed universe and at late time it approaches to zero, which indicates the second law of thermodynamics remains impact-less on our model.

5.3.3 Case : II

For

$$\Lambda \propto [a(t)]^{-n} \quad i.e., \;\; \Lambda = \Lambda_2 \left(\frac{\alpha t}{2m - \alpha t}\right)^{-\frac{2n}{m}} . \tag{5.3.20}$$

The model parameters can be calculated as

$$\rho = \frac{12}{t^2(\alpha t - 2m)^2} + \frac{3k}{a_1^2}\left(\frac{\alpha t}{2m - \alpha t}\right)^{-\frac{2}{m}} - \Lambda_2 \left(\frac{\alpha t}{2m - \alpha t}\right)^{-\frac{2n}{m}} , \tag{5.3.21}$$

$$p = \frac{8(m-\alpha t)-12}{t^2(\alpha t-2m)^2} - \frac{k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2}{m}} + \Lambda_2\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2n}{m}}, \tag{5.3.22}$$

$$\gamma = \frac{8(m-\alpha t)+\Lambda_2\alpha^2 t^4\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2(n+m)}{m}} - \frac{k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2}{m}} t^2(\alpha t-2m)^2 - 12}{12-\Lambda_2\alpha^2 t^4\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2(n+m)}{m}} + \frac{k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{\frac{2}{m}} t^2(\alpha t-2m)^2}. \tag{5.3.23}$$

Temperature

$$\tau = \left(\frac{12}{t^2(\alpha t-2m)^2} + \frac{3k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2}{m}} - \Lambda_2\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2n}{m}}\right)^{\eta}, \tag{5.3.24}$$

where

$$\eta = \left(\frac{\frac{8(m-\alpha t)+\Lambda_2\alpha^2 t^4\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2(n+m)}{m}} - \frac{k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2}{m}} t^2(\alpha t-2m)^2 - 12}{12-\Lambda_2\alpha^2 t^4\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2(n+m)}{m}} + \frac{k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{\frac{2}{m}} t^2(\alpha t-2m)^2}}{1+\frac{8(m-\alpha t)+\Lambda_2\alpha^2 t^4\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2(n+m)}{m}} - \frac{k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2}{m}} t^2(\alpha t-2m)^2 - 12}{12-\Lambda_2\alpha^2 t^4\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2(n+m)}{m}} + \frac{k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{\frac{2}{m}} t^2(\alpha t-2m)^2}}\right). \tag{5.3.25}$$

The entropy density

$$\begin{aligned} S_1 = &\left(1+\frac{8(m-\alpha t)+\Lambda_2\alpha^2 t^4\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2(n+m)}{m}} - \frac{k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2}{m}} t^2(\alpha t-2m)^2 - 12}{12-\Lambda_2\alpha^2 t^4\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2(n+m)}{m}} + \frac{k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{\frac{2}{m}} t^2(\alpha t-2m)^2}\right) \\ &\left(\frac{12}{t^2(\alpha t-2m)^2} + \frac{3k}{a_1^2}\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2}{m}} - \Lambda_2\left(\frac{\alpha t}{2m-\alpha t}\right)^{-\frac{2n}{m}}\right)^{1-\eta}. \end{aligned} \tag{5.3.26}$$

As above here also we got same results for pressure but for energy density here the open universe also obey the recent observations.

5.4 Energy conditions

As described in the previous section here also we got similar results in view of energy conditions. that indicates the assumption made for dark energy in Case-II remain valid in energy condition aspects.

