DARK ENERGY IN HIGHER DIMENSIONAL SPHERICALLY SYMMETRIC SPACE-TIME

A thesis submitted to Bodoland University in partial fulfilment of the requirements for the award of the degree of

Doctor of Philosophy

Submitted by

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Under the Supervision of

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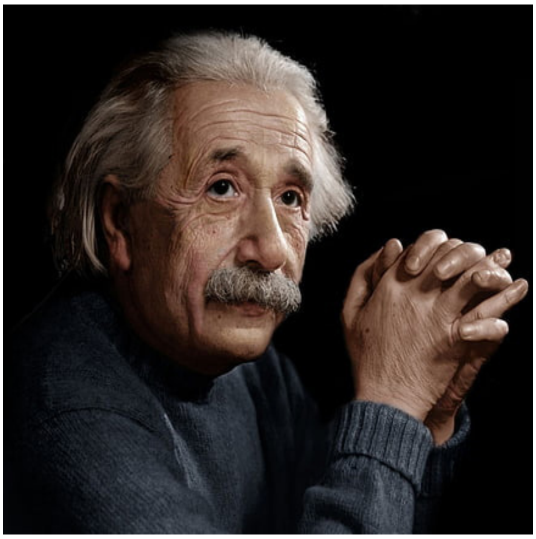
This thesis is dedicated to the memory of my parents, Shri. Ph. Kameshwar Singh and Smt. N. Ibempishak Devi. I am blessed to have the honour of being their son. This work is also dedicated to my wife, Ms. H. Dayapati Devi, and my angel daughter, Jodalin Pheiorijam for their patience, love, and understanding throughout the journey, which have been a constant source of motivation for me.



Life is like riding a bicycle.
To keep your balance,
you must keep moving.

(Albert Einstein)
14 March 1879 – 18 April 1955

DECLARATION

I hereby declare that the work presented in this thesis titled **Dark Energy in Higher Dimensional Spherically Symmetric Space-time** submitted in partial fulfilment of the requirements for the award of the Degree of **Doctor of Philosophy in Mathematics** to Bodoland University, Kokrajhar, Assam, India is an authentic record of my own research carried out under the supervision and guidance of **Dr. Kangujam Priyokumar Singh**, Professor, Department of Mathematical Sciences, Bodoland University, Kokrajhar.

The work presented in this thesis has not been submitted, in parts or full, for the award of any other degree or diploma to this or any other institute/university, except for some chapters that have been published in international journals of repute.

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CERTIFICATE

This is to certify that the work presented in the thesis titled **Dark Energy in Higher Dimensional Spherically Symmetric Space-time** submitted in partial fulfilment of the requirements for the award of the Degree of **Doctor of Philosophy in Mathematics** by **Mr. Pheiroijam Suranjoy Singh**, **Ph.D. Final Registration Number-FINAL/09MAT0015/2017-18**, to Bodoland University, Kokrajhar, Assam, India is an authentic record of the work carried out by him under my supervision in the Department of Mathematical Sciences, Bodoland University, Kokrajhar.

The work presented in this thesis has not been submitted for the award of any other degree or diploma to this or any other institute/university.

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Contents

List of Acronyms/Abbreviations

3D	Three dimension
4D	Four dimension
5D	Five dimension
BDT	Brans-Dicke Theory
CC	Cosmological constant
DE	Dark energy
DM	Dark matter
EFE	Einstein's field equations
ETG	Einstein's theory of gravitation
EoS	Equation of state
GR	General relativity/Theory of general relativity
HDE	Holographic dark energy
KK	Kaluza-Klein
LM	Lyra manifold
MM	Mixed method
NED	Negative energy density
SBT	Saez-Ballester Theory
SCT	Scale Covariant Theory
SS	Spherically symmetric
VE	Vacuum energy

List of Symbols

$a(t)$	Average scale factor
A_h	Anisotropic parameter
c	speed of light in vacuum
G	Gravitational constant
G_{ij}	Einstein tensor
H	Hubble parameter
$j(t)$	Jerk parameter
p	Pressure
p_d	Pressure of dark energy
r, s	Statefinder parameters
$r - s$ plane	Statefinder plane
R	Ricci scalar
R_{ij}	Ricci tensor
T_{ij}, S_{ij}	Energy momentum tensors
V	Volume
Λ	Cosmological constant
ω	Equation of state parameter
Ω_d	Dark energy density parameter
Ω_m	Matter density parameter
Ω	Overall density parameter
ω_{SB}	Saez-Ballester coupling parameter
ω_{BD}	Brans-Dicke coupling parameter
μ, δ	Cosmic scale factors
ρ	Energy density
ρ_d	Energy density of dark energy

ρ_m	Energy density of matter
σ^2	Shear scalar
φ	Scalar field
θ	Scalar expansion

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Referred journals:

[**1**] Kangujam Priyokumar Singh and **Pheiroijam Suranjoy Singh**, Higher dimensional phantom dark energy model ending at the de-Sitter phase, ***Chinese Journal of Physics***, 77 (2022) 1732-1741, DOI: 10.1016/j.cjph.2021.05.022 **(IF-3.237)**.

[**2**] **Pheiroijam Suranjoy Singh** and Kangujam Priyokumar Singh, Vacuum energy in Saez-Ballester theory and stabilization of extra dimensions, ***Universe***, 8 (2022) 60, DOI: 10.3390/universe8020060 **(IF- 2.278)**.

[**3**] **Pheiroijam Suranjoy Singh** and Kangujam Priyokumar Singh, A higher dimensional cosmological model for the search of dark energy source, ***International Journal of Geometric Methods in Modern Physics***, 18 (2021) 2150026, DOI: 10.1142/S0219887821500262 **(IF-1.874)**.

[**4**] **Pheiroijam Suranjoy Singh** and Kangujam Priyokumar Singh, $f(R,T)$ gravity model behaving as a dark energy source, ***New Astronomy***, 84 (2021) 101542, DOI: 10.1016/j.newast.2020.101542 **(IF-1.325)**.

[**5**] Kangujam Priyokumar Singh and **Pheiroijam Suranjoy Singh**, Dark energy on higher dimensional spherically symmetric Brans-Dicke universe, ***Chinese Journal of Physics***, 60 (2019) 239-247, DOI: 10.1016/j.cjph.2019.05.003 **(IF-3.237)**.

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International:

1. ***Is the big rip singularity the ultimate fate of the universe?*, 4th International Conference of Arts and Sciences, orgd. by Cebu Normal University, Philippines on September 21-22, 2022 (virtual)**.

2. *Will the universe rip apart in the phantom scenario?*, The 15th International Online Conference of IMBIC on "Mathematical Sciences for Advancement of Science and Technology", orgd. by Institute for Mathematics, Bioinformatics, Information Technology and Computer Science, Kolkata, India on December 21-23, 2021. (virtual).

3. ***Is the cosmic doomsday inevitable when the dark energy EoS parameter is less than -1?*, The 30th Workshop on General Relativity and Gravitation in Japan, jointly orgd. by Dept. of Physics, Waseda University, Japan, Waseda Institute for Advanced Study, Japan and Institute of Theoretical Physics, Rikkyo University, Japan on December 6-10, 2021 (virtual).**

4. *Dynamics of a dark energy cosmological model in higher dimensional setting*, International Webinar on Foundations for Contemporary Mathematical Research, orgd. by Dept. of Mathematics, Manipur University, India on November 8-10, 2021.

5. *Phantom Dark Energy Model Avoiding Big Rip Singularity*, International Conference on Advanced Physics 2021, orgd. by Dept. of Basic Science and Humanities, Institute of Engineering and Management, Kolkata, India on April 1-3, 2021 (virtual).

6. *Scale Covariant Theory Behaving as a Phantom Dark Energy Model*, 2nd International Conference on Mathematical Techniques and Applications, orgd. by Dept. of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Tamil Nadu, India on March 24-26, 2021 (virtual).

7. *Developing a Dark Energy Model from Modified Gravity Theory*, International Conference on Emerging Trends in Mathematical Sciences and Computing, orgd. by Dept. of Basic Science and Humanities Institute of Engineering & Management, Kolkata, India on February 5-7, 2021 (virtual).

8. *Dark Energy Model with the Estimation of the Variation of the Gravitational Constant*, Virtual International Conference on Physical Sciences, jointly orgd. by Dept. of Physics, Chemistry and Applied Mathematics and Humanities, Sardar Vallabhbhai National Institute of Technology, Surat, India on February 5-6, 2021.

9. *Phantom Dark Energy Model Ending at the de-Sitter Phase*, International Web Seminar on Applied Mathematics & Mathematical Physics, Number Theory and Analysis, orgd. by Calcutta Mathematical Society, India on January 16-17, 2021.

10. *A Search for the Dark Energy Source*, 14th International Conference MSAST 2020, orgd. by IMBIC, India on December 21-23, 2020 (virtual).

11. *Whether Scale Covariant Theory Itself Can Behave as a Dark Energy Model?*, 26th International Conference of International Academy of Physical Sciences (CONIAPS XXVI) on Advances in Relativistic Astrophysics and Cosmology, orgd. by Dept. of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal, Punjab, India on December 18-20, 2020 (virtual).

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13. *A Dark Energy Model with Big Crunch Singularity as the Ultimate End*, International Webinar on Mathematics and Application, orgd. by Dept. of Mathematics, The University of Burdwan, West Bengal, India on September 25-27, 2020.

14. ***Dark Energy in Five Dimensional Spherically Symmetric Universe with Future Singularity*, Cosmología en Colombia (CoCo) 2020, orgd. by Universidad Antonio Narino, Colombia in collaboration with Universidad del Valle, Colombia, Universidad Industrial de Santander, Colombia and Universidad Nacional de Colombia, Colombia on September 23-25, 2020 (virtual).**

15. *A Dark Energy Model with the Estimation of Hubble's Parameter and Energy Density Parameters*, International Conference on Recent Trends in Multidisciplinary Research, orgd. by Eudoxia Research Centre, India on September 19-20, 2020 (virtual).

16. *A Discussion on the Interacting Scenario of Dark Matter and Dark Energy*, International Conference on Present Scenario of Mathematical Sciences, orgd. by Dept. of Mathematics, Karnatak University's Karnatak Arts College, Dharwad, Karnataka, India on September 12-13, 2020 (virtual).

17. *An Increasingly Dark Energy Dominated Model in Brans-Dicke Theory*, International Conference on Mathematics and Computer Applications, orgd. by Dept. of Mathematics, Don Bosco College, Yelagiri Hills, Tamil Nadu, India on September 10-12, 2020 (virtual).

18. *A Dark Energy Model Satisfying the Findings of the Latest Planck 2018 Results*, International Webinar on Recent Advancements in Mathematical Sciences and its Applications, orgd. by Dept. of Mathematics, Chakdaha College, West Bengal on September 3-4, 2020.

19. *Interacting Matter and Dark Energy Model in Higher Dimensional Space Time*, 1st International Conference on Advances in Mathematics, Science and Technology, orgd. by Dept. of Mathematics, Rajiv Gandhi University, Arunachal Pradesh, India on September 1-3, 2020 (virtual).

20. *Dark Energy Model with Negative Constant Deceleration Parameter*, Two-days International Webinar on Recent Advances in Pure and Applied Mathematics, orgd. by Dept. of Mathematics, Kurseong College, Darjeeling, West Bengal, India on August 24-25, 2020.

21. *Higher Dimensional Holographic Dark Energy Model in Brans-Dicke Universe*, International Conference on Mathematical Advances & Applications, orgd. by Ratna Prasad Multidisciplinary Research & Educational Institute, Vijayawada, AP, India on August 7-8, 2020 (virtual).

22. *Interacting Matter and Holographic Dark Energy Model in Scalar-Tensor Theory of Gravitation*, International Webinar on Recent Trends in Mathematical Sciences, orgd. by Govt. Arts & Science P.G. College, Ratlam (M.P.), India on August 6-7, 2020.

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ABSTRACT

The thesis titled "**Dark Energy in Higher Dimensional Spherically Symmetric Space-time**" consists of 9 chapters, for which the abstracts of each chapter is given below.

Chapter 1 is an introductory chapter. The chapter explains the motivation for conducting the research and introduces the works presented in the thesis. In this chapter, we provide the basic idea of the foundation and formulation of cosmological problems of general relativity. We also present the brief highlights of various concepts, space-time, and theories of gravitations discussed in the following chapters of the thesis. We also present the review of various literature from different sources to obtain a better grasp of the past and present works.

Chapter 2 presents the research method and methodology employed to acquire knowledge and data to answer the research problem. The logic behind the method used to acquire and analyse the results of the study is discussed in this chapter.

Chapter 3 presents a research problem titled "**Higher dimensional phantom dark energy model ending at a de-Sitter phase**". The work discussed in this chapter is published in **Chinese Journal of Physics, 77 (2022) 1732-1741, DOI: 10.1016/ j.cjph.2021.05.022 (IF-3.237)**. In this chapter, using a spherically symmetric metric, we investigate a minimally interacting holographic dark energy model within the framework of Saez-Ballester Theory. We predict that the dark energy component dominating the universe is of phantom type, which will lead the model universe to cosmic doomsday (big rip singularity). As big rip and holographic dark energy are incompatible with each other, we employ a higher dimensional scenario so that the cosmic doomsday is replaced by the de-Sitter phase. The model expands with a slow and uniform change of size during the early evolution, whereas the change becomes faster, agreeing with the present observation of the accelerated expansion. The present values of Hubble parameter and the dark energy EoS parameter are measured to be $H = 67$ and $\lambda = -1.00011$, which agree with the respective values $H_0 = 67.36 \pm 0.54 kms^{-1}Mpc^{-1}$ and $\lambda = -1.03 \pm 0.03$ of the most recent Planck 2018 result. Discussions on the parameters obtained are also presented in details with graphs.

Chapter 4 presents a research problem titled "**Vacuum energy in Saez-Ballester Theory and stabilization of extra dimensions**". The work discussed in this chapter is

is published in **Universe, 8 (2022) 60, DOI: 10.3390/universe8020060 (IF-2.278)**. In this chapter, we study a spherically symmetric metric in 5D within the framework of Saez-Ballester Theory, where minimal dark energy-matter interaction occurs. We predict that the expanding isotropic universe will be progressively DE dominated. We estimate few values of the deceleration parameter, very close to the recently predicted values. We obtain the value of the DE EoS parameter as $\omega = -1$. Additionally, we measure the value of the overall density parameter as $\Omega = 0.97(\approx 1)$, in line with the notion of a close to or nearly (not exactly) flat universe. We predict that the model universe starts with the Big-Bang and ends at the Big Freeze singularity. In general, we cannot find conditions for stabilization of extra dimensions in general relativity, and all dimensions want to be dynamical. Here, we present two possible conditions to solve this stabilization problem in general relativity.

Chapter 5 presents a research problem titled "**A higher dimensional cosmological model for the search of dark energy source**". The work discussed in this chapter is published in **International Journal of Geometric Methods in Modern Physics, 18 (2021) 2150026, DOI: 10.1142/S0219887821500262 (IF-1.874)**. In this chapter, with due consideration of reasonable cosmological assumptions within the limit of the present cosmological scenario, we analyse a spherically symmetric metric in 5D setting within the framework of Lyra manifold. The model universe is predicted to be a DE model, dominated by vacuum energy. The model represents an oscillating model, each cycle evolving with a big bang and ending at a big crunch, undergoing a series of bounces. The universe is isotropic and undergoes super-exponential expansion. The value of Hubble parameter is measured to be $H = 67.0691$ which is very close to $H_0 = 67.36 \pm 0.54 kms^{-1}Mpc^{-1}$, the value estimated by the latest Planck 2018 result. A detailed discussion on the cosmological parameters obtained is also presented with graphs.

Chapter 6 presents a research problem titled "$f(R,T)$ **gravity model behaving as a dark energy source**". The work discussed in this chapter is published in **New Astronomy, 84 (2021) 101542, DOI: 10.1016/j.newast.2020.101542 (IF-1.325)**. In this chapter, within the limits of the present cosmological observations in $f(R,T)$ gravity theory, we analyse a spherically symmetric space-time in 5D setting. The field equations have been carefully studied considering reasonable cosmological assumptions to obtain exact solutions. It is predicted that the isotropic model universe behaves like a dark energy (vacuum energy) model. In the present scenario, the model evolves with a slow and uniform change of shape. It is observed that the universe is close to or nearly flat. The model is free from initial singularity and is predicted to approach the de-Sitter phase dominated by vacuum energy or cosmological constant in the finite-time future. A comprehensive discussion on the cosmological parameters obtained in view of the recent studies is presented in detail with graphs.

Chapter 7 presents a research problem titled "**Scale Covariant Theory as a dark energy model**". The work discussed in this chapter is under review at **International Journal of Geometric Methods in Modern Physics**. In this chapter, we study a spherically symmetric space-time in 5D with the consideration of Scale Covariant Theory. The Scale Covariant Theory model is found to be isotropic and behaves as a phantom dark energy model, which tends to the de-Sitter phase avoiding finite time future singularity (big rip). The gravitational constant G decreases with a variation of -7.2×10^{-11} yr^{-1} and the Hubble's parameter is estimated to be $H = 68$. A detailed interpretation of the cosmological findings is also provided with graphical representations.

Chapter 8 presents a research problem titled "**Dark energy on higher dimensional spherically symmetric Brans-Dicke universe**". The work discussed in this chapter is published in **Chinese Journal of Physics, 60 (2019) 239-247, DOI: 10.1016/j.cjph.2019.05.003 (IF-3.237)**. In this chapter, we present a cosmological model in 5D spherically symmetric space-time with energy momentum tensors of minimally interacting fields of dark matter and holographic dark energy in Brans–Dicke Theory. Under some realistic assumptions in consistent with the present cosmological observations, we have analyse the field equations to obtain their exact solutions. With particular choices of the constants involved, the values of the overall density parameter and the Hubble parameter are obtained to be very close to the latest observational values. We obtain a model universe which will be increasingly dark energy dominated in the far future. A comprehensive presentation of the physical as well as kinematical aspects of the parameters, including future singularity, in comparison with the present observational findings is also provided.

Chapter 9 is a concluding chapter. This chapter aims to provide a thorough summary of the major findings and the arguments of the research. The chapter summarizes the works presented in the thesis and conveys the relevance of the research. The chapter also explains what new knowledge has been brought to light and suggests future research ideas on the subject.

