CHAPTER-3

MATHEMATICAL ANALYSIS ON ANISOTROPIC BIANCHI TYPE-III INFLATIONARY STRING COSMOLOGICAL MODELS IN LYRA GEOMETRY

The content of this chapter has been published in *"Indian Journal of Science and Technology",* WOS indexed journal (UGC care listed group-II) ISSN:0974-6846(P); 0974-5645(E), Volume-14, Issue-1, pp. 46-54, (2021).

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3.1 INTRODUCTION

It is still an exciting area of research to discover its unknown phenomena that have yet to be observed in order to study the ultimate fate of the Universe. However, cosmologists have yet to reach a final and comprehensive conclusion about the origin and evolution of the Universe based on strong evidence. As a result, more and more investigations are needed to discover and understand the unknown phenomena of the Universe, as well as the many mysterious particles that must be observed in order to study the ultimate fate of Universe. String theory was developed by cosmologists or researchers to describe the Universe, its early stages, and its evolution over time. As a result, the study of string cosmology is becoming a very interesting area for cosmologists, due to its importance in the early stages of formation and evolution of the Universe, as well as understanding future evolution. The investigation of strings in the field of GR was generally initiated by prominent authors Stachel(1980) and Letelier (1979,1983). Many prominent authors have investigated cosmic strings in the context of Lyra geometry in recent years because they can play a significant role in describing the Universe in its early stages of evolution [Kibble(1976,1980)] and can give rise to density perturbations that can lead to the formation of large scale structures (galaxies) [Zel'dovich(1974,1980)].

The strings are critical topological stable defects that occurred as a result of phase transitions in the early days of the Universe, when the temperature is lower than a specific temperature, known as the critical temperature. The presence of strings in the Universe causes anisotropy in space-time, even though the strings aren't visible in the current epoch. Soon after the big bang, there was a break in symmetry during the phase transition, and the cosmic temperature dropped below some critical temperatures, resulting in the emergence of strings, as predicted by grand unified theories [Everett(1981), Vilenkin(1981, 1981a)].

Though Einstein's GR is one of the most accepted theories in the modern era for describing the Universe, it is unable to explain some of the major unknowns about the Universe, such as the accelerated expansion of the Universe, the reason for the expansion, and so on. As a result, several researchers are attempting to solve and explain those aspects of the Universe using various modified theories of Einstein's GTR, such as Weyl's theory, Brans-Dicke theory, f(R) gravity theory, f(R,T) theory, Lyra geometry, scalar tensor theory, and so on. Lyra geometry is one of the most important of these theories. Weyl(1918) developed Weyl's theory by geometrizing electromagnetism and gravitation, which was inspired by the geometrization of gravitation. However, this theory was criticized and not accepted due to non-integrability of length of vector under parallel displacement. Lyra (1951) proposed a modification of Riemannian geometry, which is also considered a modification of Weyl's geometry, by introducing a Gauge function (or scale function) into the structure less manifold, which eliminates the non-integrability status of the length of a vector under parallel transport (i.e. the metricity condition is restored) and naturally introduces a cosmological constant from the geometry. Some of the prominent authors who have already constructed various cosmological models in Lyra geometry include Halford(1970), Bhamra(1974), Beesham(1988), T. Singh & G.P. Singh(1991,1993), Rahaman et al.(2002), Reddy(2005), Reddy and Rao(2006, 2006a), Rao et al.(2008), Adhav et al.(2009), A. K. Yadav(2010), V. K. Yadav et al.(2010). Recently, Singh et al.(2016), Singh & Mollah(2018), W. D. R. Jesus & A. F. Santos(2018), Mollah et al.(2018), Yadav & Bhardwaj(2018), and Maurya & Zia(2019) investigated various cosmological models in various contexts while taking Lyra's geometry into account.

Inspired by the above discussions, here a string cosmological model with particles connected to them in Bianchi type-III Universe considering Lyra geometry has been discussed. The work and findings in this chapter differ slightly from those in previous chapters.

