CHAPTER 3

MATHEMATICAL TOOLS AND TECHNIQUES

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3.1. Introduction

We have utilized various aspects of mathematics and mathematical tools during the course of our research. Mathematics and Engineering are two fields, which are bonded, as there can be hardly any engineering / technology without the contribution of mathematics. Likewise, Mathematics becomes more meaningful and practical oriented with various engineering fields. In this chapter we cover in brief some field of applied mathematics which we have used in some form or the other during our research. Operations Research (OR), Mathematical Optimization, Convex Optimization, Stochastic modelling and Markov chain are some of the fields we have encountered during the research. A brief of all these fields is covered in the subsequent paragraphs.

3.2 Operation Research

Operation Research is a comparatively new field in applied mathematics. As is it a relatively new field, various aspects of the subject are yet to be crystallized including the contents. Hence formally defining OR will be a difficult thing. When there is a utilization of various mathematical methods and quantitative techniques as a tool to assist in decision-making, we start using OR.

We make decisions in almost every step in our life, some decisions are easy and routine like what to have for breakfast; and some decisions are complex involving multiple parameters and factors, like how to simulate a complex real-life problem in a virtual environment. The routine and easy decision is made based on experience, mood, common sense without using any mathematical or any other model in simple situations. But the decisions we are concerned with here are complex and heavily responsible, for example planning for a large communication network in a city to include layout mobile communication network, internet connectivity and public telephone network; complex business operations including multiple product portfolios, production and distribution system, profit and financial consideration etc.

Operations Research tools are multi-disciplinary in nature. OR utilized tools from various fields such as mathematics, engineering, statistics, psychology, economics etc. These tools are then utilized to make a new set of knowledge, which helps in optimal decision-making. Since the last so many years, Operations Research has evolved a professional field dealing with the application of scientific methods for rational making decision, and especially so for allocation of scarce resources. The main objective of OR is to offer a rational decision-making tool in the absence of complete information, since the systems composed of human, machine, and procedures may do not have complete information. OR should be consider a scientific field as it involves describing, understanding and predicting the systems behaviour, especially man-machine system. Thus, operation research involves the following three classical aspect of science:

(a) Determining the systems behaviour
(b) Analyzing the systems behaviour by developing appropriate models
(c) Predict the future behaviour using these models

Operation Research gave maximum emphasis / importance on analysis of processes / operations as a whole, this is what distinguished Operation research from other research or engineering subject. Lately Operation Research among others, is very popular with business applications, where various optimal alternatives actions are explored and analysed.

3.2.1 Stages of Development of Operations Research

The stages of development of OR are also known as phases and process of OR, which has six important steps. The six steps can be are under numerated:-

First Step: Observation of the problem atmosphere / environment

Second Step: Analysis of the problem and subsequent definition

Third Step: Development of a suitable model

Fourth Step: Selection of appropriate / suitable data input

Fifth Step: Suggestion of a possible solution and subsequently test it

Sixth Step: Implementation of the solution

Step I: Observation of the problem atmosphere / environment

The first step in the process of OR development is the problem environment observation. Problem environment observation includes various activities like attending conferences, studying related literature, site visit, research, observations etc. These activities provide sufficient information to the OR specialists to formulate the problem.

Step II: Analysis of the problem and subsequent definition

In this step we carry out analysis of the problem at hand and subsequently try to define the problem. We also define the possible objective and its utilization, limitation of our problem definition, if any etc. This step help us in clear understanding of the problem in hand, also the requirement of looking for a solution.

Step III: Development of a suitable model

The step involves developing a model, which will represent some abstract or real situation. This model is a mathematical model, which will describe systems and processes may be in the form of equations/ formula/relationships etc. There can be various activities in this step such as defining variables, formulating equations etc. This model is then rigorously tested in the field under different constraints and if required subsequently modified in order to function. We may sometimes modify the model to satisfy the management with the results.