Chapter 1

Introduction

This chapter explains the motivation for conducting the research and introduces the works presented in the thesis. In this chapter, we provide the basic idea of the foundation and formulation of cosmological problems of general relativity. We also present the brief highlights of various concepts, space-time, and theories of gravitations discussed in the following chapters of the thesis. Above all, we also present the literature review of articles to get a better understanding of similar works from the past and present. The review is helpful in assisting to identifying the knowledge gap as well as potential research ideas.

1.1 Motivation

The Earth looks like a beautiful blue marble due to its 70% water content. Even though the dark under the deep blue ocean remains unexplored to a large extent, we at least know water, and its stretch since ocean exploration started a long time ago. But, what if we are visiting an ocean for the first time. The first experience of the amazing view, the sound of the endless cold waves, and the beautiful shoreline will draw us into the magnificent ocean. This will be a magical experience, possibly igniting a spark of interest leading to the birth of a new scientific study. Presently, cosmologists are at the dawn of such a new scientific study since the discovery of the 70% content of the universe called dark energy (DE). The term “dark energy” was coined by Michael S. Turner in 1998. DE is the dominant component of the immense universe, according to literature and observations. This classifies DE as the perfect humour in that the dominating part of the universe is also the least studied. The late distinguished professor of IUCAA and a renowned astrophysicist, Thanu Padmanabhan labelled this dark component as the “Mystery of the Millennium” (Padmanabhan 2006). This dark entity is believed to be the driving force behind the late-time accelerated expansion of the universe. Cosmologists and theorological physicists all over the map, despite investing tremendous scientific efforts, details of its origin, nature, and application to modern cosmology are still up for grabs. Similarly, understanding precisely the origin of the universe, its evolution, and the ultimate fate are no less challenging for modern cosmology. As an effort to broaden our knowledge about the enigmatic DE and the dynamics of the mysterious universe, we have considered an investigation using a 5D spherically symmetric metric paired with some modified theories of gravity which is presented in this thesis titled "**Dark Energy in Higher Dimensional Spherically Symmetric Space-time**".

1.2 Dark energy

Gazing toward the night sky uncovers a little piece of the universe. Despite the fact that the universe seems static to the unaided eye, it is expanding at an expedited rate. Researchers attribute this expanding paradigm to a hypothetical form of energy called dark energy (DE). DE is a natural property of space with a constant energy density and a large negative pressure exerting a gravitationally repulsive effect driving the late-time accelerated expansion of the universe. It is the dominant component making up 70% of the universe. DE isn't straightforwardly noticed but instead deduced from perceptions of gravitational interactions between cosmic objects.

The four meetings of the Prussian Academy of Science during November 1915 can be set apart as the most memorable minutes in the life of the famous Einstein. On the fourth, eleventh, eighteenth, and twenty-fifth of the month, he introduced four of his outstanding communications (Einstein 1915a, 1915b, 1915c, 1915d) at the meetings, which prompted the establishment of the Theory of General Relativity, or simply General Relativity (GR). In 1917, he introduced the cosmological constant into his theory as a repulsive force to act against the attractive gravity to maintain a static universe (Einstein 1917). The cosmological constant is denoted by the Greek alphabet Λ. In 1929, Hubble made ground-breaking discoveries that showed the universe is expanding (Hubble 1929), defying the concept of a static universe. Einstein considered the introduction of Λ to his theory as the "greatest blunder". Finally, he dropped the constant from his work (Einstein 1931). Over the years, Λ went in and out of favour as new observational findings seemed to necessitate it time and again. There were suggestions in the early '90s that Λ might be needed once more.

In 1998, the astronomical observations of distant Type Ia supernovae by two independent teams of astronomers discovered that the rate of expansion of the universe was accelerating, rather than slowing down (Riess et al. 1998; Perlmutter et al. 1999). This discovery was based on the observation that the supernovae appear fainter than expected for a universe decelerating under gravity. For this, the supernovae must be farther away, and the expansion rate should be slower in the past.

Finally, researchers concocted three ways to explain the accelerated expansion. Perhaps it is a consequence of the abandoned Λ term. Perhaps there is some bizarre sort of energy-fluid filling up space. Possibly there is a mistake with the relativity theory, and another optimized theory could incorporate some sort of field that leads to the expedited expansion. Researchers don't have the foggiest idea of the right explanation, yet they termed the possible answer as "dark energy". DE exerts a gravitationally repulsive effect that pushes rather than pulls, driving the miraculous expanding phenomenon. The cosmological constant Λ is considered to be the most natural candidate for DE.

Cosmologists consider the equation of state (EoS) parameter ω a good choice to classify DE into specific categories. ω is defined as the ratio of the pressure of DE to its energy density. The value $\omega = -1$ represents the cosmological constant (CC), or in other words, vacuum energy (VE). Phantom energy has $\omega < -1$, whereas the range $-1 < \omega < \frac{-1}{3}$ signifies quintessence. To construct a cosmological model universe undergoing late-time expedited expansion, one should obtain the range $\omega < -\frac{1}{3}$ (Tripathi et al. 2017). According to the most recent Planck 2018 results (Collaboration et al. 2020), $\omega = -1.03 \pm 0.03$, which is an indication that the form of DE in the present universe is highly likely to be of phantom type.

As mentioned above, the cosmological constant Λ is the most natural candidate for DE. However, it falls short of explaining the enigma of the coincidence problem (Zlatev et al. 1999). After numerous attempts, researchers proposed different candidates of DE (Copeland et al. 2006). One such proposed candidate worth considering is the holographic dark energy (HDE) introduced by Gerard 't Hooft (Hooft 2009). As a result of the holographic principle (Bousso 2002) being applied to DE, HDE is formed. The work of Wang et al. (2017) provides a peek of HDE's fundamental nature and properties.

1.3 Higher dimensional cosmological model

According to our daily experience and observations, it is obvious that we are living in a 3D space with one time dimension i.e., 4D space-time. We can only move forward and backward, left and right, and upward and downward. We notice that our physical laws solely rely on just three spatial dimensions to explain the movements of living and non-living things around us. Then, why is all the fuss about this extra dimension? This question might appear valid, nevertheless, there is no solid logical justification that space-time should have no more four dimensions (Zumino 1986; Overduin & Wesson 1997; Rubakov 2001; Brax & Bruck 2003; Bruck & Longden 2019). The study on extra-dimensions began in order to explain some of the challenges that had arisen in physics, for instance, the cosmological constant problem (Zel'dovich 1967, 1968).

A cosmological model equipped with at least one extra dimension beyond the standard four dimensions is termed a higher dimensional cosmological model. According to Bahrehbakhsh et al. (2011), the first study that presented the construction of a unified theory based on extra dimensions can be found in the work by Nordstorm (1914). There was a time when the hunt for a unified explanation of gravity and particle interaction resulted in a large number of astrophysical works getting stuck. However, the problem was solved in 1921 when Kaluza extend GR from 4D to 5D, uniting gravity with electromagnetism (Kaluza 1921). In 1926, by considering the small size of the extra dimension, Klein modified the method to include quantum effects (Klein 1926). These led to the attribution of the introduction of the higher dimensional cosmological model in GR to Kaluza and Klein. In the following years, researchers have proposed numerous options for the possibility of having more than one extra dimension. As anyone might expect, a conspiracy seems to start to humiliate the advocates of extra dimension. For instance, the theoretical physicist Lee Smolin strongly criticizes string theory which employs extra dimensions, whereas Peter Woit claims that the theory is not even science (Woit 2006; Smolin 2006). Nonetheless, numerous studies have successfully developed compelling justifications for the existence and practical importance of employing extra dimensions.

The higher-dimensional model emerges as one of the good choices among cosmologists and theorological physicists. Such a model can explain both the early inflation and the late time expanding phenomenon of the universe (Farajollahi & Amiri 2010; Banik & Bhuyan K. 2017; Aly 2019). Marciano (1984) discusses a study to validate the existence of the extra dimension. Questions about the nature of DM and DE may find answers in theories involving extra dimensions (Bruck & Longden 2019). According to Zhang (2010), the employment of an extra dimension makes HDE models more complete and consistent. Extra dimensions help to solve the hierarchy problem in a natural way (Randall 2007). Wesson (2015) asserts that the fifth dimension has made a significant contribution to our understanding and the logical consistency of physics. Perhaps our lives would have been less interesting if we haven't been concerned with extra-dimensions.

1.4 Stabilization of extra dimensions

The study on the stabilization of extra dimensions is considered a phenomenological necessity in higher-dimensional models. Generally, we witness the discussion on stabilization in the field of particle physics, supersymmetry, supergravity, string theory, and braneworld models. We require a stabilization mechanism to prevent modification of gravity to an experimentally undesirable manner (Kribs 2006). The stabilization also makes sure the visible 4D universe with a long lifetime (Ketov 2019). Another benefit of stabilization is that we can ignore any unwanted outcomes of quantum gravity at Planck length distances (Hamed et al. 2002). One of the most classic solutions for stabilization is the Goldberger-Wise mechanism (Goldberger & Wise 1999), where stabilization is achieved in the presence of an additional scalar field. The works in this thesis are based on cosmological models in GR. In GR, generally, we cannot find conditions for stabilization, and all dimensions want to be dynamical (Bruck & Longden 2019). In an accelerating model with the cosmological constant, stabilization cannot be obtained (Rador 2007). Notwithstanding this stabilization problem in GR, we have presented, in Chapter 4, a trial to solve the issue in GR. Our work is most likely the first to establish the possible stability condition in GR.

1.5 Spherically symmetric space-time

Since the outset of GR in 1915, spherically symmetric (SS) space-time has garnered ample attention and praise. We can witness works on SS space-time as early as in the papers of 1916 and 1917 by renowned authors (Szenthe 2004b). With the progress of the research on SS space-time, the study on the relativistic theory of cosmology in GR has also been developed (Takeno 1952a). SS space-time can be considered as one of the important tools for studying GR owing to its comparative simplicity and useful applications to both astrophysics and cosmology. It simplifies the study of a system's dynamics by allowing the transformation of a 4D solution to 2D (Parry 2014). The space-time used in relativistic cosmology, including the space-time of the de-Sitter and the Einstein universes, is also SS (Takeno 1952a). The Robertson-Walker space-time model depicting the expanding cosmos is also SS (Karade 1980). To discuss a problem in GR, SS space-time is an excellent option to start with. Deriving non-trivial SS space-time as the exact solution of the Einstein equation is one of the first tasks taken up in GR, a crucial solution in terms of experimental verification of GR (Das & DeBenedictis 2012; Parry 2014). This led to the development of Schwarzschild space-time, which is perhaps the most significant SS solution, and then Birkhoff's theorem along with some of its generalizations (Birkhoff 1923; Wald 1984; Bronnikov & Melnikov 1995; Szenthe 2004a; Jebsen 2005). Some of the remarkable works with a great deal of information about SS space-time can be found in the articles of Takeno (1951, 1952a, 1952b, 1952c, 1952d, 1952e, 1952f, 1953, 1966), Takeno & Ikeda (1953), Kunzle (1967), Clark (1972), Foyster & Mclntosh (1973), Szenthe (2004a), Ferrando & Saez (2010), Tupper et al. (2012), Parry (2014) and Bagde et al. (2021). Since SS space-time is still noteworthy, and there is a lot of content about it spread across the literature, a discussion on it within the framework of GR to understand DE and the accelerating universe would be valuable.

If the isometry group of a space-time contains a subgroup that is isomorphic to the rotation group $SO(3)$, then the space-time is referred to as SS. By expressing space-time in terms of scalars and vectors, Takeno (1951) put forward the definition of a SS space-time in 4D. However, because the thesis mainly looks at higher-dimensional models, we won't go into great length concerning 4D. Takeno (1952e) further defines an n ($n \geq 5$) dimensional SS space-time with the metric tensor g_{ij} as an n dimensional Riemannian space with the following properties:

i. Its curvature tensor satisfies

$$K_{ijlm} = \rho^1 \alpha_{[i} \alpha_{[l} \beta_{j]} \beta_{m]} + \rho^2 g_{[i[l} \alpha_{j]} \alpha_{m]} + \rho^3 g_{[i[l} \beta_{j]} \beta_{m]} + \rho^4 g_{[i[l} g_{j]m]} \tag{1.5.1}$$

where $i, j, ... = 1, ..., n, \alpha_i$ and β_i are mutually orthogonal unit vectors satisfying

$$\nabla_i \alpha_j = \sigma \alpha_i \beta_j + \kappa \left(g_{ij} - \alpha_i \alpha_j - \beta_i \beta_j \right) - \overline{\sigma} \beta_i \beta_j \tag{1.5.2}$$

$$\nabla_i \beta_j = \overline{\sigma} \beta_i \alpha_j + \overline{\kappa} \left(g_{ij} - \alpha_i \alpha_j - \beta_i \beta_j - \right) \sigma \alpha_i \alpha_j \tag{1.5.3}$$

and $\rho^a, (a = 1, ..., 4); \sigma, \overline{\sigma}; \kappa, \overline{\kappa}$ are scalars determined from these equations.

ii. One of the five scalars $\rho^a, (a = 1, ..., 4)$ and $K \equiv K_{ij}^{ji}$ is such that its gradient vector is a linear combination of α_i and β_i.

iii.

$$\rho^4 + 2 \left(\kappa^2 + \overline{\kappa}^2 \right) \neq 0 \tag{1.5.4}$$

iv. Moreover, for the sake of simplicity and symmetry, the fundamental form is taken positive and $\alpha_s \alpha^s = \beta_s \beta^s = 1$.

In the above properties, ∇ is the usual Riemannian covariant derivative, whereas the ( ) and [ ] respectively denote the usual symmetric and antisymmetric relations. Corresponding theory based on g_{ij} of the type $(- - ... +)$ and $-\alpha_s \alpha^s = \beta_s \beta^s = 1$ can be derived with minor modifications.

Abolghasem et al. (1998) present a general form of the 5D SS metric provided in Eq. (1.5.5) and obtain solutions with potential applications to astrophysics and cosmology.

$$ds^2 = -P^2(t,r,y)\,dt^2 + Q^2(t,r,y)\left(dr^2 + r^2 A^2(t,y)\,d\overline{\Omega}^2\right) + R^2(t,r,y)\,dy^2 \tag{1.5.5}$$

where y is the coordinate corresponding to the extra dimension and $d\overline{\Omega}^2$ is the metric of the 2-sphere given by

$$d\overline{\Omega}^2 = d\Theta^2 + sin^2\Theta d\phi^2 \tag{1.5.6}$$

In this thesis, we pair some modified theories of gravity with a 5D SS metric (Samanta & Dhal 2013) provided in Eq. (1.5.7) in order to broaden our knowledge about the enigmatic DE and the dynamics of the mysterious universe.

$$ds^2 = dt^2 - e^{\mu}\left(dr^2 + r^2 d\Theta^2 + r^2\sin^2\Theta d\phi^2\right) - e^{\delta}dy^2 \tag{1.5.7}$$

where $\mu = \mu(t)$ and $\delta = \delta(t)$ are cosmic scale factors.

1.6 Cosmology

Cosmology, the scientific investigation of the beginning of the universe and its progression by and large, is as old as humankind. A bone fragment depicting a lunar calendar unearthed in Sub-Saharan Africa circa 20,000 BC is the earliest known proof for cosmological reasoning among humans, and the Nebra sky disc, which dates back to roughly 1,600 BC, is the oldest record of cosmic observation (Corneanu & Corneanu 2016). The term cosmology is derived from the Greek words "kosmos" and "logia" which mean "world" and "study of" respectively. The term first appeared in English in 1656 and was later adopted in Latin by German philosopher C. Wolff in 1731, but it was after WWII that it became scientifically mainstream (Kragh 2007; Hetherington 2014).

Humans have gazed at the stars for centuries, pondering about the mysteries of the universe's dynamics and evolution. However, it wasn't until the '90s when scientists developed modern observational tools and theory, which revolutionizes cosmology forever. This marked the era of the birth of modern cosmology (Topper 2013; Nussbaumer 2014). The present-day modern cosmology has entered a beautiful era - The Golden Age of Cosmological Physics, thanks to the raw data derived from the accurate calculations of different cosmological parameters from several experiments (Garcia-Bellido 2000). The enormous increase in observational approach explicitly committed to cosmological problems demonstrates that modern cosmology is becoming a mature physical science with its own subject and method (Baryshev et al. 2008).

The hot Big Bang model, which explains the universe's evolution from the first fraction of a second to the current era, about 13.8 billion years later, is the foundation of our current knowledge of the cosmos. This model was developed in 1931 by Lemaitre, in which he assumed that the universe expanded from an initial point - "primeval atom" (Lemaitre 1931). It is the most widely accepted theory about how the universe began. It states that the universe began with an infinitely hot and dense singularity, which subsequently inflated, initially, at an extremely high speed, then at a more quantifiable rate throughout the following years to become the universe we see today. The model is homogeneous and isotropic, with matter and radiation fluids as its major components, and kinematic properties agree with those measured in the actual universe. Furthermore, the radiation component of the energy density is considered to be of cosmological origin, which is why the model is referred to as "hot" (Coles & Lucchin 2002). Undoubtedly, our actual universe isn't perfectly homogeneous and isotropic, so that this model has some flaws. However, this standard model offers us a platform to explore the creation of objects like galaxies and their clusters from minor alterations in the density of the early cosmos.

The Big Bang cosmology is based upon four solid foundations, a GR-based theoretical basis, presented by Einstein (1917, 1922) and Friedmann (1922), as well as three fundamental observational facts. The first is Hubble's discovery of the universe's expansion (Hubble 1929). Second, the elucidation of the relative abundance of light elements in the '40s by Gamow (1946, 1948). The third is the cosmic microwave background (CMB), discovered by Penzias & Wilson (1965) as the afterglow of the Big Bang. These findings contributed to the hot Big Bang becoming the most favoured model, and they have been verified to near-perfect accuracy (Garcia-Bellido 2005).