3.2 THE METRIC AND FIELD EQUATIONS

The Bianchi type-III metric is considered as

$$ds^2 = a^2dx^2 + b^2e^{-2x}dy^2 + c^2dz^2 - dt^2 \tag{3.1}$$

In this case, a, b, and c are functions of 't' alone. For the above metric let

$$x^1 = x,\ x^2 = y,\ x^3 = z \text{ and } x^4 = t \tag{3.2}$$

The field equation ($8\pi G = 1$, C= 1) in Lyra manifold is

$$R_{ij} - \frac{1}{2}Rg_{ij} + \frac{3}{2}\phi_i\phi_j - \frac{3}{4}g_{ij}\phi_k\phi^k = -T_{ij} \tag{3.3}$$

Where ϕ_i is the displacement field vector given by

$$\phi_i = (0,0,0,\beta) \tag{3.4}$$

Here β is the function of time.

The energy-momentum tensor for a cosmic string is taken as

$$T_{ij} = \rho u_i u_j - \lambda x_i x_j \tag{3.5}$$

Where $\lambda = \rho - \rho_p$ is the tension density, the ρ is the energy density, ρ_p is the particle density. Also u^i is the four velocity vector of particles and x^i is the unit space like vector representing the direction of the string given by

$$x^i = (0,0,c^{-1},0) \text{ and } u^i = (0,0,0,1)$$

$$\text{Such that} \quad u_iu^i = -1 = -x_ix^i \text{ and } u_ix^i = 0 \tag{3.6}$$

If r(t) is the scale factor of the Universe then volume V is given by

$$V = abc = r^3 \tag{3.7}$$

The expansion scalar is given by

$$\theta = u^i_{,i} = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} \tag{3.8}$$

Hubble parameter is given by

$$H = \frac{1}{3}\left(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right) \tag{3.9}$$

The shear scalar is given by

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{3}\left[\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\dot{b}}{b}\right)^2 + \left(\frac{\dot{c}}{c}\right)^2 - \frac{\dot{a}\dot{b}}{ab} - \frac{\dot{b}\dot{c}}{bc} - \frac{\dot{a}\dot{c}}{ac}\right] \tag{3.10}$$

Mean anisotropy parameter is

$$\Delta = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{H_i - H}{H}\right)^2 \tag{3.11}$$

Where H_i $(i = x, y, z)$ denotes the directional Hubble factors which are given by $H_x = \frac{\dot{a}}{a}$, $H_y = \frac{\dot{b}}{b}$ and $H_z = \frac{\dot{c}}{c}$. for the metric (3.1).

The field equation (3.3) with the equations (3.4)-(3.6) for the equation (3.1) takes the form

$$\frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{b}\dot{c}}{bc} + \frac{3}{4}\beta^2 = 0 \tag{3.12}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{c}}{c} + \frac{\dot{a}\dot{c}}{ac} + \frac{3}{4}\beta^2 = 0 \tag{3.13}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\dot{a}\dot{b}}{ab} + \frac{3}{4}\beta^2 - \frac{1}{a^2} = \lambda \tag{3.14}$$

$$\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} + \frac{\dot{a}\dot{c}}{ac} - \frac{3}{4}\beta^2 - \frac{1}{a^2} = \rho \tag{3.15}$$

$$\frac{\dot{a}}{a} - \frac{\dot{b}}{b} = 0 \tag{3.16}$$

The overhead dots in this case represent the order of differentiation with time 't'.

3.3 SOLUTIONS OF THE FIELD EQUATIONS

When the equation (3.16) is solved, we get

$$a = Lb \tag{3.17}$$

The integration constant L is used here. We can take $L = 1$ for the sake of generality.