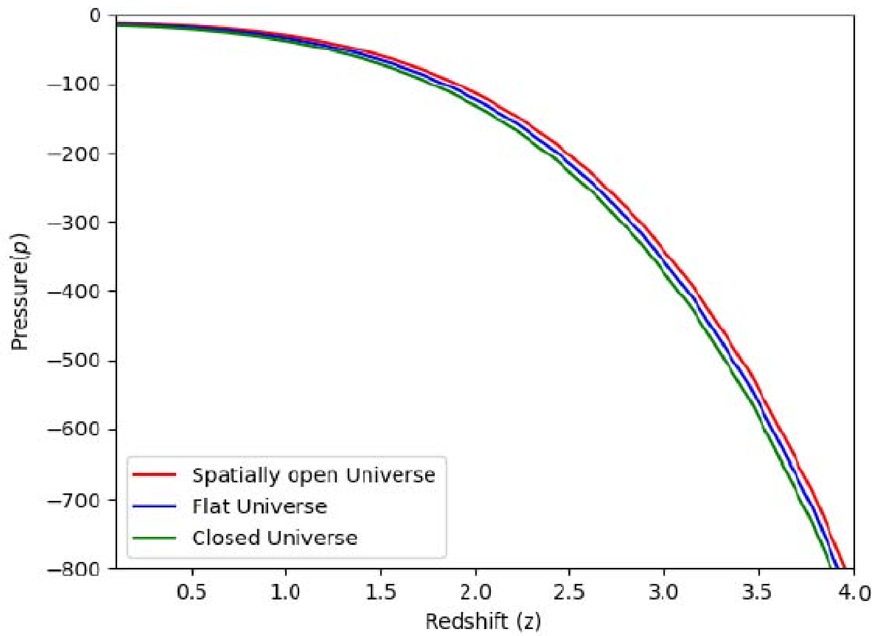

Figure 5.8: Pressure vs. Redshift for case-II

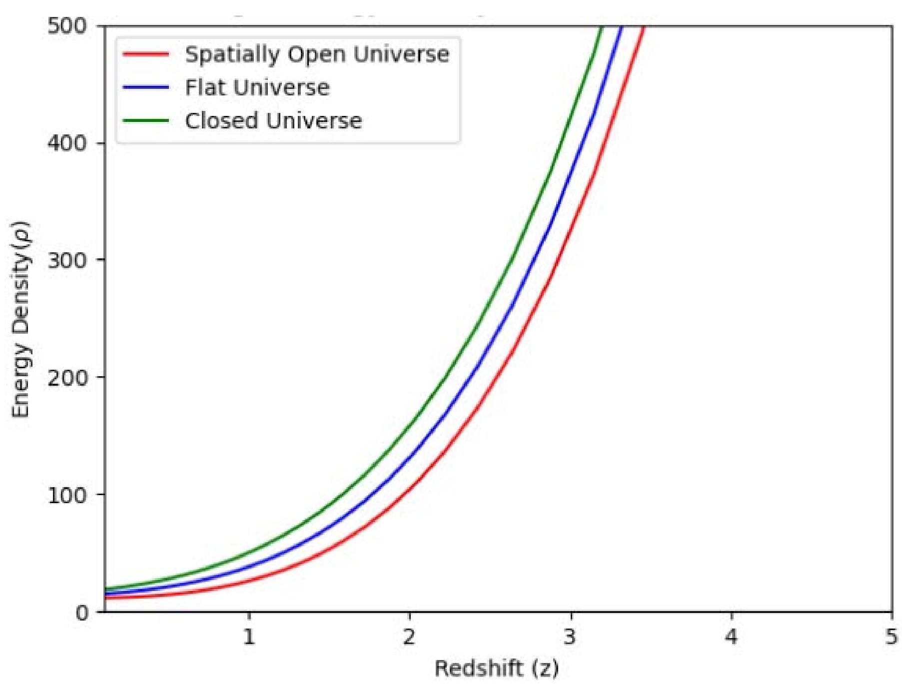

Figure 5.9: Energy Density vs. Redshift for case-II

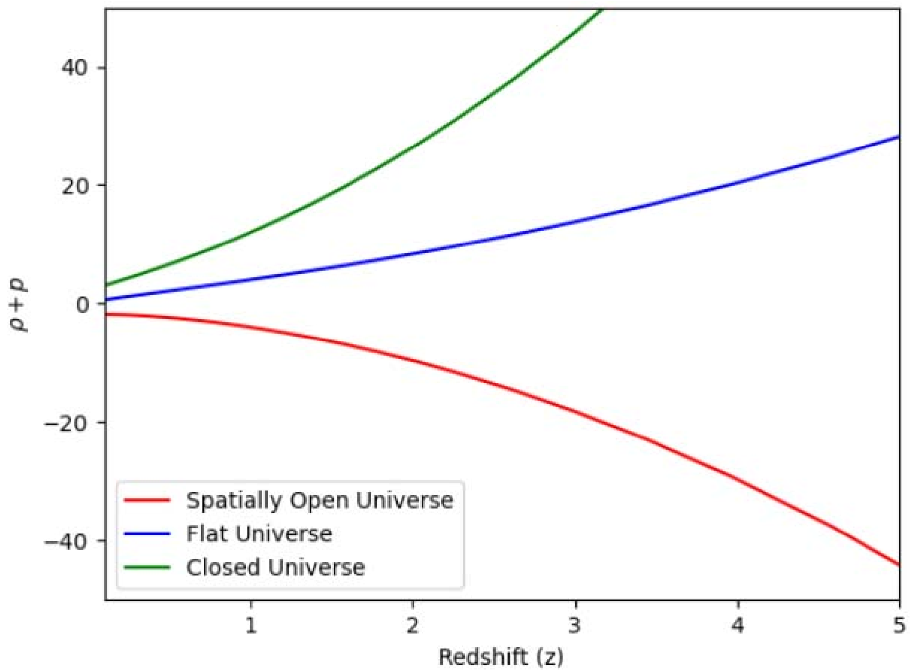

Figure 5.10: $\rho + p$ vs. Redshift for case-II

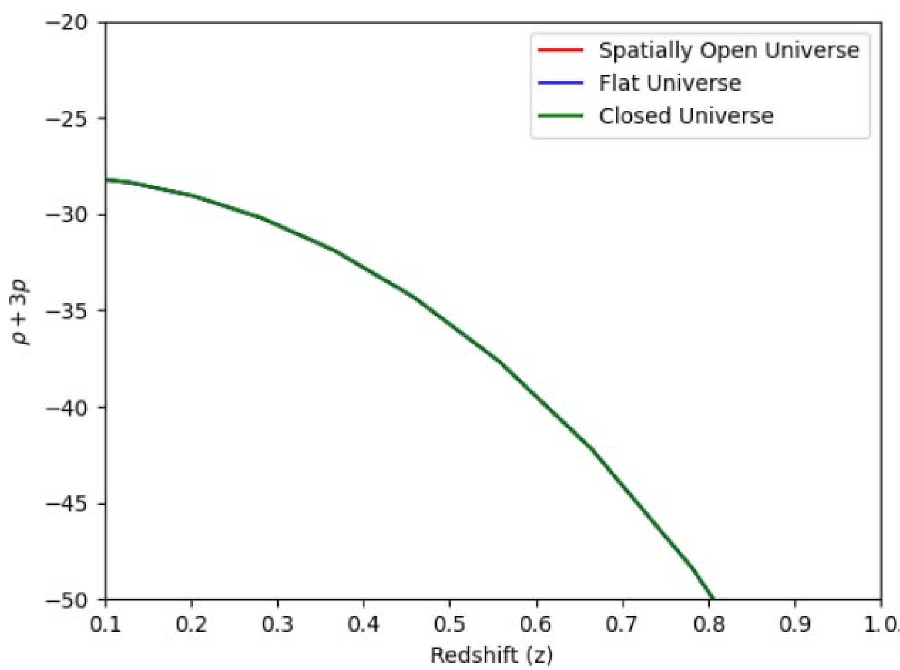

Figure 5.11: $\rho + 3p$ vs. Redshift for case-II

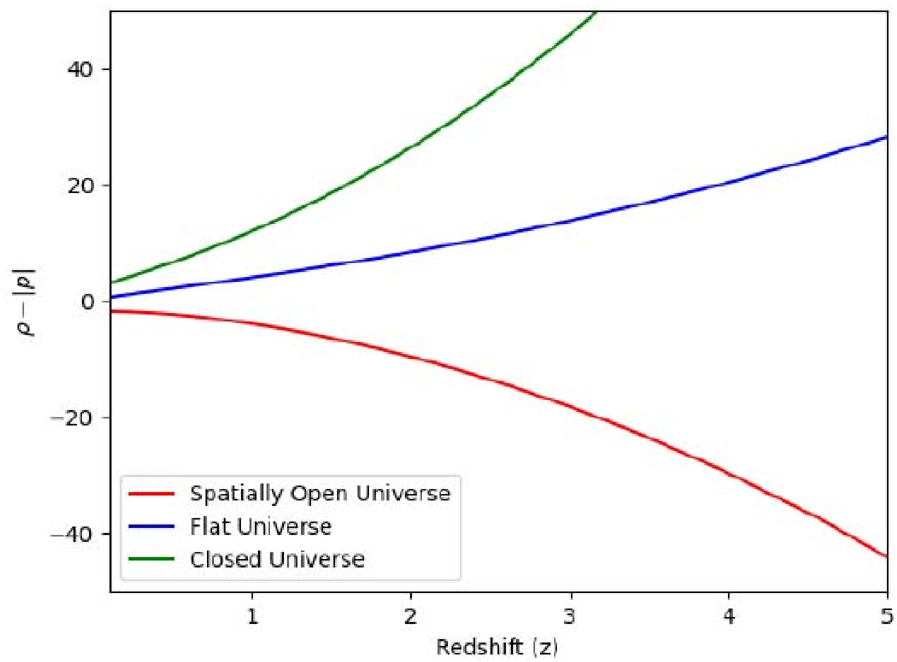

Figure 5.12: $\rho-\mid \rho \mid$ vs. Redshift for case-II

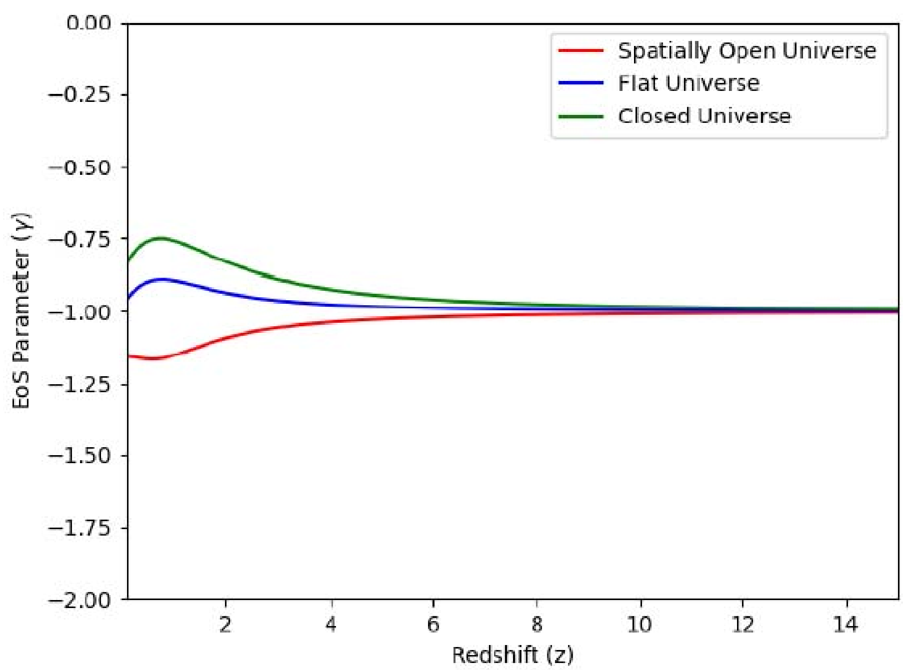

Figure 5.13: EoS vs. Redshift for case-II

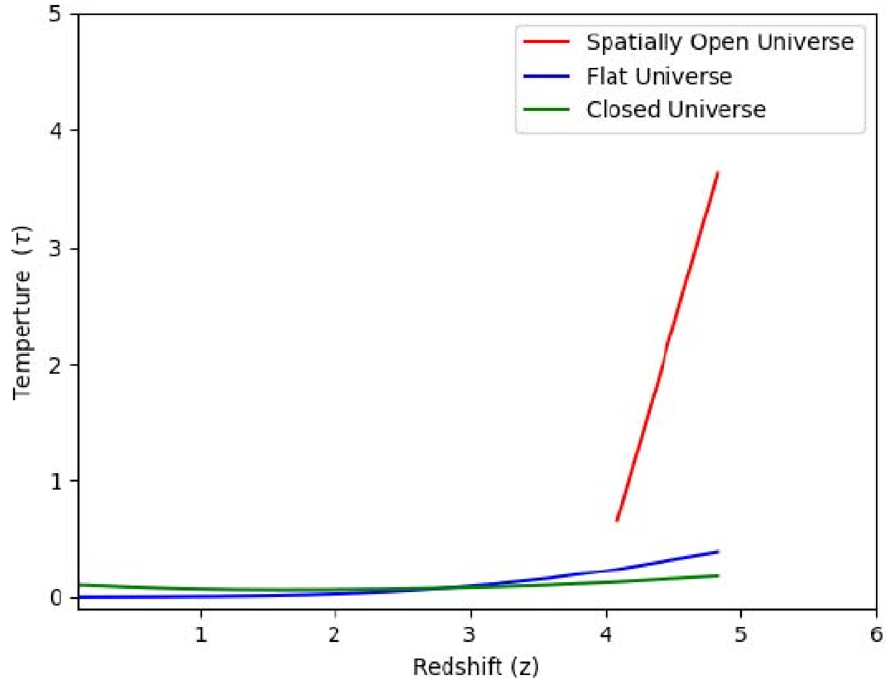

Figure 5.14: Temperature vs. Redshift for case-II

Fig. 5.13 depicts the model enters into quantissence phase for flat and closed universe where as for open universe it indicates a phantom phase.