Despite its widespread acceptance, the hot Big Bang model is not free from drawbacks. In the Big Bang scenario, the universe is homogeneous and isotropic. However, these conditions, which appear to be self-evident at first glance, are not fully addressed by the theory. This is a major flaw in this theory, or perhaps, numerous flaws that are interconnected. The inflationary theory is one viable solution. The inflationary theory, in particular, proposes a method to give rise to cosmic perturbations. The Big Bang lacks such a method, which is also a major flaw of the theory. This is one of the reasons that the inflation theory is so appealing. Guth (1981) is credited for introducing the theory of cosmic inflation. The scenario that triggered inflation involves a scalar field in a local (but not global) minimum of its potential energy function (Guth 2004). Starobinsky (1979, 1980) introduced a similar concept a little earlier as a (failed) effort to address the initial singularity problem. Later, the Norwegian Academy of Science and Letters awarded the 2014 Kavli Prize in Astrophysics to Alan Guth, Andrei Linde, and Alexei Starobinsky for pioneering the theory of cosmic inflation. According to the inflationary theory, there was a brief period of extremely rapid cosmological expansion preceding the more gradual Big Bang expansion. During the period, the universe's energy density was dominated by a CC type of VE, which then decayed, resulting in the formation of matter and radiation. An in-depth explanation of inflationary theory as a possible solution to shortcomings of Big Bang theory, viz. horizon problem, flatness problem, entropy problem, and primordial perturbation problem, is presented in the book authored by Gorbunov & Rubakov (2011).

1.7 Theory of general relativity

The four meetings of the Prussian Academy of Science in November 1915 are among Einstein's most memorable moments. At the sessions, he gave four remarkable presentations (Einstein 1915a, 1915b, 1915c, 1915d) that led to the establishment of GR. His intellect reimagined space and time, foreshadowing a universe so strange and vast that it defied human imagination. GR is a fundamental concept in modern physical science. It correlates gravity to curvature of space-time geometry, or, to put it another way, it explains gravity in the context of bending space. Einstein came up with GR after a decade he put forward the special theory of relativity (Einstein 1905), which asserts that space and time are inextricably linked but did not address the presence of gravity. To be specific, GR, as the name implies, is the generalized form of the special theory. The mathematical equations of GR, which have been confirmed repeatedly, are by far the most accurate tool to describe gravitational interactions, effectively replacing those proposed by Newton (Newton 1687) hundreds of years ago. GR is a remarkable achievement. It is now commonly regarded as one of the two foundations of modern physics, alongside quantum field theory. Despite the benefits of GR, due to certain incompatibilities, we don't yet have a quantum field theory counterpart of GR. Harmonizing GR with quantum physics is still a work in progress in modern physics (Kiefer & Weber 2005; Alfonso-Faus 2007; Mamedov 2015; Jakobsen 2020).

In 1918, Einstein put forward the following three principles on which the establishment of GR rests (Einstein 1918).

(a) ***Principle of relativity:*** The laws of nature are only assertions of time-space coincidences; therefore they find their unique, natural expression in generally covariant equations.

(b) ***Principle of equivalence:*** Inertia and weight are identical in essence. From this and from the results of the special theory of relativity, it follows necessarily that the symmetric "fundamental tensor" (g_{ij}) determines the metric properties of space, the inertial relations of bodies in it, as well as gravitational effects. We will call the condition of space, described by the fundamental tensor, the "G-field".

(c) ***Mach's principle:*** The G-field is determined without residue by the masses of bodies. Since mass and energy are equivalent according to the results of the special theory of relativity and since energy is described formally by the symmetric energy tensor (T_{ij}), this means that the G-field is conditioned and determined by the energy tensor.

Further, in the footnote of his work (Einstein 1918), he wrote that he was introducing the term "Mach's principle" for the first time. The principle (a) can also be referred to as the principle of general covariance since the latter is a generalization of principle (a) (Norton 1993). Ellis & Williams (1988) extended principle (b) by stating that "the laws of physics are the same for all observers, no matter what their state of motion".

Carmeli (1982) defined three versions of the general covariance principle as given below, which, he mentioned, were "not quite equivalent".

- All coordinate systems are equally good for stating the laws of physics. Hence, all coordinate systems should be treated on the same footing, too.
- The equations that describe the laws of physics should have tensorial forms and be expressed in a four-dimensional Riemannian space-time.
- The equations describing the laws of physics should have the same form in all coordinate systems.

In 1907, Einstein observed that an object in a free fall doesn't feel its weight, later established as the principle of equivalence (Samaroo 2020). The principle of equivalence incorporates gravity's effects into the formation of GR. It establishes the equivalence of the forces exerted by gravity and acceleration. Accordingly, a physical experiment cannot differentiate between gravitational and acceleration forces. As presented by Pauli (1958), the equivalence principle would be satisfied even though the coordinate system is not physically realizable. In addition, the principle is satisfied if and only if a manifold is physically realizable.

Mach's proposal that inertial motion is regulated by the whole of masses in the universe, rather than Newton's absolute space and time (Mach 1872, 1883) was one of the main influences to Einstein's formulation of GR as it hinted to a relationship between geometry and matter. Later in 1918, Einstein presented a specific statement of it in the framework of GR (Einstein 1918). Einstein later dropped the principle (Einstein 1949) when he established that inertia is implicit in the geodesic equation of motion and doesn't rely on the presence of matter somewhere else in the cosmos. In the literature, there are various ways in which Mach's principle is formalized, particularly in the framework of GR (Barbour & Pfister 1995; Barbour 2010; Putz 2019).

The 4D line element in special theory of relativity is given by

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2dt^2 \tag{1.7.1}$$

where x, y and z are Cartesian coordinates.

The space-time in GR is described by the pseudo-Riemannian metric given by

$$ds^2 = g_{ij}dx^i dx^j, \ i, j = 1, ..., 4 \tag{1.7.2}$$

This is the generalization of the 4D space-time in special relativity. The symmetric metric tensor g_{ij} acts as gravitational potential. In GR, the space-time is 4D and the gravitational orbits are geodesics. Einstein's field equations (EFE) which describe the behaviour of space and time are given by

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} = -\frac{8\pi G}{c^4}T_{ij} \tag{1.7.3}$$

where G_{ij} is the Einstein tensor, R_{ij} is the Ricci tensor, R is the Ricci scalar (Scalar curvature), c is speed of light in vacuum, G is the Newtonian constant of gravitation and T_{ij} is the energy-momentum tensor due to matter.

In order to maintain a static universe (Einstein 1917), Einstein modified the field equations by introducing the Λ term as follows.

$$G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = -8\pi T_{ij} \tag{1.7.4}$$

In 1929, Hubble provided breakthrough discoveries to indicate that the universe is expanding, contradicting the idea of a static universe (Hubble 1929). Einstein considered the introduction of the Λ term as the "greatest blunder". Finally, he dropped the constant term from his work (Einstein 1931). The left-hand sides of both the Eqs. (1.7.3) and (1.7.4) represent the geometry of space-time determined by the metric, and the right-hand sides, the matter distribution/energy content of the space-time.

1.8 Modified theory of gravity

Newton proposed the mathematical formalization of gravity in 1687 and presented one of the most important results in physics (Newton 1687), as given below.

$$F = G\frac{m_1 m_2}{\bar{r}^2} \tag{1.8.1}$$

Newton's work, however, ended as just half of the picture. The second half is GR, postulated by Einstein more than two centuries later. Einstein's GR is regarded as one of the greatest achievements of twentieth-century physics. It has far-reaching implications for many cosmological phenomena. Besides describing the anomalous precession of planetary orbits, it also explains the origin and evolution of the universe, the physics of black holes, and gravitational lensing. In recent years, scientists and engineers have created modern methods and technologies that accurately depict the effects of GR, for instance, the effect of GR ensures Global Positioning System (GPS) gadget detect a location precisely within a few meters (Ashby 1995, 2003).

However, in recent decades, researchers have raised issues that cannot be effectively addressed by GR alone. GR, although its accomplishment in characterising the universe and the solar system, falls short of being the ultimate theory of gravity. With the introduction of the dark universe scenario, the constraints of GR have come into prominence. There has been indications for nearly three decades that if gravity is governed by EFE, then the cosmos should contain a significant quantity of DM, and DE has just been discovered to be required to explain the universe's purported accelerated expansion. If GR is accurate, it appears that approximately 96 percent of the cosmos is made up of energy densities that do not interact electromagnetically (Clifton et al. 2012). Because of such an unusual content, researchers have suggested that GR may not be the right gravity theory to address the universe. The emergence of a dark universe could be another clue that we need to explore outside the scope GR. We may also witness a discussion about the limitations of GR in Krogdahl's work (Krogdahl 2007). As a result, researchers devised alternate theories to extend GR, employing various approaches to generate different field equations and cosmic consequences. Such theories are known as modified theories of gravity or alternatives to GR.

Over the last decade, the concept of modifying gravity theory has exploded in popularity. Such modification has been partly motivated by the introduction of higher-dimensional cosmological models and the advancement in the formulation of renormalizable gravity theories (Clifton et al. 2012). Modifying GR, in general, brings in additional degrees of freedom, which must be effectively filtered on terrestrial and cosmic scales for the modified theory to be credible (Sbisa 2014). Furthermore, one can evaluate the reliability of such theories by reviewing the theory's results with solar system tests and observational findings (Nojiri & Odintsov 2007; Clifton et al. 2012). Notwithstanding the strict limitations of the solar system tests, there are many forms of modified theory that could challenge GR. However, to reconcile such theories with a range of observational data and solar system experiments, a more detailed analysis is required.

Modified theories are a generalization of GR in which a set of curvature invariants substitutes or is introduced to the classical Einstein–Hilbert action. Thus, in this perspective, the early and late-time acceleration of the universe may be induced. In applications for late-time accelerating universe and DE, the modified gravity technique is highly appealing. Moreover, the mathematical framework of modified theories, as well as their features, is a fascinating area of study. A few of the worth mentioning benefits of the modified theory of gravity are listed below.

- A modified theory provides a natural gravitational substitute for DE.
- Such a theory unifies the early inflation and late-time acceleration in a very natural way.
- The transition from decelerating to accelerating universe is well explained by modified theory.
- It could be the foundation for a unified theory of DE and DM. It can also describe some cosmic phenomena, such as galaxy rotation curve.
- Without the need to add any exotic matter, such a theory might naturally characterize the shift from non-phantom to phantom phase. Generally, with modified theory, the cosmic doomsday can be avoided in the phantom type DE model (Nojiri & Odintsov 2007).
- Modified gravity is found to help address the coincidence problem.
- Such a modified theory can explain the source of DE.
- Modified gravity theory also serves as a helpful tool in the field of high-energy physics.

In 1922, Whitehead proposed the Whitehead's theory of gravity, a simpler alternative to GR that does not require any arbitrary parameters (Whitehead 1922) and probably, the first modified theory of gravity. Notwithstanding the arguments revealed in 1971 by Will that Whitehead's theory contradicts experimental results (Will 1971), academics remained interested in it. Further in 2008, Will teamed up with Gibbons and pointed out that the theory falls short in explaining its validity in five different experimental tests, finding that Whitehead's theory is essentially a failure, despite its solid intellectual roots (Gibbons & Will 2008). Since then, many researchers have proposed various fascinating modified theories of gravity that have effectively and convincingly captured the attention of cosmologists. A handful of such theories that have not escaped our attention are Brans-Dicke theory, Saez-Ballester theory, Lyra manifold, scale covariant theory, and $f(R,T)$ gravity. In the upcoming sub-sections, we'll go over these modified theories in more detail.

1.8.1 Scale covariant theory

Many cosmologists have successfully proposed many well-appreciated optimised modified theories of gravity throughout the years, which firmly match with current cosmic trends. One such modification that has caught our interest is the scale covariant theory (SCT) introduced by Canuto et al. (1977a) and Canuto et al. (1977b). They formulated the theory by applying the mathematical operation of scale transformation with the physics of using different dynamical systems to measure space-time distances Canuto et al. (1977a). According to them, the generalized Einstein's field equations are invariant under scale transformation and they successfully investigated many astrophysical tests with SCT Canuto et al. (1977b). In this theory, EFE are valid in gravitational units whereas atomic units are used for physical quantities. The metric tensors associated with these two systems of units are connected by a conformal or scale transformation $\overline{g_{ij}} = \varphi^2(x^k)g_{ij}$, where bar denotes gravitational units and the unbar denotes atomic quantities whereas φ is a gauge function which is a homogeneous function of all space-time coordinates satisfying $0 < \varphi < \infty$, without possessing any wave equation. Using this transformation, Canuto et al. (1977a) and Canuto et al. (1977b) transform the usual Einstein equations into

$$R_{ij} - \frac{1}{2}g_{ij}R + f_{ij}\left(\varphi\right) = -8\pi G(\varphi)T_{ij} + \Lambda\left(\varphi\right)g_{ij} \tag{1.8.2}$$

such that

$$\varphi^2 f_{ij} = 2\varphi\varphi_{i;j} - 4\varphi_{,i}\varphi_{,j} - g_{ij}\left(\varphi\varphi^{,k}_{;k} - \varphi^{,k}\varphi_{,k}\right) \tag{1.8.3}$$

where all the symbols have their usual meanings.

According to Katore et al. (2014), SCT is one of the best alternatives to ETG. This theory permits the variation of the gravitational constant G (Wesson 1980; Will 1984). The ambiguous DE and the mysterious expanding phenomenon have been successfully studied by many authors within the framework of SCT. In the recent study by Singh et al. (2020), it is asserted that SCT might be one of the probable contributors to the late time accelerated expanding phenomenon. Zeyauddin et al. (2020) present a cosmological model in SCT that decelerates during the initial phase and accelerates during the present evolution. Ram et al. (2015) present a forever expanding DE-dominated universe in SCT which tends to the de sitter universe in the future. Naidu et al. (2015) present a DE model with early inflation and late-time acceleration. Katore et al. (2014) investigate three Bianchi space-times involving magnetized anisotropic DE. Zeyauddin & Saha (2013) study an endlessly expanding and shearing model with an initial singularity within the theory. Reddy et al. (2012) construct an expanding DE model in SCT, which doesn't evolve from a singularity in the initial epoch. In the present scenario, SCT paired with DE is considered to align with cosmological observations.

1.8.2 $f(R,T)$ gravity theory

The modified theory of gravity, $f(R,T)$ gravity, introduced by Harko et al. (2011) has the gravitational Lagrangian expressed by an arbitrary function of the Ricci scalar R and the trace T of the energy-momentum tensor. The action of $f(R,T)$ gravity theory is given by

$$S = \int \left(\frac{1}{16\pi} f(R,T) + \mathcal{L}_m \right) \sqrt{-g} d^4x \tag{1.8.4}$$

where $g \equiv det(g_{ij})$, f is an arbitrary function of the Ricci scalar $R = R(g)$ and the trace $T = g^{ij} T_{ij}$ of the energy-momentum tensor of matter T_{ij} defined by Koivisto (2006) as

$$T_{ij} = -\frac{2}{\sqrt{-g}} \frac{\delta \left(\sqrt{-g} \mathcal{L}_m \right)}{\delta \, g^{ij}} \tag{1.8.5}$$

The matter Lagrangian density $\mathcal{L}_m$ is assumed to rely solely on g_{ij}, and hence

$$T_{ij} = g_{ij} \mathcal{L}_m - 2 \frac{\partial \mathcal{L}_m}{\partial g^{ij}} \tag{1.8.6}$$

The action S is varied w.r.t. the metric tensor g^{ij}, so that the field equations of $f(R,T)$ gravity is given by

$$f_R\left(R,T\right)R_{ij} - \frac{1}{2} f\left(R,T\right) g_{ij} + \left(g_{ij}\Box - \nabla_i \nabla_j\right) f_R\left(R,T\right) = 8\pi T_{ij} - f_T\left(R,T\right) T_{ij} - f_T\left(R,T\right) \theta_{ij} \tag{1.8.7}$$

where

$$\theta_{ij} = -2\,T_{ij} + g_{ij}\mathcal{L}_m - 2g^{lk}\frac{\partial^2 \mathcal{L}_m}{\partial g^{ij}\partial g^{lk}} \tag{1.8.8}$$

The subscripts appearing in f represent the partial derivative w.r.t. R or T and $\Box \equiv \nabla^i\nabla_i, \nabla_i$ being the covariant derivative.

Taking ρ and p respectively as the energy density and pressure such that the five velocity u^i satisfies $u^i u_i = 1$ and $u^i \nabla_j u_i = 0$, we opt to consider the perfect fluid energy-momentum tensor of the following form

$$T_{ij} = (p + \rho)\, u_i u_j - p g_{ij} \tag{1.8.9}$$

We let $\mathcal{L}_m = -p$ so that Eq. (1.8.8) becomes

$$\theta_{ij} = -2T_{ij} - p g_{ij} \tag{1.8.10}$$

The field equations of $f(R,T)$ gravity, in general, rely on the physical aspect of the matter field too, and therefore there are three types of field equations given by.

$$f(R,T) = \begin{cases} R + 2f(T) \\ f_1(R) + f_2(T) \\ f_1(R) + f_2(R) f_3(T) \end{cases} \tag{1.8.11}$$

Our research will be focused on the type $f(R,T) = R + 2f(T)$, with $f(T)$ as an arbitrary function. Now, the field equations of the theory is reduced to

$$R_{ij} - \frac{1}{2} R g_{ij} = 8\pi\, T_{ij} + 2f'(T)\, T_{ij} + \left\{\, 2p\, f'(T) + f(T)\,\right\}\, g_{ij} \tag{1.8.12}$$

where the prime indicates differentiation w.r.t. T, and we consider that $f(T) = \lambda T$, where λ is an arbitrary constant.

The $f(R,T)$ gravity theory has fascinated many cosmologists in recent years since it proposes natural gravitational alternatives for DE (Chirde & Shekh 2019). Myrzakulov (2020) has recently looked into the theory and predicted the requirements for an expanding universe without DE. Mishra et al. (2016b) and Singh & Kumar (2016) study the link of $f(R,T)$ gravity theory with DE. The discussion of model within the theory with DE driven acceleration can be found in the work of Mishra et al. (2016a). Sun & Huang (2016) explore expanding models within the framework of the theory in the absence of DE. Sahoo et al. (2020) investigate a mixture of barotropic fluid and DE within the theory where the universe starts from the Einstein static era and attains ΛCDM. Zia et al. (2018) discuss the theory studying future singularities in a DE model. Fayaz et al. (2016) present a discussion of a DE model with phantom or quintessence scenario. Houndjo and Piattella (2012) redevelop the theory from HDE. It won't be a bad assumption to conclude that the combination of DE and $f(R,T)$ gravity must have some form of hidden relationship.