And by employing it, (3.17) can be written as

$$a = b \tag{3.18}$$

Thus using relation (3.18) the field equations (3.12) to (3.15) reduces to

$$\frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} + \frac{\dot{b}\dot{c}}{bc} + \frac{3}{4}\beta^2 = 0 \tag{3.19}$$

$$2\frac{\ddot{b}}{b} + \left(\frac{\dot{b}}{b}\right)^2 + \frac{3}{4}\beta^2 - \frac{1}{b^2} = \lambda \tag{3.20}$$

$$\left(\frac{\dot{b}}{b}\right)^2 + 2\frac{\dot{b}\dot{c}}{bc} - \frac{3}{4}\beta^2 - \frac{1}{b^2} = \rho \tag{3.21}$$

There are three highly nonlinear independent equations (3.19)-(3.21) with unknown variables b, c, β , ρ and λ. So, in order to obtain the exact solutions to the above equations, we must satisfy two additional conditions. So, in this case, the following two physically plausible conditions are used:

Here, we assume that the shear and expansion scalar are proportional to one another ($\sigma \propto \theta$), yielding the equation

$$b = c^m \tag{3.22}$$

Here $m \neq 0$ is a constant.

The preceding assumption is based on observations of the velocity and red-shift relationship for an extragalactic source, which predicted that the Hubble expansion is

30% isotropic, as supported by Thorne(1967), Kantowski & Sachs(1966), Kristian & Sachs(1966), and Collins et al.(1980).

Second, we use Berman's (1983) assumption about the variation of Hubble's parameter H, which results a constant DP as

$$q = -\frac{r\ddot{r}}{\dot{r}^2} = (constant) \tag{3.23}$$

The scale factor "r" allows the solution-

$$r = (ht + k)^{\frac{1}{1+q}}, \quad q \neq -1 \tag{3.24}$$

Here $h \neq 0$ and k are constants of integration.

Using the equations (3.7), (3.18), (3.22) and (3.24), we get

$$a = b = (ht + k)^{\frac{3m}{(1+q)(2m+1)}} \tag{3.25}$$

$$c = (ht + k)^{\frac{3}{(1+q)(2m+1)}} \tag{3.26}$$

Without sacrificing generality, we use $h = 1$ and $k = 0$ then (3.25), (3.26) becomes

$$a = b = t^{\frac{3m}{(1+q)(2m+1)}}, \quad c = t^{\frac{3}{(1+q)(2m+1)}} \tag{3.27}$$

This (3.27) can be used to reduce the metric (3.1) to

$$ds^2 = t^{\frac{6m}{(1+q)(2m+1)}}(dx^2 + e^{-2x}dy^2) + t^{\frac{6}{(1+q)(2m+1)}}dz^2 - dt^2 \tag{3.28}$$

This gives the geometry of the metric (3.1).

3.4 PHYSICAL AND GEOMETRICAL PARAMETERS

In this section, some of the important physical and geometrical parameters are obtained.

Using (3.27) in (3.21), ρ can be obtained as

$$\rho = \frac{3(m+1)(2-q)}{(1+q)^2(3m+1)t^2} - t^{-\frac{6m}{(1+q)(2m+1)}} \tag{3.29}$$

From (3.18) and (3.19) using (3.27), we obtained

$$\lambda = \frac{3(m-1)(2-q)}{(1+q)^2(3m+1)t^2} - t^{-\frac{6m}{(1+q)(2m+1)}} \tag{3.30}$$

From (3.29), (3.30), ρ_p can be obtained as

$$\rho_p = \frac{6(2-q)}{(1+q)^2(3m+1)t^2} \tag{3.31}$$

The gauge function β is obtained as

$$\beta^2 = \frac{4[(m+1)(2m+1)(1+q)-3(m^2+m+1)]}{(1+q)^2(3m+1)^2t^2} \tag{3.32}$$