Step IV: Selection of appropriate / suitable data input

Having developed a model, we know that a model will work appropriately when there exist suitable appropriate input data. Hence, selecting appropriate input data is important step in the OR development stage or process. The activities in this step include internal/external data analysis, fact analysis, and collection of opinions and use of computer data banks. The aim of this step is providing sufficient input data to operate and then test the model developed earlier in Step III.

Step V: Suggestion of a possible solution and subsequently test it

In this step we try to home down to a solution with the help of step III and step IV. Having find a solution, we rigorously test the solution to find out any short coming and any such limitation is studied and suitable amendments, if any, is made to the model. Now we try and test implement the solution. So here in this step we find out a solution which is in sync with our requirements.

Step VI: Implementation of the solution

Having obtained the solution in the previous step, we implement the solution in this step. Here the implementation may involve many issues, all these issues have to be identified and rectified before complete implementation of the solution.

3.3 OR Tools and Techniques

There are numerous OR tools and techniques. Some of the popular OR tools/ techniques are linear programming, game theory, decision theory, queuing theory, inventory models and simulation. There are also others tools/ techniques like non-linear programming, integer programming, dynamic programming, sequencing theory, Markov process, network scheduling (PERT/CPM), symbolic Model, information theory etc. In our research, we have simulation and Markov process for further analysis of model and solution, as part of various step followed on OR.

Simulation. Simulation involves creation of a model according to our problem at hand. For creation of the model, we should first carefully observe the problem for which we are looking for a solution. Next, we, after detailed analysis of the problem, defined the problem. We also define the possible objective and its utilization, limitation of our problem definition, if any etc. This step helps us in clear understanding of the problem in hand, also the requirement of looking for a solution. Outputs of the step will be clear understanding/ idea for need of a solution. Now after creation of the model, since we know that a model will work appropriately when there exist suitable appropriate input data, we start selecting appropriate input data. Then we conduct several trial runs using the input data. The output of these trial runs is analysed and checked whether the output data is in agreement with our theoretical results. Simulation generally requires throughout planning and understanding of the subject. It is generally use when actual experimentation is infeasible or not preferred due to various constrains like time, resources, risk involved etc.

Markov Process. Markov process or sometimes-called Markov chain is a type of random/ stochastic process in which the future is independent of the past given the present state. Markov process allows prediction of the future state having known the present state but do not depend on the previous state. Hence, Markov process can be utilized for decision-making process where present state is defined. The probability from present state to the future state solely on the present state and not on the past state. There different type of Markov process, to name the few Discrete-time Markov chain and Continuous time Markov chain. We can use Markov chain/process to model some of the entities during our simulation and this will help in understanding how the entire network will perform in presence of various types to entities.

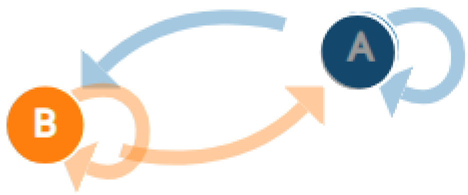

Fig 3.1: A simple two-step Markov Chain

3.4 MATHEMATICAL OPTIMIZATION

Optimization can be defined generally, as a process of finding the best possible solution also called as optimal solution from a set of feasible solutions. The basic form of optimization problem as be defined as under:-

$$(P)\ Min\ f(x)$$

$$Subject\ to\ x \in C$$

Where $f{:}R^n \rightarrow R$, *and* $C \subseteq R^n$

In the above equation the problem (P) is the basic mathematical programming problem; function f is the objective function where set C is the constraint set or feasible set. A point $x \in C$ is called the feasible. The feasible point for the above problem P where it attains maxima or minima is called the optimal point or optimal solution. If C = ø, then the problem is infeasible. Also if $C=R^n$ then, the optimization problem is unconstraint, otherwise it is constraint optimization problem.

As brought out in the above paragraph, optimization is nothing but selection of best element from some set, the elements of this set is the available data, under some constraints. The solution to the optimization problem consist of maximizing or minimizing of a real function by choosing input data from a defined set and then finding the value of the function. We have utilized mathematical optimization to obtain an optimal synchronization precision and accuracy in our algorithm. A detail analysis is conduct in a pursue to obtain the tight synchronization in our algorithms. Concept of sliding window, weighting averaging and theoretical analysis is conduct in developing the algorithm. Details on this will be covered in the subsequent chapters.