1.8.3 Brans-Dicke theory

As an alternative to GR, scalar-tensor theories of gravitation are intensively researched. Brans-Dicke theory (BDT) is one such theory that has effectively challenged ETG. BDT was originally formulated as a simple modified theory by Brans & Dicke (1961) with regard to an action developed from a metric g_{ij} and a scalar field φ, exclusively relying on dimensional assertions, where the matter Lagrangian is minimally coupled. In the theory, φ describes the dynamics of gravity, whereas g_{ij} depicts the space-time geometry. The gravity interacts with φ through a dimensionless parameter ω_{BD}, known as the BD coupling parameter, and $G \sim \frac{1}{\varphi}$.

The action for the BDT is given by

$$S = \int d^5x\sqrt{-g}\left[\varphi R - \frac{\omega_{BD}}{\varphi} g^{ij}\varphi_{,i}\varphi_{,j}\right] + \frac{16\pi}{c^4}\int d^5x\sqrt{-g}L_m \tag{1.8.13}$$

where φ is the scalar field, R is the curvature scalar corresponding to the 5D metric g_{ij}, ω_{BD} is the BD coupling parameter, and L_m is the 5D Lagrangian of matter fields. The field equations of g_{ij} from Eq. (1.8.13) is given by

$$R_{ij} - \frac{1}{2}g_{ij}R + \omega_{BD}\varphi^{-2}\left(\varphi_{,i}\varphi_{,j} - \frac{1}{2}g_{ij}\varphi_{,k}\varphi^{,k}\right) + \varphi^{-1}\left(\varphi_{i;j} - g_{ij}\varphi^{k}_{;k}\right) = -8\pi\varphi^{-1}T \tag{1.8.14}$$

where T is the energy momentum tensors for matter field, and R_{ij} is the Ricci tensor. Here, we consider $G = 1 = c$.

The BDT appears to be an intriguing approach to constructing a much more accurate account of the universe, one that provides an account as to why the accelerated expansion is observed only in the present era (Hrycyna & Szydlowski 2013a). BDT is a viable alternative to GR for explaining the accelerated expansion of the universe, and it also passes the solar system tests (Dubey et al. 2021). Among all the known modified theories, the BDT is perhaps the most favourable, since it has effectively handled the difficulties of inflation as well as the early and late time dynamics of the cosmos (Kumar et al. 2020). BD scalar field can be considered as a DE candidate (Zia & Maurya 2018). Higher-dimensional BDTs are being considered as possible options for studying cosmic acceleration (Qiang et al. 2005). According to Tripathy et al. (2015), BDT has proven to be a preferable option to study GR, thus it's worth discussing DE models within the theory.

1.8.4 Saez-Ballester theory

The Saez-Ballester theory (SBT) is also one of the scalar-tensor theories of gravitation that many authors prefer to investigate GR. The theory was introduced by Saez & Ballester (1986). Its applications to cosmology yield reasonable findings. In the SBT, unlike in BDT, the scalar field does not serve the part of varying G. Rather, it's regarded as a dimensionless field that doesn't have to adhere to any restrictions imposed by observations. The intensity of the coupling between gravity and the scalar field is defined by a dimensionless parameter ω_{SB}, known as the SB coupling parameter. Weak fields are adequately described by this coupling (Singh & Shriram 2003).

The action for the SBT is given by

$$S = \int d^5x\sqrt{-g}\left[\varphi R - \omega_{SB}\varphi^n g^{ij}\varphi_{,i}\varphi_{,j}\right] + 8\pi L_m \tag{1.8.15}$$

where φ is the scalar field, R is the curvature scalar corresponding to the 5D metric g_{ij}, ω_{SB} is the SB coupling parameter, and L_m is the 5D Lagrangian of matter fields. The field equations of g_{ij} from Eq. (1.8.15) is given by

$$R_{ij} - \frac{1}{2}g_{ij}R - \omega_{SB}\varphi^n\left(\varphi_{,i}\varphi_{,j} - \frac{1}{2}g_{ij}\varphi_{,k}\varphi^{,k}\right) = -T \tag{1.8.16}$$

where T is the energy momentum tensors for matter field, and R_{ij} is the Ricci tensor. Here, φ satisfies

$$2\varphi^n\varphi^{;i}_{;i} + n\varphi^{n-1}\varphi_{,k}\varphi^{,k} = 0 \tag{1.8.17}$$

where n is an arbitrary constant.

The dimensionless scalar field in SBT can lead to the emergence of an anti-gravity phase (Singh & Shriram 2003), which can be related to the anti-gravity DE. Rao et al. (2012) present a DE model in SBT, obtaining results agreeing with recent observations. It has been proved that the missing matter problem in cosmology can be solved by SBT (Rasouli & Moniz 201). The investigation of SBT draws the attention of numerous researchers because of its importance in explaining the initial phases of evolution (Mohanty & Sahu 2003). The SBT scalar field is crucial when considering DE models and the initial phases of the evolution (Naidu et al. 2012). Within cosmology, different consequences from the SBT are frequently used to derive solutions in either 4D or 5D by adopting different line elements (Mohanty et al. 2007; Naidu et al. 2012; Pimentel 1987; Rao et al. 2012; Rao et al. 2015; Singh & Agrawal 1991; Singh & Shriram 2003; Yadav 2013). Owing to its useful applications in cosmology, SBT has recently been investigated by several authors to study DE and the accelerating universe, both in 4D and 5D (Aditya & Reddy 2018; Mishra & Chand 2020; Naidu et al. 2012, 2021; Pradhan et al. 2013; Raju et al. 2016; Ramesh & Umadevi 2016; Rao et al. 2015, 2018a, 2018b; Reddy 2017; Reddy et al. 2016b; Santhi & Sobhanbabu 2020; Shaikh et al. 2019; Sharma et al. 2019; Vinutha et al. 2019).

1.8.5 Lyra manifold

The sessions of the Prussian Academy of Science held during November 1915 might be considered as the most memorable moments of Albert Einstein's life. During the event, he revealed four of his most important works (Einstein 1915a, 1915b, 1915c, 1915d), which contributed to the formation of GR. Since then, different authors have investigated gravity in various contexts. Weyl (1918) was the first to try to extend GR to combine gravity and electromagnetic forces geometrically. Lyra's modification (Lyra 1951), similar to Weyl's, introduces a gauge function into the structureless manifold, resulting in one of the well-known modified theories of gravity. The modified EFE based on Lyra manifold (LM) were obtained by Sen (1957) and Sen & Dunn (1971).

The field equations are derived from the Lagrangian density

$$L = K\sqrt{-g}\left(x^0\right)^4 \tag{1.8.18}$$

where K is the contracted curvature scalar (Sen 1957). The simplification of Eq. (1.8.18), with the consideration of the natural gauge $x^0 = 1$ yields the field equations given by

$$R_{ij} - \frac{1}{2}g_{ij}R + \frac{3}{2}\varphi_i\varphi_j - \frac{3}{4}g_{ij}\varphi_k\varphi^k = -T_{ij} \tag{1.8.19}$$

where φ_i is the displacement vector and other symbols have their usual meaning as in Riemannian geometry. The displacement vector φ_i takes the time dependent form

$$\varphi_i = (\beta(t),\ 0,\ 0,\ 0,\ 0) \tag{1.8.20}$$

The notion that φ_i is time-independent, i.e. constant, is ambiguous because no particular scientific justification exists for how a constant displacement vector aids to the late-time expansion of the universe at an expedited rate. (Yadav 2020). Above everything, considering a constant displacement vector field is purely for simplicity's sake and has no scientific basis (Singh & Desikan 1997).

In recent years, cosmologists have become increasingly interested in studying the enigmatic DE coupled with the LM. We can see a DE universe in LM in an article by Hova (2013), which demonstrates that the expansion can be attained without any negative pressure energy element. Khurshudyan et al. (2014) present a paper on the examination of a two-component DE model in LM. Bhardwaj & Rana (2020) look at the presence of LM in the context of normal matter and DE interaction. In agreement with the current observation, Ram et al. (2020) propose a DE cosmological model coupled with LM. The study of Patra et al. (2019) discusses the influence of DE on models with linearly changing deceleration parameters in LM. Aditya et al. (2019) investigate a KK DE model in LM, resulting in an exponentially expanding cosmos. According to Singh & Sharma (2014a), a DE model in LM with a constant deceleration parameter may be developed. We may conclude from these noteworthy research works that the LM may be one of the viable candidates for studying DE and the expansion of the universe.

1.9 Literature review

A literature review is an examination of scholarly sources on a particular subject. It gives us a broad perspective of current knowledge, helping us spot pertinent ideas, methodologies, and future research. It is a vital chapter of the thesis, with the objective of providing context and rationale for the study conducted (Bruce 1994). A literature review, in principle, recognizes, analyses, and reconstructs significant literature in a particular area of study. It elucidates how knowledge has progressed in the domain, the previously performed research, the widely recognized concept, the new focus of interest, and the present state of knowledge on the subject. It is critical for researchers to be able to determine what is known about a specific topic and, by extension, what is unknown. A substantive, thorough, and sophisticated literature review is a precondition for doing substantive, thorough, and sophisticated research (Boote & Beile 2005).

1.9.1 Literature review on related works

In this section, a total of 108 research articles have been examined, with the key methodology and findings of the investigations highlighted. Most of them are articles published in the last few years.

Ali et al. (2015) present a comprehensive categorization of SS space-time through Noether symmetries. According to the authors, SS solutions to EFE are crucial in GR. The study of SS space-times is intriguing as it aids in expanding our knowledge on gravitational collapse and black holes. The quest for SS space-times is a crucial undertaking, given their importance in comprehending the universe.

Adhav et al. (2015) investigate a Bianchi type-V universe, which is spatially homogeneous and anisotropic, where DM and HDE interaction occurs. The authors use a particular form of deceleration parameter and special law of variation of Hubble parameter to obtain exact solutions. There is no coincidence problem when the interaction between DM and DE is suitably defined. In addition, the anisotropy fades fast and is replaced by isotropy within a short period. The authors obtain the largest value of the Hubble parameter and the fastest expansion of the universe.

Adhav et al. (2014) investigate a Bianchi type-I universe, which is anisotropic and homogeneous, where DM and HDE interaction occurs. The authors study models with two forms of deceleration parameter, one is of a fixed value and the other is of a particular form. There is no coincidence problem when the interaction between DM and DE is suitably defined. The anisotropy fades fast and is replaced by isotropy within a short period. The authors use the statefinder parameters to differentiate their DE models from the models developed by other authors.

Aditya et al. (2019) study the behaviour of a KK DE model in the presence of a large scalar field in the LM. The DE model corresponds to ΛCDM. The energy density of the model is positive and decreases during evolution. The values of Hubble parameter, scalar expansion, and shear scalar are finite at $t = 0$ and tend to infinity at $t \to \infty$. The enormous scalar field impacts all of the parameters at the minimum scale.

Aditya et al. (2021) study a Bianchi type-VI_0 DE cosmological model in the presence of a large scalar field. The model is non-singular and undergoes early inflation. The cosmological parameters H, θ, and σ^2 are finite at $t = 0$ and tend to infinity at $t \to \infty$. The anisotropy fades and is replaced by isotropy at late time. The authors obtain a phantom DE model which approaches ΛCDM at late time.

Aditya & Reddy (2018) investigate anisotropic HDE models in SBT. The model universe undergoes a transition from decelerating to accelerating phase during evolution. The EoS parameter of the interacting model crosses the phantom divide, whereas the EoS parameter of the non-interacting model tends to -1. The DE density parameter is obtained to be $\Omega = 0.73$.

Adler & Overduin (2005) investigate the shape of the universe. They claim that the universe is nearly flat (not exactly flat). They also provide three interpretations of a nearly flat universe. They claim that all three interpretations are equivalent and are based on a particular constant.

Agarwal (2011) discusses a Bianchi Type-II cosmological model in LM. The authors obtain models which evolved from an initial singularity, which are expanding and shearing. The displacement vector corresponds to the cosmological constant. The authors also discuss the entropy of the universe.

Ahmed & Pradhan (2020a) explore the accelerated expansion of an FRW universe and the evolution of DE across the cosmological constant boundary in universal extra dimensions. The model is homogeneous and anisotropic. The model universe undergoes a transition from decelerating to accelerating phase during evolution. The authors assert that in the present epoch, the DE of the model is of phantom type. During evolution, the DE EoS parameter crosses the phantom divide.

Ahmed et al. (2016) investigate a Bianchi type-V model universe within the framework of $f(R,T)$ theory, with field equation of the class $f(R,T) = f_1(R) + f_2(T)$. The model universe is expanding, shearing, and non-rotating. The cosmological constant of the model decreases and tends to a small positive value in the far future.

Alcaniz (2006) states that the so-called DE, a pretty absurd content of the universe, offers one of the biggest struggles cosmologists have ever faced. The authors proposed three different possible forms of DE. They explored all of these possible forms and appear to be capable to describe some of the present cosmological observations, but no definite judgement on the present characteristics of DE can be reached.

Aly (2019) investigate a HDE model in a $n+1$ dimensional FRW universe. The model accelerates driven by DE of phantom nature. A higher dimensional model is a good choice to explain the late time expanding phenomenon. The constructed model doesn't show any ΛCDM character.

Amirhashchi (2017) investigates the behaviour of DE in the context of an anisotropic Bianchi type-V universe. The DE EoS parameter is compared for viscous and non-viscous cases, and a correlation of DE with quintessence is established. Finally, the author looks at the circumstances within which the Bianchi type-V universe may be transformed to the FRW universe.

Arapoğlu et al. (2018) study the dynamics of a 5D universe in the context of dynamical system analysis. The authors predict that with EoS $\omega < -\frac{1}{3}$, one can obtain a flat universe, however, stabilization of the extra dimension is not achieved. With $\omega > -\frac{1}{3}$, the stabilization problem can be solved.

Araujo (2005) presents a discussion on the dynamics of a DE-dominated universe. The author examines and analyses cosmological models with a DE component, with a unique property of unending accelerating. It is mentioned that the universe might start evolving with and ends at an inflationary epoch. Further, if the DE is of CC type, the universe will ultimately go through an exponential expansion scenario.

Baushev (2010) states that the DE component that pervades the present universe is of phantom type and its density increases as the universe expand. This will the density to tend to an infinite quantity, ultimately leading to the cosmic doomsday, the big rip. However, with certain arguments and explanations, the author claims that the universe is free from any type of singularity.

Benvenuto et al. (2004) present the cosmological constraints on the variation of G. It is believed that G is a function of cosmic time. Here, in the study, the authors predicts two possible bounds on the variation. According to the authors, if $\dot{G} < 0$, the allowed bound on the variation is $-2.5 \times 10^{-10} \leq \frac{\dot{G}}{G} \leq 0$ yr^{-1}, whereas for $\dot{G} > 0$, we have $0 \leq \frac{\dot{G}}{G} \leq 4 \times 10^{-10}$ yr^{-1}.

Berezhiani et al. (2017) attempt to explain the mechanism behind the accelerated expansion of the universe. It is believed that the expanding phenomenon can be explained by two approaches - the dark energy approach, and the other is the modified gravity approach. However, the authors present an approach that explains the expanding paradigm by DM-baryon interactions, in the absence of DE.

Biswas et al. (2019) discuss the dynamics of a generalized ghost DE model in the FRW universe. The cosmological parameter ρ_d is assumed to be in the form $\rho_d = aH + bH^2$, where a and b are constants. It is observed that ρ_d decreases with expansion. The value of the Hubble parameter is obtained to be $H \approx 68.9$. The DE EoS parameter lies with the phantom region and tends to -1 in the future. The model undergoes a transition from decelerating to accelerating phase.

Bruck & Longden (2019) investigate a theory of modified gravity with extra dimensions. The study on the stabilization of extra dimensions is considered a phenomenological necessity in higher-dimensional models. However, according to the authors, in GR, generally, we cannot find conditions for stabilization, and all dimensions want to be dynamical.

Calder & Lahav (2008) state that one of science's greatest puzzles is DE. The origin and the idea of DE are tracked historically to Newton and Hooke of the seventeenth century in the work. The authors also discuss a hypothetical relationship between the CC and the total mass of the universe.

Capolupo (2018) investigate the dynamics of vacuum condensates, which describe a wide range of physical processes. The author mentions that many attempts have been undertaken to learn more about the properties and origins of DE. According to the author, vacuum condensate may lead the way to the DE source.

Carroll (2001a) put forward a review of cosmology of the existence of CC and the physics of a minimal VE. It is an obvious fact that the universe is dominated by the cryptic DE with negative pressure and positive energy density. However, the author asserts that NED is possible only if the DE is in the form of VE.

Chakraborty & Debnath (2010) construct a $4+d$ dimensional EFE in a 4D space-time with a FRW metric. The model is anisotropic. The authors claim that the unknown extra dimensions might be related to two unseen DE and DM.

Chan (2015a) presents an interesting study to highlight the DE problem and discuss a natural approach to solve it. According to the author, recent data suggest that the expansion rate of our universe is decreasing, casting doubt on the standard ΛCDM model. The author further claims that the presence of particles with imaginary energy densities can explain the decreasing rate and give a comprehensive answer to the root of DE.

Collaboration et al. (2020) discuss and present the values of various cosmological parameters. This is the most recent Planck results and considered to be one of the most standard results of astrophysics. The values of some of the cosmological parameters predicted in the work are $\Omega_d = 0.679 \pm 0.013$, $\Omega_m = 0.315 \pm 0.007$, $H_0 = 67.4 \pm 0.5$ km^{-1} Mpc^{-1} and $\omega = -1.03 \pm 0.03$.

Copeland et al. (2006) address the dynamics of DE. The authors present the explanation in favour of DE, as well as current advancements in determining its characteristics. They examine the observable information for the universe's current rapid expansion and propose a variety of DE models. They present different aspects of the possible future singularities and approaches to avoid cosmic doomsday. They also propose methods of modifying gravity that can induce accelerated expansion in the absence of DE.

Dasunaidu et al. (2018) study cosmic string in $f(R,T)$ gravity theory with the consideration of a 5D SS space-time. The model doesn't evolve from an initial singularity and is anisotropic all through. All the cosmological parameters, except the increasing volume, vanish at $t \to \infty$. The value of the deceleration parameter is obtained to be $q = -0.73$.