The model's spatial volume, expansion scalar, Hubble parameter, shear scalar and anisotropy parameter are obtained as

$$V = t^{\frac{3}{1+q}} \tag{3.33}$$

$$\theta = \frac{3}{(1+q)t} \tag{3.34}$$

$$H = \frac{1}{(1+q)t} \tag{3.35}$$

$$\sigma = \frac{\sqrt{3}(m-1)}{(1+q)(2m+1)t} \tag{3.36}$$

$$\Delta = \frac{2(m-1)^2}{(2m+1)^2} = \text{Constant.} \tag{3.37}$$

3.5 INTERPRETATIONS OF THE SOLUTIONS

The Bianchi type-III anisotropic cosmological model with strings in Lyra geometry is represented by equation (3.28). The physical and geometrical behaviour of the model for $-1 < q < 0$ is discussed as follows.

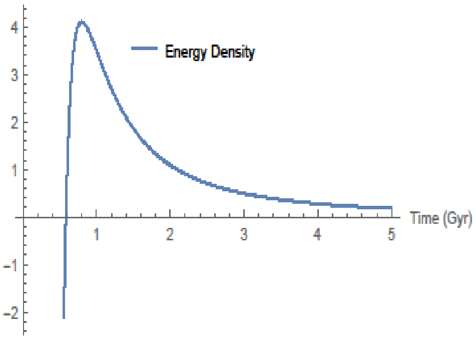

Figure 3.1: Energy density Vs. Time

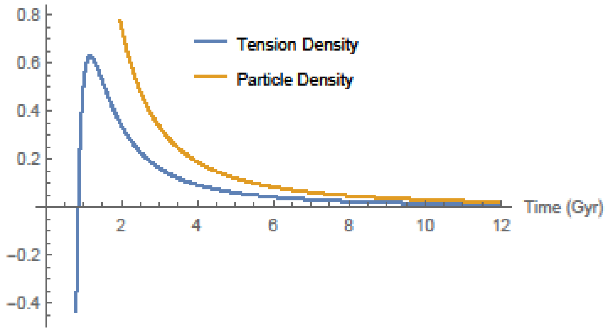

Figure3.2:Tension density, Particle density Vs. Time

We can see from the expressions of energy density and tension density given by eqns.(3.29) and (3.30), that both of them are negative at $t = 0$, but as time passes, they change sign from negative to positive, then gradually decrease until they reach zero at $t \rightarrow \infty$. Figure-3.1 depicts the variations in energy density with time t, demonstrating clearly that $\rho \rightarrow 0$ at infinite time. Also, Figure-3.2 depicts the nature of the variations in tension density λ vs. time t, from which it can be deduce that at $t \rightarrow 0$, λ is negative, but as cosmic time passes, it also changes sign from negative to positive and finally, at $t \rightarrow \infty$, it becomes zero, which agrees the Letelier (1979, 1983).

For the model Universe, the expression of particle density ρ_p is found as the Eqn.(3.31) and its variations versus cosmic time is shown in Figure-3.2., which shows that ρ_p is always positive that decreases from $\rho_p = \infty$ as t = 0 to $\rho_p = 0$ whenever $t \rightarrow \infty$. Also, Figure-3.2 depicts that the tension density diminishes more quickly than the particle density. Therefore, with the passage of time string will disappear leaving the particles only. Hence the model obtained in this chapter is realistic one. It can also be seen that $\frac{\rho_p}{|\lambda|} > 1$, indicating that string tension density decreases faster than particle density. This implies that the late Universe is dominated by particles.