3.5 STOCHASTIC MODELLING

A stochastic process, also called random process, is a mathematical process dealing with family of various types of random variables. Random Variables are those types of variables which may change with time, prediction of such variables are difficult. Applications in many disciplines including sciences and technology.

Classifications

Stochastic process can be classified in various ways, one of these classifications is based on the cardinality of index set and also on the state space.
The classification are Discrete time Stochastic Process and Continuous Time Stochastic Process.

Types of Stochastic Process.

Stochastic Process is an important subject in Applied Mathematics. There are various type of Stochastic Process and some of the examples of stochastic processes are as under

(a) **Bernoulli process** – Bernoulli process is one of the most basic type of Stochastic Process. Here the process is link to IID (Independent & Identically Distributed) type of Random Variables. In Bernoulli Process the random variables are in binary state i.e., either One or Zero, therefore if the possibility of the variable being One is, say, P then the probability of being Zero will be (P-1). Hence, we can see the process is akin to tossing a coin, fair or not, where it can be assumed that obtaining Head may mean One or otherwise.

(b) **Random walks** - The next Stochastic Process is Random Walks, it is a type of Discrete time Stochastic process, where the random variables associated are IID. Here the value of the next random variable is independent of the current value. Random Walks can be defined in various dimensions also.

(c) **Wiener process** - The concept of Wiener Process was first suggested by Professor Norbert Wiener, He establish the existence of such process mathematically. This is a type of Stochastic process which associated with Stationary but independent increments. The same is based on Normal Distribution and is dependent on the size of the increments. This process is also called Brownian Motion Process. As the same is associated with Brownian Fluid motion, it is sometimes also called Brownian Motion only.

(d) **Poisson process** - Another type of Stochastic Process and a popular type is the Poisson Process. It is some ward similar to the counting process. It is a type of Stochastic Process which represent Random No of events or points. It is discrete stochastic process were we know the average possibility of an event but do not know the exact possibility of the event happening. Here the events are independent of each other but the overall average of the events happening is constant.

3.6 Random-walks

In our research we have use the concept of random walks to model an important type of parameter which is utilized in simulating both type algorithms. Let us assume that $\{X_i; i \geq 1\}$ be a sequence of IID random variables, and also let $S_n = X_1 + X_2 + \cdots + X_n$.

Random walk is nothing but a discrete random process or stochastic process where each step is positive integer time based $\{S_n; n \geq 1\}$, and since the it hugs the single dimension number line $\{Xi; i \geq 1\}$, it is one dimensional Random Walk process.

For any given n, S_n is simply a sum of IID random variables, but here the behaviour of the entire random walk process, $\{S_n; n \geq 1\}$, is of interest. Hence, for any real number $\eta > 0$, we can find the probability that the sequence $\{S_n; n \geq 1\}$ contains any term for which $S_n \geq \eta$ or we can also find the distribution of least n for which $S_n \geq \eta$.

As an illustration let us assume that position of the moving dot at any time *t* be X_t. Since the dot can only move in discrete time, *t* takes only integer values (i.e., 0, 1, 2, ….). The process of random walk of the dot on the number line or line of integers will start with $X_0 = 0$. Before taking each step, in order to decide, if we move left or right, a coin (fair) is tossed. If we get heads, then the dot moves to the right, otherwise, it moves to the left. Hence, the position of the dot at $t=k$ can be written as:

$$X_K = Y_1 + \cdots\cdots + Y_K,$$

where *Yi* can be regarded as the outcome of the coin flip before taking the *ith* step, and it can only take on values of +1 or -1. Since a fair coin is used, it is safe to assumed all coin flips are independent events. Our interest is to find the location of the dot at any given time, *t*. It is obvious that it is impossible to exactly find out where the dot will be since every movement is decided by a probabilistic event, and hence we use probability distribution function to describe the location of the dot at any time *t*.