Dikshit (2019) presents a study discussing different aspects of the universe. The author put forward a pure quantum mechanical approach to explain DE. Additionally, the work also discusses the shape, size, and age of the universe.

Dubey et al. (2021) construct an interacting HDE model in BDT in FRW universe. The model constructed can induce an early decelerating phase, followed by an expedited expansion of the universe at late time. The DE component of the model starts from the quintessence region and crossed the phantom divide line. The value of the parameter H attains the value 70 in the far future. According to the authors, BDT is a viable alternative to GR for explaining the accelerated expansion of the universe, and it also passes the solar system tests.

Dubey & Sharma (2021) consider studying different HDE models in their work. They compare and contrast their newly defined DE models using $r - s$ and $r - q$ trajectories. Some of the DE models agree with the standard ΛCDM model. They also discuss the stability of the newly defined model with squared sound speed.

Farajollahi & Amiri (2010) study a 5D cosmology within the framework of KK cosmology. The 4D part of the model is taken to be FRW, while the fifth dimensional part consists of DE density. The authors use the model to describe the early inflation and late time acceleration of the universe.

Gontijo (2012) tries to find out a possible DE source. In the study, the author presents a physical mechanism as one of the origins of DE. The author asserts that the study could offer up new possibilities in cosmology by reinterpreting the dark entities as a scalar field contained in the space-time metric.

Gutierrez (2015) presents a discussion on the evolution of DE. The author explains that among the most significant unresolved problems among the cosmological society is the expansion of the universe at an expedited, which is induced by the component DE. This enigmatic DE makes up around 70% of the universe. The author goes through the present state of DE experimental results and gives a brief overview of the future studies that will make us understand in detail about this dark component.

Hrycyna & Szydłowski (2013a) discuss BDT using a scalar field potential function. They show that emergence of the ΛCDM scenario from BDT. According to them, BDT appears to be an intriguing approach to go toward constructing a much more accurate account of the universe, one that provides an account as to why the accelerated expansion is observed only in the present era.

Huterer & Shafer (2018) briefly recap the events that revealed the presence of DE. The parametric representations of DE and the cosmological tests that assist us in familiarizing ourselves with its characteristics are discussed. The cosmic investigations of dark energy are also presented. The authors also discuss the underlying mechanism of each investigation. Finally, they go through the present state of DE research.

Joyce et al. (2016) address the comparison of DE with modified gravity theory. Knowing the cause of the apparent expansion of the universe at an expedited rate is one most basic unanswered issues of science. According to the authors, a distinction of DE from modified gravity has been developed among physical theories for this expedition. They present a summary of models in both cases, and also about their behaviour and nature. They also make a clear difference between DE and modified gravity.

Knop et al. (2003) measure the values of the cosmological parameter Ω_d, Ω_m and ω with WFPC2 on the Hubble Space Telescope. Under two conditions, they measure two different bound on -1.61< ω <-0.78 and -1.67 < ω < -0.62. The values of Ω_d and Ω_m are measured to be $\Omega_d = 0.75^{+0.06}_{-0.07}$ (statistical) ± 0.04 (identified systematics) and $\Omega_m = 0.25^{+0.07}_{-0.06}$ (statistical) ± 0.04 (identified systematics), respectively.

Kolb et al. (2006) address the universe's late-time acceleration at an expedited rate. The expanding phenomena is caused by the DE component of the cosmos, which is widely acknowledged. In the absence of DE, however, the authors anticipate a circumstance that will result in acceleration. The back-reaction of cosmic perturbations, they say, is responsible for the rapid expanding phenomena.

Korunur (2019) discusses a Bianchi type-III universe with a DE component as the Tsallis HDE. It is found that under two conditions, the DE component corresponds to quintessence nature and phantom nature. The one with phantom nature attains the standard ΛCDM during evolution. The author also establishes a link between the model and few popularly used scalar fields.

Kumar et al. (2020) investigate anisotropic DE models in BDT considering an LRS metric. The assert that among all the known modified theories, the BDT is perhaps the most favourable, since it has effectively handled the difficulties of inflation as well as the early and late time dynamics of the cosmos.

Kumar & Suresh (2005) study a fascinating topic to discuss the validity of a higher-dimensional universe. The authors present a quick run-down of the theories incorporating the concept of extra dimensions, spanning earlier periods to the current day. The work ends with some visualizing examples and a brief explanation of the astrophysical consequences and probable existence of extra dimensions.

Macorrav & German (2004) discuss the cosmology of scalar fields. In the cosmological society, it is an accepted fact that the universe is dominated by the DE with negative pressure and positive energy density. However, the authors present an explanation of energy density with negative value with equation of state parameter (EoS) $\omega < -1$.

Mishra et al. (2016a) study an anisotropic universe in $f(R,T)$ gravity theory. They consider a Bianchi type-VI_h space-time, where $h = -1, 0, 1$. When $h = -1, 0$, the CC starts with a divergent nature and tends to become small at late time. The EoS parameters of these two models are also found to be negative. Both the models show quintessence under certain conditions. However, $h = 1$ doesn't yield a reliable model.

Mishra & Chand (2020) investigate a Bianchi type-I universe with perfect fluid in SBT.

The model universe undergoes a transition from accelerating to decelerating phase. The parameter ρ decrease with cosmic time. It is seen that Ω_m and Ω_d have the same values during the early evolution, whereas Ω increases from negative to positive and tends to a constant value. The model is also found to be very close to the standard ΛCDM model.

Mohanty et al. (2008) discuss 5D models in LM. In one model, it is seen that the LM perishes, and the metric coefficients tend to remain unchanged. Additionally, the gauge function also tends to become constant at $t = 0$, and vanishes at $t \to \infty$. In another model, the extra dimension shrinks with the increase of cosmic time. Further, the LM scenario will fade away quickly.

Mollah et al. (2018) consider a Bianchi type-II model universe in LM. In the study, the authors use a quadratic EoS. The deceleration parameter of the model tends to -1 at $t \to \infty$. The model is anisotropic all through. The parameter ρ is always positive, and σ^2 vanished at $t \to \infty$. The expansion scalar θ evolved with a very large value and becomes constant with the increase of cosmic time.

Moradpour et al. (2013) emphasize the presence of DE in the universe. Thermodynamic reasons are used to support their assertion. They assert that the universe with require a DE component with the EoS parameter $\omega < -\frac{1}{3}$. The presence of a DE component in the universe would cause it to achieve a thermal equilibrium.

Muley & Nagpure (2016) attempt to study the dynamics of a homogeneous cosmological model in LM considering a SS space-time. The expanding model evolves from an initial singularity. At $t \to \infty$, the cosmic parameters V, θ, and σ^2 tend to vanish, which is an indication that the model universe will end at the big crunch singularity. The anisotropy of the model fades and is replaced by isotropy at infinite time.

Narain & Li (2018) investigate the late-time acceleration of the universe at an expedited rate. They believe DE is the driving force behind the expanding phenomenon. However, the authors predict a condition to obtain acceleration, not because of DE. They present an interesting work to obtain acceleration from an Ultraviolet Complete Theory.

Neiser (2020) develops a cosmological model associated with an antineutrino star to search the origin of DE. The author asserts that the degenerate remains of an antineutrino star might have a mass density that is comparable to the DE density in the standard ΛCDM. Further, it is mentioned that the developed model could explain to us the root of

DE.

Parry (2014) presents a survey on SS space-times. SS space-time can be considered as one of the important tools for studying GR owing to its comparative simplicity and useful applications to both astrophysics and cosmology. It simplifies the study of a system's dynamics by allowing the transformation of a 4D solution to 2D. As a result, it's a good idea to start exploring GR with SS space-time.

Rao et al. (2018a) study an anisotropic cosmological model filled with matter and holographic Ricci DE in SBT. To obtain exact solutions, the authors consider two cosmological assumptions. The model universe undergoes accelerated expansion. The cosmological parameters H, θ, ρ_m, ρ_d and σ^2 diverge at $t = 0$, and tend to become constant at $t \to \infty$. The authors conclude by mentioning that scalar field φ is indeed an important parameter in DE cosmology.

Rao et al. (2018b) investigate a plane-symmetric model universe in the presence of matter and DE in SBT. The model universe undergoes a transition from decelerating to accelerating phase during evolution. The $r - s$ plane of the model corresponds to ΛCDM limit. The EoS parameter of the model implies a quintom scenario.

Rao & Jaysudha (2015) consider a 5D SS space-time in BDT of gravitation. The exact solutions are obtained under the assumption of two certain cosmological conditions. The expanding model universe is found to be isotropic. The expanding model evolved from an initial singularity. The cosmological parameters H, p and ρ diverge and vanish at $t \to \infty$.

Rao & Rao (2015) study a 5D anisotropic DE model in $f(R,T)$ gravity. The model undergoes early inflation and late-time acceleration. The model universe is found to be expanding, shearing, and non-rotating. The anisotropy fades and is replaced by isotropy at late time. The DE EoS parameter is obtained to be $\omega = -1$.

Rasouli & Moniz (2017) attempt to construct a 4D modified SBT from 5D SBT with the application of an intrinsic dimensional reduction. On contrary to usual SBT, the constructed 4D modified SBT is found to have significant new characteristics. According to the authors, the extra dimensions shrink with cosmic time.

Reddy (2017) studies a spatially homogeneous and anisotropic Bianchi type-V model universe in the presence of matter and DE in SBT. The author obtains three cosmological models that undergo a transition from decelerating to accelerating phase during evolution. The models evolved from an initial singularity. One of the models shows a constant value of H, implying a continuously expanding universe at a constant rate throughout evolution.

Reddy (2018) investigate a cosmological model considering a 5D SS metric within the framework of LM. The model doesn't come across an initial singularity during evolution. The displacement vector of the model is found to diverge. The constructed model universe is found to be isotropic experiencing expansion at an expedited rate.

Reddy et al. (2016a) present an expanding 5D model universe within the framework of DBT, where DE-DM interactions occur. To obtain exact solutions, the authors apply two reasonable cosmological assumptions. It is found that the H, θ and σ^2 diverge at $t = 0$, and tend to vanish at $t \to \infty$. The model universe is anisotropic throughout evolution. The DE EoS parameter corresponds to the phantom scenario.

Reddy et al. (2016b) study a Bianchi type-VI_0 model universe in the presence of interacting matter and DE in SBT. The authors apply a hybrid expansion law to obtain exact solutions. The authors obtain an expanding universe, evolving from an initial singularity. The DE EoS parameter crosses the phantom divide. The model undergoes a transition from decelerating to accelerating phase during evolution.

Reddy et al. (2012) discuss a Bianchi type-I DE cosmological model universe in SCT. The model doesn't evolve from an initial singularity. It is found that the H, θ and σ^2 diverge at $t = 0$, and tend to vanish at $t \to \infty$. The model universe is anisotropic throughout evolution.

Reddy et al. (2016) construct a 5D universe in the presence of interacting matter and DE within the framework of BDT. The authors consider a 5D SS space-time in their work. The model doesn't evolve from an initial singularity. It is found that H, θ, ρ_m, ρ_d, p_d and σ^2 diverge at $t = 0$ and vanish at $t \to \infty$. Under certain conditions, the model reduces to the standard ΛCDM model.

Sadjadi & Vadood (2008) present a note on an interacting HDE model in the FRW universe and study the nature of DE density in an expanding scenario. They discuss the characteristics and dynamics of HDE. They investigate the EoS of the model crossing the phantom divide. They predict some conditions that will lead to a transition from quintessence to the phantom scenario. These conditions might also help in alleviating the coincidence problem.

Sadri & Khurshudyan (2019) study both interacting and non-interacting new NHDE models within the framework of a spatially flat FRW universe. The EoS parameter and the parameter q describe an accelerating universe. The $r - s$ diagnosis reveals that the DE component of the interacting and non-interacting models correspond to quintessence and phantom nature, respectively.

Saha & Ghose (2020) explore the Tsallis HDE model experiencing accelerated expansion in 5D. In the context of Compact KK gravity, an interacting DE is presented using Generalized Chaplygin gas. The authors point out that the DE dominating the model universe might have evolved from the phantom phase during the early evolution.

Sahoo & Singh (2003) investigate a homogeneous and isotropic cosmological model within the framework of a generalized BDT. The BD scalar field decreases with cosmic time. The authors also found that the variational of gravitation constant G is safely below 4×10^{-10} yr^{-1}.

Sahoo & Mishra (2014a) study an anisotropic DE cosmological model considering a 4D space-time. It is predicted that the model universe can attain isotropy during evolution. The authors obtain the largest value of the Hubble parameter and the fastest expansion of the universe. The cosmological parameters ρ and ω diverge when $t \to \infty$ and remain constant at $t = 0$.

Sahoo & Mishra (2014b) construct an accelerated expanding 5D KK space-time with wet dark fluid in $f(R,T)$ gravity theory. The authors use a new DE EoS in the form of wet dark fluid. The model undergoes the early inflation and late-time acceleration. The authors claim that accelerated expansion depends on geometric contribution and matter content. The model attains isotropy during evolution. The cosmological parameter θ is constant, whereas σ^2 is finite and vanishes at $t \to \infty$.

Sahoo et al. (2020) discuss a model in $f(R,T)$ filled with barotropic fluid and DE. The authors claim that accelerated expansion depends on geometric contribution and matter content. The model evolved with large positive ρ and large negative p. However, these two parameters vanish at $t \to \infty$. The model universe is anisotropic throughout evolution. During evolution, the model universe attains ΛCDM in the future.

Samanta & Dhal (2013) present a 5D expanding cosmological model in $f(R,T)$ gravity theory, considering a 5D SS space-time. The model universe is isotropic throughout evolution. According to the authors, the extra dimensions shrink with cosmic time. The model evolved with large positive ρ and large negative p. However, these two parameters vanish at $t \to \infty$.

Samanta et al. (2014) investigate an accelerated expanding 5D space-time with DE in the form of wet dark fluid in $f(R,T)$ gravity theory. The model universe doesn't attain the standard ΛCDM model during evolution. The model universe is anisotropic all through. The extra dimensions shrink with cosmic time. The value of $j(t)$ coincides with that of flat ΛCDM model.

Santhi et al. (2016) an interacting HDE Model with generalized Chaplygin gas in SBT. The model stars evolving from a point-type singularity. At $t \to \infty$, the comic parameters V tends to infinity, whereas the other parameters θ and H vanish. The DE component of the model shows a quintessence nature. The model undergoes a transition from deceleration to acceleration phase.

Santhi et al. (2019) study a Bianchi type-I universe in $f(R,T)$ gravity theory. The model is isotropic and non-shearing universe. The model evolves from an initial singularity. The authors claim that their model might approach de Sitter expansion under a certain condition. The model universe undergoes a transition from decelerating to accelerating phase during evolution. The model evolved with a large negative p.

Sarkar (2014a) considers work on Bianchi type-I universe with interacting DM and HDE. The anisotropic parameter of the model vanishes at $t \to \infty$. The model universe ends at the cosmic doomsday. The ratio $\frac{\rho_d}{\rho_m}$ diverges with cosmic time. The author considers an equivalence between the energy density of DE and that of Chaplygin gas DE.

Sarkar (2014b) investigate an expanding Bianchi type-V universe with interacting DM and HDE. At $t \to \infty$, the anisotropic parameter of the model vanishes, and the shape of the model universe becomes flat. The ratio $\frac{\rho_d}{\rho_m}$ diverges with cosmic time. In the far future, the DE EoS parameter tends to -1. Lastly, the author explains the evolution of black holes, interacting with a mixture of DE and DM.

Sarkar (2015) presents an FRW model universe with interacting HDE. The ratio $\frac{\rho_m}{\rho_d}$ of the model decreases with time. The model universe undergoes a transition from decelerating to accelerating phase during evolution. The DE of the universe is of phantom type, leading the model to the cosmic doomsday. Before the occurrence of the cosmic doomsday, during evolution, the model encounters a phase where ρ_m and ρ_d are almost equal.

Satheeshkumar & Suresh (2011) explain the dynamics of gravity and consider extra dimensions in their study. The authors explain the ways human knowledge of gravity is rapidly evolving, and how prior theories have impacted contemporary advancements in the area such as superstrings and braneworlds. The authors assert that with an infinite-volume extra dimension, one doesn't need stabilization.

Sharif & Nawazish (2018) present interacting and non-interacting DE cosmological models in $f(R)$ gravity theory. They discuss the evolution and the expansion of the cosmos. They claim that at the observational scale, we can find proof confirming the existence of interaction between DE and DM or cold DM.

Sharif & Ikram (2019) study the dynamics of HDE in an accelerated expanding FRW universe within the framework of $f(G, T)$ gravity. The authors mention that accelerated expansion depends on geometric contribution and matter content. The DE EoS parameter of the model corresponds to phantom energy, whereas the $r - s$ plane corresponds to the Chaplygin gas model.

Sharma et al. (2019) investigate a homogeneous and anisotropic Bianchi-V universe considering SBT. The model universe undergoes a transition from decelerating to accelerating phase during evolution. The value of the deceleration parameter is obtained to be $q = -0.63$. At $t \to \infty$, the anisotropic parameter of the model vanishes. The cosmological parameters H and θ decreases with cosmic time.

Singh & Kumar (2015) discuss HDE models in a homogeneous and isotropic FRW universe within the framework of $f(R, T)$ gravity. The authors mention that an interacting HDE model can explain the accelerated expansion of the universe. The authors also discuss the models with the consideration of $r - s$ and $r - q$ trajectories.

Singh & Kumar (2016) present non-viscous and viscous HDE models in an FRW universe within the framework of $f(R, T)$ gravity. The authors mention that an interacting HDE model can explain the accelerated expansion of the universe. The authors try to find out if infrared cut-off could describe the expansion of the universe at an expedited rate. In the case of the non-viscous model, during evolution, the author obtains the fixed ΛCDM point under certain conditions, whereas in the viscous case, the model remains fixed in ΛCDM.

Singh & Kar (2019) try out to predict a source of DE. DE is the component of the universe that is responsible to drive the expansion of the universe at an expedited rate. The authors claim that an emergent D-instanton could be a possible source of DE.