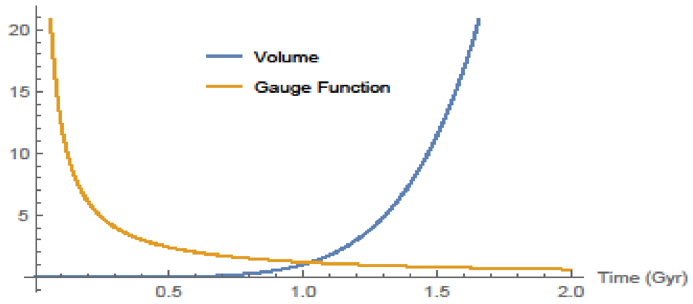

Figure 3.3 : Gauge Function, Volume Vs. Time

The gauge function β^2 is found to be infinite at the initial epoch of time in this model Universe and it decreases with the passage of time. Finally, the gauge function $\beta^2 \to 0$ at $t \to \infty$.

As the time passes, the volume of this model increases. The expression of volume V in Eqn.(3.33) shows that the model Universe begins with an initial singularity at $t = 0$ from $V = 0$ and when $t \to \infty$, $V \to \infty$. As a result, the model suggests that the Universe is expanding at a faster rate for $1+q > 0$.

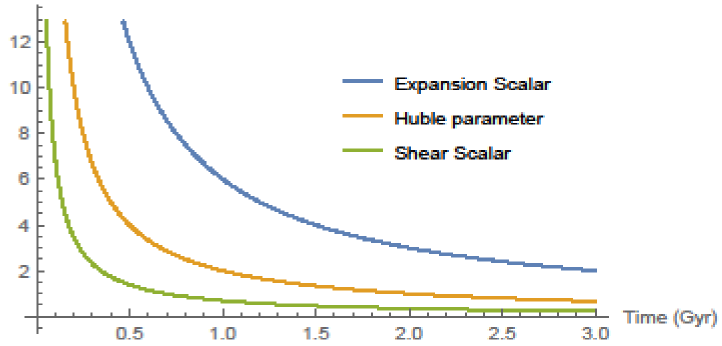

Figure 3.4 : Expansion Scalar, Shear Scalar, Hubble parameter Vs. Time

According to the scalar expansion and Hubble parameter for the model (3.28), both θ and H are infinite at $t = 0$, gradually decrease as time passes and finally become 0 at

$t \to \infty$. As a result, the Universe expands with time, but at a slower rate as time passes, eventually coming to a halt at $t \to \infty$. When t approaches infinity, it is seen that $\frac{dH}{dt} = -\frac{1}{(1+q)t^2} = 0$, implying the greatest value of Hubble's Parameter and accelerated expansion of the Universe. Figure 3.4 illustrates these behaviours.

From the equation (3.36) & Figure-3.4 it is seen that the shear scalar $\sigma \to \infty$ at initial epoch and as time passes, it decreases until it reaches zero in the late Universe, indicating that the Universe obtained here is shear free in the late time.

The mean anisotropy parameter $\Delta = \text{constant}(\neq 0)$ for $m \neq 1$ and $\Delta = 0$ for $m = 1$. Also as $t \to \infty$ the value of $\frac{\sigma^2}{\theta^2} = \frac{(m-1)^2}{3(2m+1)^2} = \text{constant}(\neq 0)$ for $m \neq 1$ and $\frac{\sigma^2}{\theta^2} = 0$ for $m = 1$. It can be deduce from both the statements that this model is anisotropic for large value of t when, $m \neq 1$ but it becomes isotropic for $m = 1$.

3.6 CONCLUSIONS

In Lyra geometry, a new solution to the field equations derived for the Bianchi type-III Universe has been investigated using Hubble's law of variation of parameter H, which yields constant DP. The values of the average scale factors ‘r’ are precisely determined by this variational law for H. The model begins at $t = 0$ with volume 0 and expands with acceleration until the strings vanish, leaving only the particles in the late Universe, resulting in a particle-dominated Universe that matches the current observational data. The model is accelerating, expanding, anisotropic for $m \neq 1$ at the late Universe, non-shearing, and admits an initial singularity at time $t = 0$, all of which are consistent with current observational data. Through this chapter, we aim to propose a finer scholarship of the cosmological transformation of the existing Universe with the aid of Bianchi type-III Universe in Lyra geometry.

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