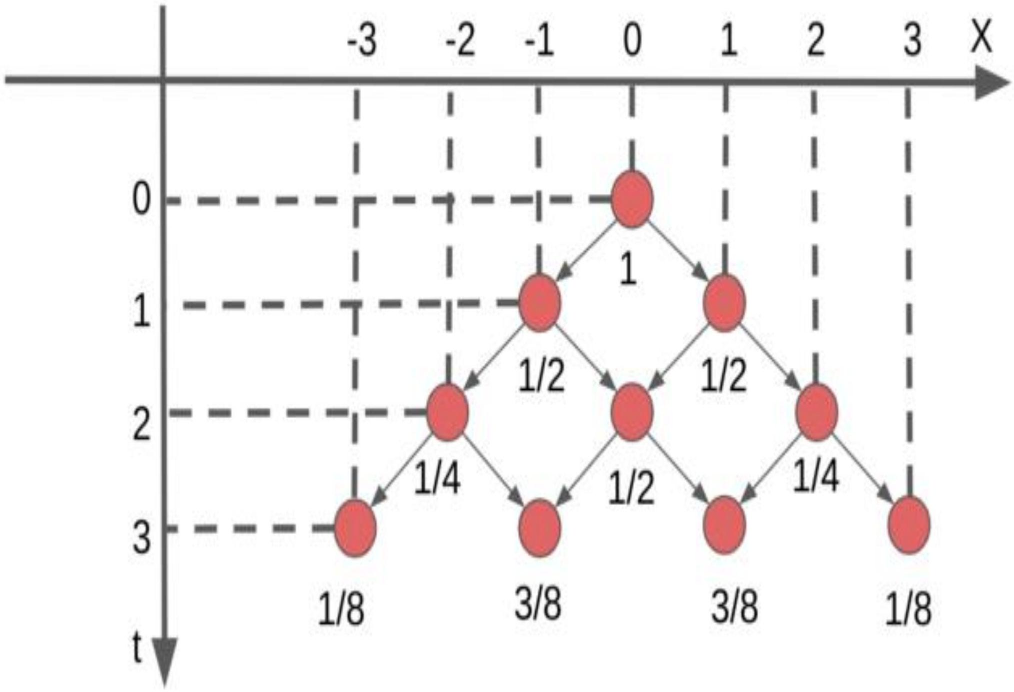

Fig. 3.2: Random Walk Tree

We can visualize the movement with a tree. In the tree in Fig. 3.2 above the horizontal axis is the line of integers where the dot is moving on, the vertical axis is the discrete time line, and the number below each red dot is the probability of the dot being at that particular location and time. Sine the dot at t=0 is at 0, its probability is 1. The remaining probabilities can be easily worked out. In the figure if we observed carefully, we can see If we factor out the denominators of the probabilities on each row (each time step), the remaining numbers form the Pascal's triangle, i.e., for t=0, we have 1, for t=1, we have 1, 1, for t=2, we have 1, 2, 1, for t=3, we have 1, 3, 3, 1, and so on. We also know that Pascal's triangle gives the coefficients of binomial expansions, and also since the sum of all probabilities at each time step is 1, it should be instinctively clear that the probability distribution of the dot's location should be following the binomial distribution at each time step.

We can show that, mathematically, the probability of the location of the dot at $X=k$ for $t=n$ is:

$$p(k,n) = \left|\frac{(-1)^n + (-1)^k}{2}\right| \binom{n}{\frac{n-k}{2}} \left(\frac{1}{2}\right)^{\frac{n-k}{2}} \left(\frac{1}{2}\right)^{n-\frac{n-k}{2}}$$

which simplifies to:

$$p(k,n) = \left|\frac{(-1)^n + (-1)^k}{2}\right| \binom{n}{\frac{n-k}{2}} \left(\frac{1}{2}\right)^{n}$$

(Probability Distribution of a Random Walker)