Singh & Bishi (2017) present FRW models with modified Chaplygin gas considering BDT. The exact solutions are obtained by applying a particular form of deceleration parameter. For particular choices of the values of the constants involved, the cosmological parameters of the models obtained are found to align with the previous cosmological findings.

Singh et al. (2004) study a spatially flat 5D universe in LM. The authors consider a time G in their study. They claim that the extra dimensions either shrink or expand slowly with cosmic time. The authors also briefly discuss the variation of the gravitational constant. They mention that G can be either decreasing or increasing.

Singh & Sharma (2014a) consider a spatially homogeneous and anisotropic Bianchi Type-II models universe in LM. The models undergo accelerated expansion at an expedited rate. The authors predicted that their models evolved from zero volume. In the power-law model, the DE EoS parameter is negative, whereas, in the exponential model, the DE EoS parameter tends to 1 for a small value of the cosmic time.

Singh et al. (2020) investigate FRW models in SCT and discuss the accelerated expansion of the universe. The models also discuss the past as well as the present of the universe. The DE component of the models is of CDM and quintessence nature. One of the models ends at the cosmic doomsday in the far future. The authors predict that the interaction of DE and DM is boosted by gauge function.

Singh & Samanta (2019) study two DE models in BDT considering a SS space-time. In one model, the DE component is phantom and the occurrence of negative time if possible. The DE model reduces to flat ΛCDM during evolution. In the other model, the DE component is of a quintessence nature. In this model, it is found that DE induces big bang, and it reduces to DM during evolution.

Singh et al. (2017a) emphasize the importance of DE outside the scope of astrophysics.

The work of the authors presents a fascinating explanation of the applicability of DE in solving the issue of global warming.

Singh et al. (2017b) attempt to predict a source of DE. DE is the component of the universe that is responsible to drive the expansion of the universe at an expedited rate. During the investigation of a 5D cosmological model in LM, the authors found that the LM behaves as a DE source.

Singh & Singh (2019a) emphasize the application of DE beyond the scope of astrophysics. The authors try to find the positive aspect of DE in the field of health sciences. They study a 5D universe in BDT. It is found that the DE component of the universe can aid in the treatment and healing of diseases.

Skibba (2020) presents an interesting study discussing the ultimate end of the universe. The authors explain in detail the big crunch, big rip, and big freeze singularities, one of which is considered as the possible end of the universe.

Srivastava & Singh (2018) investigate a new HDE model in $f(R,T)$ gravity theory. The authors discuss the possible future singularity of the model. The model reduces to the standard ΛCDM model in the future. It is claimed that bulk viscosity is an important aspect in the explanation of DE. Lastly, the thermodynamic aspects of the model are also studied in detail.

Srivastava et al. (2019) study a new HDE model in Bianchi type-III model universe. The model undergoes a transition from decelerating to accelerating phase during evolution. The model reduces to the flat ΛCDM during evolution. The DE of the model is made up of two components, i.e. CC and HDE.

Szenthe (2004a) presents a discussion on the global geometry of SS space-time. According to the author, ever since the inception of GR, SS space-times have been studied by many authors. Eventually, a comprehensive theory of SS space-times was developed, including basic findings and important results relating to their global geometry. The author further mentions that to this day, it appears that a broad global framework is missing. The author presents some basic details about the global geometry of SS space-times in the work.

Szenthe (2004b) asserts that ever since the inception of GR, SS space-times have been studied by many authors. In the work, the author provides a detailed compilation of the important topological aspects of SS space-time.

Takeno (1951) discusses in detail the characteristic system of SS space-times. Firstly, the author provides the definition of SS space-time. Secondly, the characterizing vectors and scalars are presented. Thirdly, it is shown the definition is equivalent to that of Einstein. Fourthly, some important properties of SS space-time are provided.

Takeno (1952a) claims that SS space-times serve a vital part in GR. Through using the idea of the characteristic system presented by the present author, a theory is constructed for assessing if a space-time described by a line element randomly defined in any coordinate system is SS.

Takeno (1952e) extends the definition of 4D SS space-time to dimensions greater than or equal to 5. The author claims that the characteristics of the 4D case apply to the later with minor adjustments. Further, the later case is simpler in certain ways than the 4D one.

Umadevi & Ramesh (2015) study an interacting HDE model in Bianchi type-III universe within the framework of BDT. The model undergoes accelerated expansion at an expedited rate. It is found that H, θ and σ^2 diverge at $t = 0$ and vanish at $t \to \infty$. The ratio $\frac{\rho_d}{\rho_m}$ tends to -1 at $t \to \infty$, whereas it tends to infinity at $t = 0$. The anisotropy fades and is replaced by isotropy at late time.

Valentino & Mena (2020) construct a cosmological model involving the interaction of the two dark components of the universe - DE and DM. The author assert that their interacting model can be helpful to alleviate the Hubble constant tension.

Yadav & Bhardwaj (2018) try to find if LM can be obtained in a hybrid universe with interacting DE in the Bianchi-V universe. The authors consider a particular form of $a(t)$ in the study. The model universe undergoes a transition from decelerating to accelerating phase during evolution. The DE dominating the universe is of quintessence type. The time-dependent displacement vector behaves as the time-dependent CC.

Wesson (2015) present a study to explain the necessity for the fifth dimension. The mention of a 5D theory can be seen that explain the origin of VE. There is a remarkable improvement in our knowledge and the logical consistency of physics by the introduction of the fifth dimension.

Zeyauddin & Saha (2013) discuss an anisotropic Bianchi type-VI universe in SCT. The authors consider certain reasonable cosmological assumptions to obtain exact solutions. The model evolves from an initial singularity. It is found that H, θ and σ^2 diverge at $t = 0$ and vanish at $t \to \infty$. The model undergoes accelerated expansion at an expedited rate.

Zeyauddin et al. (2020) investigate an anisotropic Bianchi type-V universe in SCT. The authors consider a particular form of $a(t)$ in the study. The deceleration parameter is time-dependent. The model agrees with the standard ΛCDM model. The cosmological parameters H, θ and σ^2 diverge at $t = 0$ and vanish at $t \to \infty$. The anisotropy fades and is replaced by isotropy at late time.

Zhang (2010) presents a study emphasizing the fate of the universe in the HDE scenario. In the present day, the DE component dominating the universe is considered to be of the phantom type, which will lead the universe to cosmic doomsday. However, HDE and the big rip scenario are incompatible with each other. This issue can be solved with the employment of extra dimensions. Such employment will avoid the cosmic doomsday, and ultimately lead the universe to the de Sitter phase. According to him, the employment of an extra dimension makes HDE models more complete and consistent.

Zimdahl (2012) investigate the model of interacting DE. The cosmic evolution is altered by the interaction of DE and DM. The expansion of the universe at an expedited rate is a pure manifestation of interaction. The interaction between these two dark components is a crucial aspect of the evolution of the universe. When compared to non-interacting models, interacting models provide better cosmic dynamics.

1.9.2 Summary of key findings

A total of 108 research articles have been examined, with the key methodology and findings of the investigations highlighted. Following are some findings that may be drawn after examining the works.

- DE constitutes 70% of the universe. It is the driving force behind the expansion of the universe at an expedited rate. The universe requires a DE component to achieve thermal equilibrium.
- One of the most difficult problems cosmologists have ever confronted is the so-called DE, a somewhat ridiculous content of the cosmos. They have yet to be able to accu-

rately understand the nature and properties of DE. Various authors have postulated various DE sources.

- The equation of state (EoS) parameter ω is a suitable tool for classifying DE into particular groups. The CC, often known as VE, is represented by $\omega = -1$. The range $-1 < \omega < \frac{-1}{3}$ represents quintessence, while phantom energy has $\omega < -1$. According to recent observation, phantom energy is the most probable type of DE in the current universe.

- Since the DE component dominating the universe is highly likely to be phantom type, the universe is believed to end at the cosmic doomsday, the big rip singularity. Some authors, on the other hand, provide conditions to avoid singularity.

- The mysterious DE, which has negative pressure and positive energy density, dominates the universe. Some authors, however, claim that negative energy density is possible under certain circumstances.

- Modified gravity can be presented in terms of the interaction of the two dark components in the Einstein frame. Such a model might also help in alleviating the coincidence problem and Hubble constant tension. The expansion of the universe at an expedited rate is a pure manifestation of interaction. The interaction between these two dark components is a crucial aspect of the evolution of the universe. We can find proof confirming the existence of interaction between DE and DM or cold DM. When compared to non-interacting models, interacting models provide better cosmic dynamics.

- There are two ways to explain the universe's accelerated expansion. First is the DE candidate method, and the second is the modified theories of gravity approach.

- SS space-time can be considered as one of the important tools for studying GR owing to its comparative simplicity and useful applications to both astrophysics and cosmology. It simplifies the study of a system's dynamics by allowing the transformation of a 4D solution to 2D. It serves a vital part in GR. It is so crucial in comprehending the universe. As a result, it's a good idea to start exploring GR with SS space-time.

- Numerous studies have successfully developed compelling justifications for the existence and practical importance of employing extra dimensions. The employment of an extra dimension also makes HDE models more complete and consistent. The unknown extra dimensions might be related to the two unseen DE and DM. Such a model is a good choice to explain the late time expanding phenomenon. There is a remarkable improvement in our knowledge and the logical consistency of physics by the introduction of 5D. A 5D model can describe the early inflation and late time acceleration of the universe.

- The study on the stabilization of extra dimensions is considered a phenomenological necessity in higher-dimensional models. In GR, generally, we cannot find conditions for stabilization, and all dimensions want to be dynamical. Generally, we witness the discussion on stabilization in the field of particle physics, supersymmetry, supergravity, string theory, and braneworld models.

1.9.3 Implications and ideas for further research

Following are some of the implications and ideas for further research after reviewing the research papers.

- As DE sources, many authors have proposed various theories. However, according to the literature we have come across, only the Lyra manifold is predicted to be a DE source among the modified theories of gravity in GR. It is worth seeing whether any of the other modified theories can act as DE sources.
- In most of the constructed DE models reviewed, we cannot find the calculation of the values of the cosmological parameters. We believe it is critical to test the model's reliability by calculating the cosmological parameter values and comparing them to observation data.
- We can't find criteria for stability in general in GR, and all dimensions want to be dynamical. However, we can at least attempt to find some kind of stabilizing conditions in GR.
- One of the hottest topics in cosmology is the finite-time future singularity or, in other words, the universe's ultimate fate. Because the dominant DE component is of phantom type, the big rip singularity is the most likely scenario. We feel it would be worthwhile if we could develop a reliable theory that would allow us to avoid such a terrible fate of the universe.
- Energy density should be to be positive to obtain a reliable cosmological model. On the other hand, negative energy density is claimed by some authors to be conceivable in certain conditions. We feel that constructing a reliable model involving negative energy density would be interesting.
- HDE models become more complete and consistent when an extra dimension gets involved. The two unseen DE and DM might be linked to the unknown extra dimensions. A model like this is a strong fit for explaining the universe's late-time

expansion, as well as early inflation and late-time acceleration. The introduction of 5D has made a significant advancement in our understanding and physics' logical consistency. With these points in mind, it would be a good idea to research DE and the universe's expanding phenomena in higher dimension.

- The expansion of the universe at an expedited rate is a pure manifestation of DE interaction. The interaction between DE and DM is a crucial aspect of the evolution of the universe. When compared to non-interacting models, interacting models provide better cosmic dynamics. We can find proof confirming the existence of interaction between DE and DM or cold DM. As a result, it would be more relevant to construct cosmological models that involves interaction.

- Because of its relative simplicity and practical applicability in both astrophysics and cosmology, SS space-time may be regarded one of the most significant instruments for investigating GR. It serves a vital part in GR. It is so crucial in comprehending the universe. It makes studying a system's dynamics easier. As a result, starting to explore GR with SS space-time would be a smart idea.

Chapter 2

Research Methodology

This chapter presents the research method and process employed to acquire knowledge and data to answer the research problem. The chapter takes into account the reasoning behind the method to obtain and analyse the results of the study.

2.1 Introduction

A research methodology is a structured approach to solving a research problem. It is a science that studies the way to conduct research. In simple words, it is the process by which scholars go through the task of analysing, understanding, and projecting phenomena. Its main goal is to provide a scope of the research work. On the other hand, a research method is a tactic, procedure, or technique used to collect data or information for examination to reveal new knowledge or get a deeper insight into a subject.

Let us consider an example to distinguish between a research method and research methodology. To find the roots of a quadratic equation, we can apply one of these approaches - factoring, completing the square, or the quadratic formula. Each of these approaches is the research method to find the roots. On the other hand, the research methodology explains which approach or method should be applied and the process of application and calculation of the roots.

According to Rajasekar et al. (2013), a research can be broadly divided into two categories, which are fundamental or basic research and applied research. Basic research is the study of the fundamental concepts and causes of activity or phenomenon. It is the analysis or exploration of natural phenomena or a topic in pure research. A fundamental or basic research is also known as a theoretical research. A theoretical research has a unique or fundamental identity. It allows a research an in-depth understanding of an issue, derivation of rational scientific explanations and conclusions. It contributes to the creation of new information. On the other hand, in applied research, accepted hypotheses and concepts are used to tackle specific issues. The majority of experimental study and cross-disciplinary work comes under it. It is also defined as a study that has direct relevance as a result of its findings. The results of theoretical research are the foundation for a lot of applied research. Researchers conducting applied research apply the findings of fundamental research. The works presented in the thesis are based on fundamental or basic research.

2.2 Aim and objectives

A research aim conveys the purpose or desire of the study in one sentence; it highlights what we intend to accomplish at the end of the work. Research objectives describe the activities that will lead to accomplishing the research aim. The objectives break down the aim into parts, where each part constitutes a crucial element of the study. The objectives are organized in a list, with each objective becoming a chapter of a thesis.

2.2.1 Aim

Our research aims to broaden our knowledge about the enigmatic DE and the dynamics of the mysterious universe, with the consideration of a 5D SS metric, within the framework of modified gravity theories.

2.2.2 Objectives

The objectives of our research are listed below.

- To find if modified theories of gravity can behave as DE sources.
- To calculate the present values of cosmological parameters and test the reliability by comparing them to observation data.
- To make (probably) the first attempt to find some kind of stabilizing conditions of extra dimensions in GR.
- To develop reliable cosmological models that would allow us to avoid the terrible fate of the universe, the cosmic doomsday.
- To construct a reliable model involving negative energy density.
- To research DE and the universe's expanding phenomena in higher dimension.
- To construct cosmological models that involves interaction of DE and DM.

2.3 Research design

The purpose of a research design is to offer a structure for research (Sileyew 2019). The point of consideration about investigation strategy is a highly critical choice in a research design as it defines how necessary knowledge for research will be acquired; nevertheless, a research design comprises several linked decisions (Aaker et al. 2000). In order to achieve the aim and objectives in our fundamental or basic research, the mixed method (MM) of research, which combines qualitative and quantitative approaches, will be the focus of investigation. The quantitative method is centred on determining a quantity or a measure of something. A study is explained or represented using one or maybe more quantities in this case. The conclusion of a study is a numeric figure or a group. Whereas, in qualitative research, non-numerical information is collected and analysed in order to better comprehend ideas, views, or situations. It can be utilized to get a detailed understanding of a research problem or to innovate new research findings. By the use of MM, the study in our research involves non-numerical analysis and interpretation of data, and numerical outcomes and comparisons. A detailed explanation of the qualitative and quantitative methods in the field of mathematical and physical sciences also can be found in the work of Rajasekar et al. (2013). In recent years, MM has evolved fast, becoming a well-known research method with a unique character (Denscombe 2008).

According to Creswell & Clark (2006), designs with mixed methods come in four kinds, which are triangulation, embedded, explanatory, and exploratory. Our study will deal with the explanatory and exploratory designs. Explanatory research sought to seek answers for occurrences and phenomena, such as the explanation of "Why is a thing the way it is?" or "Why is the universe expanding?". Whereas exploratory research aims to learn further about a subject or a research area, such as "Can the universe expand in the absence of dark energy?" or "Can a modified gravity theory behave as a dark energy source?". The pictorial depiction of the research design used in our study is highlighted in Fig. 2.1.

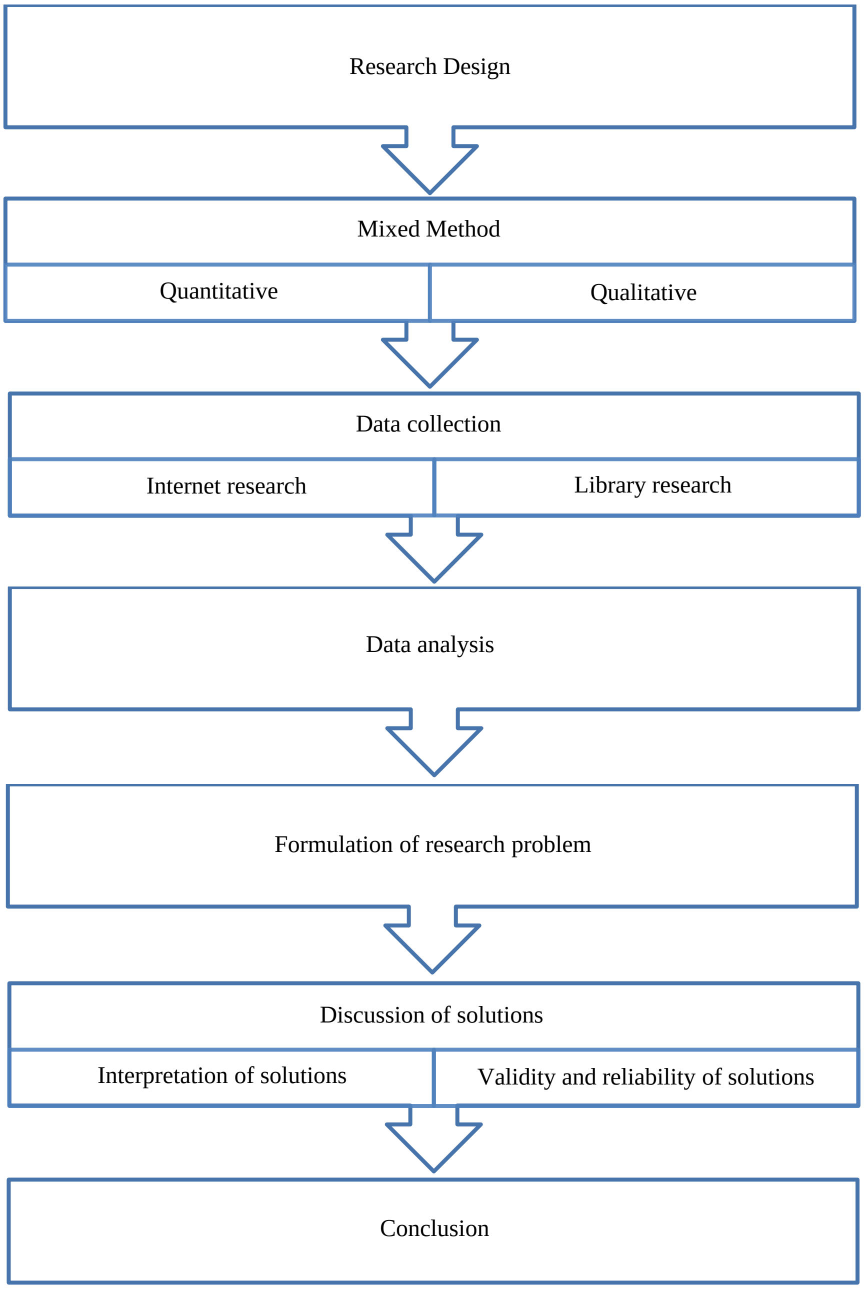

Figure 2.1: Research design.