Fig. 5.14 describes the thermodynamical temperature of the model for second assumption of dark energy behaves identical for flat and closed universe whereas the temperature of the universe increases rapidly with heavy volume during the spatially open universe. Here, we observed that the thermodynamical temperature of this model for our second assumption of cosmological constant during closed universe remain positive increasing function of time. But for closed and flat universe the temperature remain very small positive . The entropy density of the model remain positive and later on it approaches to zero for closed and flat universe. But, during open universe the entropy density remain negative and later approaches to zero, which indicates the occurrence of open universe for this model is made for some short period and the second law of thermodynamics doesn't remain impactful for the model obtained in Case-II.

5.5 Conclusion

Study of early stage of the universe with FRW cosmological models in the frame work of Einstein theory plays an important role. Also, it is well established that the mathematical formulation of different cosmological models through the laws of physics becomes an essential component in understanding the nature of the universe. Hence, in this present work, we have investigated FRW cosmological model in the context of Einstein theory of gravitation. We have studied time varying dark energy states of two different assumptions, from which we found a phantom phase during spatially open universe for $\Lambda \propto [a(t)]^n$ and all remaining results indicates a quintessence phase. we observed that the Hubble parameter approaches to infinite when time approaches to zero, this indicates the universe describes a power law inflation. The temperature and entropy density of the model remain positive for both the cases. In view of energy conditions, the assumptions yields identical results. Our study suggests the Strong Energy Condition violates for our model, that indicates an accelerating expansion of the universe. From our discussion we conclude that during both the assumptions the second law of thermodynamics remain impactless. Moreover, the study suggests our universe is of finite life time. All the obtained results are consistent with respect to observational constraints.