2.4 Data collection

Data collection is a systematic procedure of obtaining and evaluating specific information to answer related queries and assess the outcomes. It emphasizes learning everything about a particular topic. Data gathered is put through inductive reasoning, which is used to try to understand a phenomenon. Research may be classified into two categories based on the method of data collection - primary and secondary research. The main difference is that data is acquired first-hand in primary research, whereas it is collected from previous studies in secondary research. Secondary data is gathered from readily accessible sources such as books, journal articles, newspapers, etc. Our research is more of a secondary type character since it entails minimal first-hand data.

The feasibility of employing secondary data for the study has become increasingly popular since huge volumes of secondary data are being gathered and preserved by scientists throughout the globe (Johnston 2014). In our work, a desk review is undertaken to gather information from a different of sources. The secondary data of our research is collected through two main sources - internet research and library research. The collecting of data and information from library items such as books, journal articles, conference papers, dissertations, and theses is known as library research. Internet research is the collection of data through internet sources. Academic journals and online repositories provide access to research articles and books that may be downloaded or purchased. Copies of research papers are also requested from different authors through email. The internet is one of the most important sources of data collection in today's technological era. Technological advancements have resulted in massive volumes of secondary data being gathered, collated, and preserved, all of which are now readily available for research.

The minimal first-hand or primary data is collected from the expert's viewpoint on data comparisons and information on observational findings, in the form of feedback via conference presentations, informal discussions, independent observations, etc.

2.4.1 Inclusion and exclusion criteria

A set of specified features to select data or participants to be considered in a research project is referred to as inclusion criteria, whereas the features to exclude or remove after being considered for inclusion is term exclusion criteria (Salkind 2010). Inclusion criteria must be relevant to the goal of the research and must be met in order for it to be completed. The validity and reliability of the research outcomes will be improved, the relevance will be enhanced, the expenses will be minimized, and ethical issues can be avoided if the criteria are properly chosen.

The following are listed as the inclusion criteria of secondary data:

- Articles within the domain of the research.
- Articles published during the past one to two decades.

- Only the older articles that provide the basic foundations, definitions, and statements, stable results, etc.
- Secondary data which guarantee analytical quality.
- Data from reputed indexed journals or publishers.
- Articles published in the English language.

The following are the exclusion criteria of secondary data:

- Articles which are synopsis of a conference, seminar or workshop.
- Articles which are just abstract, presentation, or poster versions.
- The article that simply bears the appearance of a promotional leaflet, with no contents.
- Reviews and editorials.
- Duplicate works.

2.5 Data analysis

The research method relates to the way the scholar gathers, analyses, and understands the data (Creswell 2008). Secondary data analysis is a crucial part of the research and assessment process. Secondary data analysis has become so significant to many of the finest methodologists that they have overshadowed primary data analysis in relevance (Glass 1976). The formulation of research problems is the first phase in secondary data analysis (Johnston 2014).

Most research starts with an analysis to find what information is previously understood and what is needed to be learned (Creswell 2008), incorporating relevant and supplementary material and also taking into account already gathered secondary data on the subject. In Chapter 1, we examine the past and present work of different researchers and perform a detailed literature review of various articles on the relevant topics. The literature review aids us in recognising the various research methods and methodologies employed by different authors. Following the review, we were able to identify the knowledge gaps and the crucial concepts, leading to the formulation of research objectives based on the key results.

2.6 Formulation of problem and solutions

Formulation of a research problem entails stating the research issue in a form that can be investigated. It is the first and most crucial stage in conducting research. It is similar to deciding on the desired location before embarking on a trip. It refers to the process of shaping a research area so that it is suitable for scientific inquiry. A scholar must narrow the area and indicate explicitly what will be investigated about it. This is referred to as formulation of the problem, and it entails pinning down a wide study field into a particular topic of study and setting goals. When the research problem is defined, the topic is fully prepared to be investigated scientifically.

In our work, to formulate a research problem, keeping in mind the objectives of the research, we apply the 5D SS line element to a particular modified gravity theory. Under certain reasonable assumptions, the exact solutions of the field equations are obtained. Then, we find the expressions for the cosmic scale factors appearing in the SS line element. Using the expressions for the cosmic scale factors, the expressions/values of the related cosmological parameters are obtained as solutions to the research problem.

2.7 Research tools and techniques

The formulation and prediction of the solutions to the research problem employ certain tools and techniques. A research tool and techniques are anything that serves as a way of gathering or analysing data or that helps conduct the research. Gravity is involved in GR, which is a broad generalization of special relativity. The mathematical explanation of GR necessitates the idea of differential geometry, employing the concepts of metric, curvature, etc. The mathematical tools and techniques of calculus, differential equation, geometry, algebra, and tensor are also applied in our research. Above all, the Wolfram Mathematica software becomes handy in the graphical analysis of the solutions to the research problem.

2.8 Discussion of solutions

One of the crucial tasks of a scholar is to prepare manuscripts to analyse the results of the research, which isn't really simple (Sanli et al. 2013). The discussion section is the most essential, as well as the least appealing component to work on. In our work, we incorporate the following two components in the discussion.

- Interpretation of solutions.

- Validity and reliability of solutions.

2.8.1 Interpretation of solutions

A proper presentation of the true significance of the information provided as goals of the research being delivered, including the chapter and topic, is referred to as interpretation (Pandey & Pandey 2015). The Wolfram Mathematica software is used to analyse the research solutions of the research problem. Graphical representations of the solutions are generated by the software. In light of the current observational data and literature references, the graphical analysis aids in the interpretation of the solutions explaining the past, present, and future physical phenomena of the universe. The interpretation component of the discussion presents the fresh, unique, and insightful findings of the research. We also estimate the present values of some cosmological parameters in this section. The interpretation part is also intended to address the unanswered problems about different cosmological phenomena. It can sometimes generate new queries that can inspire further study.

2.8.2 Validity and reliability of solutions

Reliability and validity are the most crucial and essential aspects of fruitful research, and these must be presented briefly and clearly in the research methodology chapter (Mohajan 2017). These aspects improve the clarity of qualitative research and decrease the risk of bias (Singh 2014). The stability of results is termed reliability, whereas the accuracy of results is referred to as validity (Altheide & Johnson 1994). A comprehensive review of reliability and validity includes techniques employed to acquire data (Saunders et al. 2009). For validity and reliability, we discuss the solutions considering the latest results predicted by standard experiments, for example, the Planck 2018 results. We also compare our results with the findings of the latest research articles published in indexed academic journals of repute.

2.9 Research conclusions

The Conclusions portion summarises the main elements of the discussion, the crucial aspects of our model, or the most important findings of our research. Its purpose is to bring the narrative of our study to a close, it:

- is presented in a way that closely relates to the aim and objectives of our research.
- shows the degree whereby the objectives are met.

- highlights the key results, opinions, or observations of our research.
- admits the limitations and provide suggestions for further improvement (whenever applicable).
- emphasizes the importance or applicability of our research.

Chapter 3

Higher dimensional phantom dark energy model ending at a de-Sitter phase

The work presented in this chapter is published in Chinese Journal of Physics, 77 (2022) 1732-1741, DOI: 10.1016/j.cjph.2021.05.022 (IF-3.237).

3.1 Introduction

Since the profound discovery of dark energy (DE) (Riess et al. 1998; Perlmutter et al. 1999) in 1998, theoretical physicists and cosmologists consider it as one of the most important topics in modern cosmology due to its mystic nature with huge negative pressure responsible for the universe to expand at an expedited rate. This cryptic component is considered to be uniformly permeated and vary slowly or unchanged with time (Chan 2015b; Carroll 2001a, 2001b; Peebles & Ratra 2003). With a focus to investigate its nature and application to modern cosmology, cosmologists have utilized tremendous scientific efforts and are still scrabbling for a perfect answer. From literature and observations, DE is believed to dominate the massive universe. This qualifies DE as a complete irony of nature as the dominating component is also the least explored. Some worth mentioning studies on this enigmatic dark component of the universe in the last decade are briefly presented in the next paragraph.

Akarsu et al. (2020) presents the evolutionary nature of DE is presented. The discussion on the evolution of DE given the latest observational findings is presented by Wang et al. (2018). Martino (2018) studies the decaying nature of DE into photons. Josset et al. (2017) obtain DE from violation of energy conservation. The quantum-mechanical calculation of DE density is presented by Dikshit (2019). Clery (2017) predicts that galaxy clusters due to the stirring effect of DE. A theoretical investigation of DE on searching the solution of global warming is illustrated by Singh et al. (2017a). Hamilton et al. (2015) discuss the atom-interferometry constraints on DE. Chan (2015a) asserts that the presence of particles with imaginary energy density can lead us to the source of DE. Gutierrez (2015) reviews the status of the experimental data on DE. The need for DE with thermodynamic arguments is provided by Moradpour et al. (2017). Lastly, Hecht (2013) compares the speed of DE with that of the photon.

To precisely understand the underlying mechanism of the late time accelerated expansion of the universe, cosmologists have adopted two well-appreciated methods. Firstly, different possible forms of DE are developed. Secondly, modifying Einstein's theory of gravitation. Other than these two, many authors have successfully adapted other fascinating ways to explain the miraculous expanding phe nomenon. Racz et al. (2017) make a compelling attempt describing the expanding phenomenon in the absence of DE. Alfaro (2019) claims that acceleration is automatically induced by the Delta Gravity equations, other than DE. Freese (2003), Freese & Lewis (2002) and Dvali & Turner (2003) try out directly modifying the Friedmann equation empirically to explain the phenomenon. An approach is presented by Narain & Li (2018) in which the accelerating paradigm is explained by an Ultra Violet Complete Theory. Lastly, Berezhiani (2017) illustrates the expanding phenomenon by matter.

One of the possible forms of DE which has not escaped our attention is holographic dark energy (HDE), introduced by Gerard't Hooft (Hooft 2009). It is obtained by the application of holographic principle (Bousso 2002) to DE. Accordingly, all the physical quantities inside the universe including the energy density of DE can be illustrated by some quantities on the boundary of the universe (Wang et al. 2017). Models involving the interaction of HDE and dark matter (DM) or interacting holographic dark energy (IHDE) models are considered to be of paramount importance by many authors. A discussion on an expanding interacting HDE and DM model can be seen in the study of Adhav et al. (2014), where the DE component decays into pressureless DM. Nayak (2020) discusses an IHDE model asserting that, at present, the universe is dominated by quintessence DE and it will become phantom DE dominated in the near future. Kiran et al. (2014) study a minimally IHDE model in a scalar-tensor theory of gravitation experiencing cosmic re-collapse. Sarkar (2015) investigates an IHDE model undergoing accelerated expansion ending at the big rip singularity. Chirde & Shekh (2018) investigate a minimally interacting matter and HDE model with the discussion of singularity and predicting that their model universe expands with the fastest rate and the largest value of the Hubble's parameter. Umadevi & Ramesh (2015) consider an isotropic minimally IHDE model in Bianchi type-III universe exhibiting early inflation and late-time acceleration. Reddy et al. (2016a) discuss a minimally IHDE model in Brans–Dicke theory where the DE turns out to be of phantom type. In the study by Raju et al. (2016), we can witness an IHDE model expanding spatially with a constant overall density parameter. Reddy et al. (2016b) investigate an IHDE model free from initial singularity attaining isotropy at late times. In the last few years, strong arguments have been brought to light asserting that modified gravity can be explained by employing DE-DM interaction in Einstein frame (Felice & Tsujikawa 2010; He 2011; Zumalacarregui 2013; Kofinas 2016; Cai 2016). Due to the fascinating nature of such interacting models, many fundamental questions are arisen pointing out that there are a lot more physics still undiscovered.

Saez-Ballester Theory (SBT), introduced by Saez & Ballester (1986), can be considered to be the right option to study DE and the accelerating universe. It is a member of the family of Scalar Tensor Theory (STT) of gravitation. In SBT, the metric potentials are coupled with a scalar field φ. Scalar fields are considered to play key roles in gravitation and cosmology as they can illustrate prodigies like DE, DM, etc. (Aditya et al. 2021). They can be regarded as a possible contributing factor in the late time acceleration of the universe (Kim 2005). STT of are direct generalization and extension of general relativity (Panotopoulos & Rincon 2018). STT can be considered as perfect candidates for DE (Mandal et al. 2018). STT also played a key role in getting rid of the graceful exit problem in the inflationary period (Piemental 1997). Linde (1982) asserts that a scaler field might be responsible for the inflation at the initial epoch. Currently, SBT and general relativity are held to align with observation.

Recently, there has been a growing interest among cosmologists to explore the DE-DM interaction in SBT setting. Ramesh & Umadevi (2016) study interacting HDE and DM model in SBT where the expanding model starts with a big bang. The construction of an interacting new HDE model in the framework of SBT can be found in presented by Aditya & D.R.K. Reddy (2018). Reddy et al. (2016) investigate an IHDE model in SBT where they use hybrid expansion law and predict a transitioning universe. Reddy (2017) investigates an IHDE model in SBT thereby obtaining three cosmological models. Rao et al. (2018a) observe an IHDE model in SBT obtaining a transitioning model due to cosmic-recollapse can be seen. Shaikh et al. (2019) discuss a model with matter and a modified holographic Ricci DE in SBT. Lastly, Rao et al. (2018b) investigate a modified holographic Ricci DE with matter in SBT predicting a quintom-like universe.

The possibility of space-time having more than 4D has fascinated many authors. Higher-dimensional cosmological model was introduced by Kaluza and Klein (Kaluza 1921; Klein 1926). Such models are useful to describe the late time expanding paradigm (Banik & K. Bhuyan 2017). The investigation on higher dimension can be considered as an important task as the universe might have encountered a higher dimensional phase during the early evolution (Singh et al. 2004). According to Alvax & Gavela (1983) and Guth (1981), the additional dimension might provide us an explanation for the flatness and horizon problem. Marciano (1984) discusses the evidence for the existence of the additional dimension. Lastly, Chakraborty & Debnath (2010) assert that the hidden extra dimension in 5D might correspond to the unknown DE and DM.

Keeping in mind the noteworthy studies mentioned above, we consider a DM-DE interaction in SBT considering a 5D spherically symmetric (SS) space-time. In this work, we present a detailed discussion on every cosmological parameter obtained. The definition of shear scalar and its physical significance are provided. The incompatibility of big rip singularity with HDE and its elimination in phantom DE scenario by de-Sitter phase is discussed. Additionally, we calculate the present values of the Hubble's parameter and the dark energy EoS parameter. To obtain realistic results, we make assumptions in concordance with present-day cosmology. The paper is divided into sections. After the introduction, in Sect. 3.2, we present problem formulations with solutions of the cosmological parameters. In Sect. 3.3, the solutions are discussed with graphs with the consideration of the recent findings. Lastly, as a summary, a concluding remark is provided in Sect. 3.4.

3.2 Formulation of problems with solutions

We start with the consideration of a SS metric in 5D (Samanta & Dhal 2013) as given below

$$ds^2 = dt^2 - e^{\mu}\left(dr^2 + r^2 d\Theta^2 + r^2 \sin^2\Theta d\phi^2\right) - e^{\delta} dy^2 \tag{3.2.1}$$

where μ and δ are cosmic scale factors which are functions of time only.

The Saez-Ballester field equations are given by

$$R_{i\,j} - \frac{1}{2} g_{i\,j} R - \omega_{SB}\varphi^n \left(\varphi,_i \varphi,_j - \frac{1}{2} g_{i\,j}\varphi,_k \varphi,^k\right) = -\left(T_{i\,j} + S_{i\,j}\right) \tag{3.2.2}$$

where $T_{i\,j}$ and $S_{i\,j}$ are the energy momentum tensors for matter and HDE respectively, R and R_{ij} are respectively the Ricci scalar and tensors, whereas the scalar field φ satisfies

$$2\varphi^n \varphi^{;i}_{;i} + n\varphi^{n-1}\varphi_{,k}\varphi^{,k} = 0 \tag{3.2.3}$$

where n is an arbitrary constant.

T_{ij} and S_{ij} are given by

$$T_{ij} = \rho_m u_i u_j \tag{3.2.4}$$

$$S_{ij} = (\rho_d + p_d)\, u_i u_j - g_{ij} p_d, \tag{3.2.5}$$

where ρ_m and ρ_d represent the energy density of matter and HDE respectively whereas p_d is pressure of the HDE.

By conservation of energy, we have

$$T_{ij} + S_{ij} = 0 \tag{3.2.6}$$

Using co-moving coordinate system, the surviving field equations are obtained as

$$\frac{3}{4}\left(\dot{\mu}^2 + \dot{\mu}\dot{\delta}\right) + \frac{\omega_{SB}}{2}\varphi^n\dot{\varphi}^2 = \rho \tag{3.2.7}$$

$$\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 + \frac{\ddot{\delta}}{2} + \frac{\dot{\delta}^2}{4} + \frac{\dot{\mu}\dot{\delta}}{2} - \frac{\omega_{SB}}{2}\varphi^n\dot{\varphi}^2 = -p_d \tag{3.2.8}$$

$$\frac{3}{2}\left(\ddot{\mu} + \dot{\mu}^2\right) - \frac{\omega_{SB}}{2}\varphi^n\dot{\varphi}^2 = -p_d \tag{3.2.9}$$

From Eq. (3.2.6), we have

$$\ddot{\varphi} + \dot{\varphi}\left(\frac{3\dot{\mu} + \dot{\delta}}{2}\right) + \frac{n}{2}\dot{\varphi}^2\varphi^{-1} = 0 \tag{3.2.10}$$

where an overhead dot represents differentiation w.r.t. t.

We assume ω as the EoS parameter of the DE and hence, we have

$$p_d = \omega\rho_d \tag{3.2.11}$$

The conservation equation takes the obvious form as given by

$$\rho_m\left(\frac{3\dot{\mu} + \dot{\delta}}{2}\right) + \dot{\rho}_m + \dot{\rho}_d + \rho_d\,(1+\omega)\left(\frac{3\dot{\mu} + \dot{\delta}}{2}\right) = 0 \tag{3.2.12}$$

Due to their minimal interaction, HDE and matter conserve separately so that by Sarkar (2014a, 2014b), Eq. (3.2.12) can be written as

$$\rho_m\left(\frac{3\dot{\mu}+\dot{\delta}}{2}\right)+\dot{\rho}_m=0 \tag{3.2.13}$$

$$\rho_d\left(1+\omega\right)\left(\frac{3\dot{\mu}+\dot{\delta}}{2}\right)+\dot{\rho}_d=0 \tag{3.2.14}$$

Also, we have

$$(\rho+p)\left(\frac{3\dot{\mu}+\dot{\delta}}{2}\right)+\dot{\rho}=0 \tag{3.2.15}$$

From Eqs. (3.2.8) and (3.2.9), the expression for the cosmic scale factors are obtained as

$$\mu=l_1-\log\left(k-t\right)^{\frac{2}{3}} \tag{3.2.16}$$

$$\delta=m_1-\log\left(k-t\right)^{\frac{2}{3}} \tag{3.2.17}$$

where l_1, m_1 and k are arbitrary constants.

Now, from Eqs. (3.2.13), (3.2.14), (3.2.16) and (3.2.17), we have

$$\rho_m=l_0e^{-\frac{1}{2}(3l_1+m_1)}\left(k-t\right)^{\frac{4}{3}} \tag{3.2.18}$$

$$\rho_d=m_0e^{-\frac{1}{2}(1+\omega)(3l_1+m_1)}\left(k-t\right)^{\frac{4}{3}(1+\omega)} \tag{3.2.19}$$

so that the energy density of our universe is given by

$$\rho=\rho_m+\rho_d \tag{3.2.20}$$

where l_0 and m_0 are an arbitrary constants.

Again, using Eqs. (3.2.16), (3.2.17) and (3.2.20) in Eq. (3.2.15), the pressure of our universe is obtained as

$$p=\frac{1}{3}l_0e^{-\frac{1}{2}(3l_1+m_1)}(k-t)^{\frac{4}{3}}+m_0\left(\frac{4\omega+1}{3}\right)e^{-\frac{1}{2}(1+\omega)(3l_1+m_1)}(k-t)^{\frac{4}{3}(1+\omega)} \tag{3.2.21}$$

From Eqs. (3.2.11) and (3.2.19), the pressure of DE is given by

$$p_d = \omega\, m_0 e^{-\frac{1}{2}(3l_1+m_1)(1+\omega)}\,(k-t)^{\frac{4}{3}(1+\omega)} \tag{3.2.22}$$

At any time $t = t_0$, we can assume that $p = p_d$ so that from Eqs. (3.2.21) and (3.2.22), we have

$$l_0 e^{x}(k-t_0)^{\frac{4}{3}} + m_0(1+\omega)e^{(1+\omega)x}(k-t_0)^{\frac{4}{3}(1+\omega)} = 0 \tag{3.2.23}$$

where $x = -\frac{1}{2}(3l_1 + m_1)$

Eq. (3.2.23) will provide us the expression for EoS parameter ω.

Now, using Eqs. (3.2.16) and (3.2.17) in Eq. (3.2.10), the SB scalar field φ is obtained as below

$$\varphi = \left((6+3n)\,(k-t)^{\frac{7}{3}} - 14c_1\right)^{\frac{2}{2+n}} c_2 \tag{3.2.24}$$

where c_1 and c_2 are arbitrary constants.

Finally, the expressions of the different cosmological parameters are obtained as follows.

Spatial volume:

$$V = e^{\frac{3l_1+m_1}{2}}(k-t)^{-\frac{4}{3}} \tag{3.2.25}$$

Scalar expansion:

$$\theta = \frac{4}{3}\,(k-t)^{-1} \tag{3.2.26}$$

Hubble parameter:

$$H = \frac{1}{3}\,(k-t)^{-1} \tag{3.2.27}$$

Shear scalar:

$$\sigma^2 = \frac{2}{9}\left(\frac{1}{k-t} - 1\right)^2 \tag{3.2.28}$$

Anisotropic parameter:

$$A_h = 0 \tag{3.2.29}$$

Dark energy density parameter:

$$\Omega_d = \frac{\rho_d}{3H^2} = 3m_0 e^{-\frac{1}{2}(1+\omega)(3l_1+m_1)}(k-t)^{\frac{2}{3}(5+2\omega)} \tag{3.2.30}$$

Matter density parameter:

$$\Omega_m = \frac{\rho_m}{3H^2} = 3l_0 e^{-\frac{1}{2}(3l_1+m_1)}(k-t)^{\frac{10}{3}} \tag{3.2.31}$$

Overall density parameter:

$$\Omega = 3\left(l_0 e^{-\frac{1}{2}(3l_1+m_1)} + m_0 e^{-\frac{1}{2}(1+\omega)(3l_1+m_1)}(k-t)^{\frac{4\omega}{3}}\right)(k-t)^{\frac{10}{3}} \tag{3.2.32}$$

Jerk parameter:

$$j(t) = q + 2q^2 - \frac{\dot{q}}{H} = 28 \tag{3.2.33}$$

3.3 Discussion

For our convenience sake and to obtain realistic results, in this section, choose fixed values of the arbitrary constants appearing in the solutions i.e., $l_0 = l_1 = m_0 = m_1 = 1, k = 13.80497512437811$.

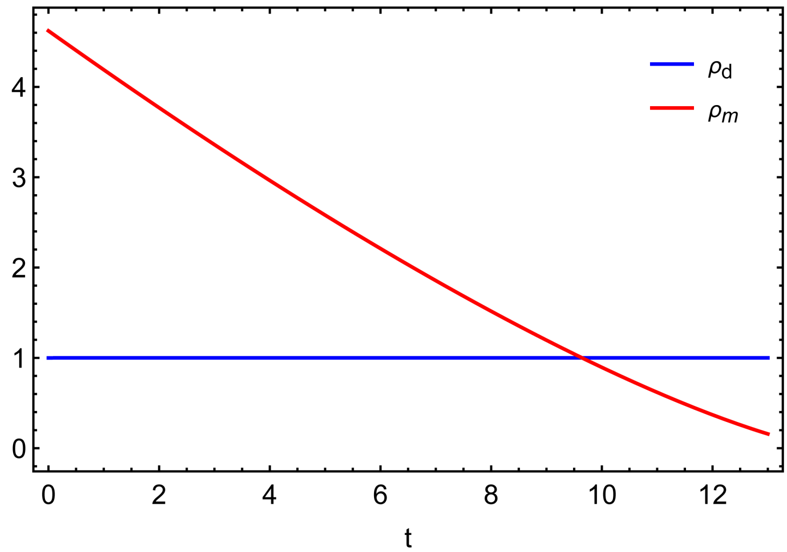

Figure 3.1: Energy densities of DE ρ_d and DM ρ_m with t when $l_0 = l_1 = m_0 = m_1 = 1, k = 13.80497512437811$.

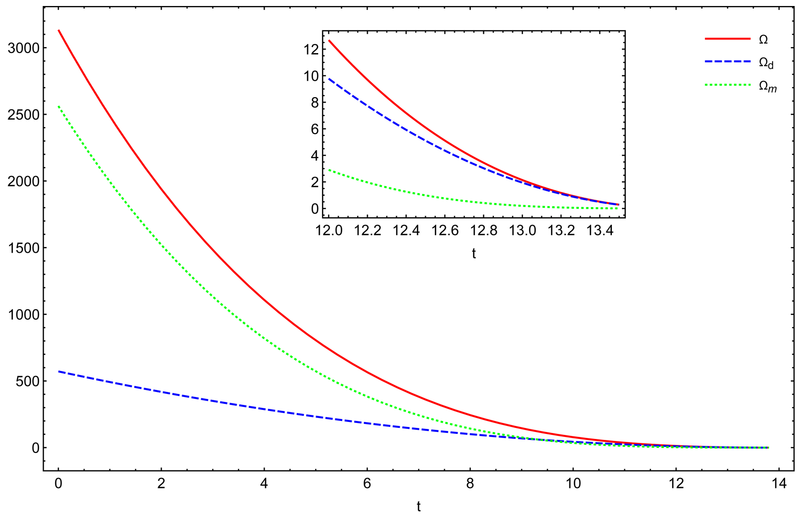

Figure 3.2: Overall density parameter Ω, DE density parameter Ω_d and DM density parameter Ω_m with t when $l_0 = l_1 = m_0 = m_1 = 1, k = 13.80497512437811$.

From Fig. 3.1, we can witness the decreasing nature of ρ_m whereas ρ_d remains consistent all through. From Fig. 3.2, we can see that Ω and Ω_d tend to become constant after decreasing for a finite period, whereas Ω_m continue to decrease to a larger extent. It may be noted that due to the expansion, galaxies move apart from each other leading DM density to diminish gradually (Carroll 2001b), whereas DE varies slowly or unchanged with time (Chan 2015b; Carroll 2001a, 2001b; Peebles & Ratra 2003). From these, we have obtained a model which is DE dominated, similar to that predicted by Carroll (2001a), Adhav et al. (2014), Araujo (2005), Ray et al. (2013), Agrawal et al. (2018), Wu & Yu (2005), Straumann (2007) and Law (2020).

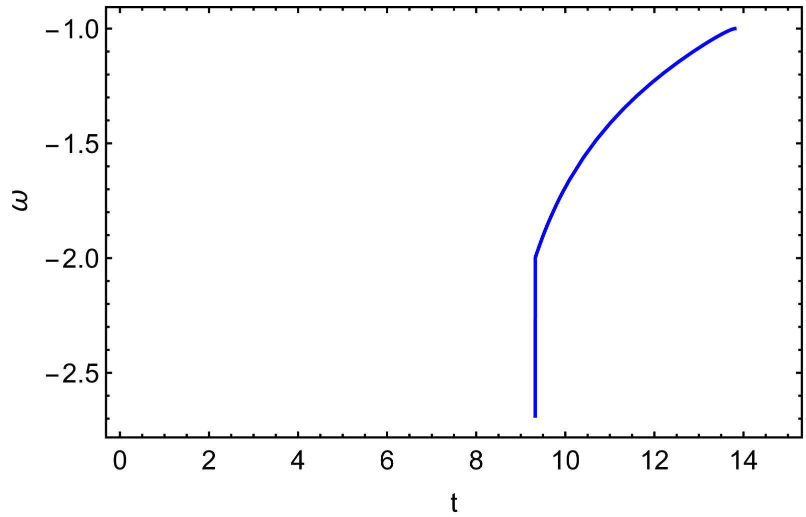

Figure 3.3: EoS parameter ω with t when $l_0 = l_1 = m_0 = m_1 = 1, k = 13.80497512437811$.

Fig. 3.3 shows the variation of time-dependent EoS parameter ω with cosmic time t. Here, it can be seen that ω starts evolving from the aggressive phantom region and tends to come very close to -1, which aligns with the recent studies (Amirhashchi 2017; V. Santhi et al. 2019). Similar observations of HDE with phantom-like nature can also be seen in the recent works (Belkacemi et al. 2020; Sharif & Ikram 2019). However, as ω appears to evolve due to time dependence, it attains the value $\omega = -1$ during evolution (Aditya et al. 2021; Basilakos & Sola 2014). Above all, a phantom model with $\omega < -1$ should reduce to $\omega = -1$ in the far future to ensure cosmological models bypass future singularity (big rip) thereby, ultimately, leading to the de-Sitter phase (Amirhashchi 2017; Carroll et al. 2003). It can also be noted that in HDE setting, the big rip singularity is not permitted, because the Planck scale excursion of UV cutoff in the effective field theory is forbidden so that the occurrence of the big rip would ruin the theoretical foundation of the HDE scenario (Zhang 2010). This issue can be solved by employing an extra dimension in HDE setting, and also the employment of an extra dimension makes HDE models more complete and consistent. The mechanism of replacing big rip singularity by de-Sitter phase with the employment of an extra dimension (higher dimension) in HDE setting can be seen in the study by Zhang (2010). Dymnikova (2019), Sakharov (1966) and Gliner (1966) also discuss on replacing big rip singularity by de-Sitter phase. According to the latest Planck 2018 result (Collaboration et al. 2020), the present age of the universe is 13.825 ± 0.037 Gyr. With $t_0 = 13.8$ Gyr and assuming $l_0 = l_1 = m_0 = m_1 = 1, k = 13.80497512437811$, from Eq. (3.2.23), the present value of EoS parameter is obtained to be $\omega = -1.00011$, which aligns with the value $\omega = -1.03 \pm 0.03$ of the of the latest Planck 2018 result (Collaboration et al. 2020).

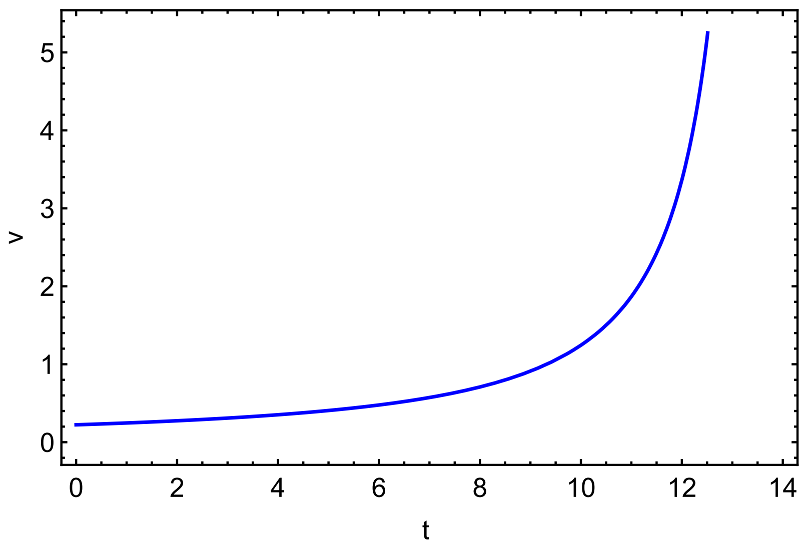

Figure 3.4: Spatial volume V with t when $l_1 = m_1 = 1, k = 13.80497512437811$.

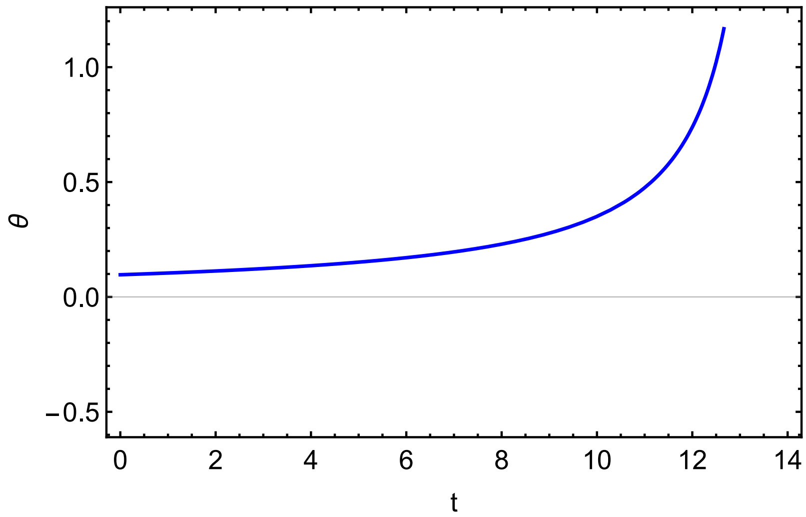

Figure 3.5: Scalar expansion θ with t when $k = 13.80497512437811$.

Fig. 3.4 and Fig. 3.5 can be considered as the perfect pieces of evidence for the spatial expansion of the universe at an expedited rate. At $t = 0$, V and other related parameters are constant which indicates that the model doesn't evolve from an initial singularity. Whereas, as discussed before, the future big rip singularity is replaced by the de-Sitter phase.

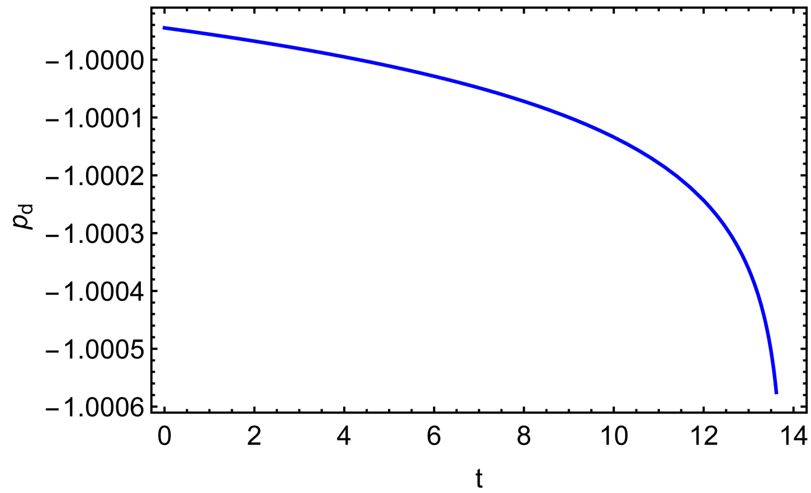

Figure 3.6: DE pressure p_d with t when $l_1 = m_0 = m_1 = 1, k = 13.80497512437811$.

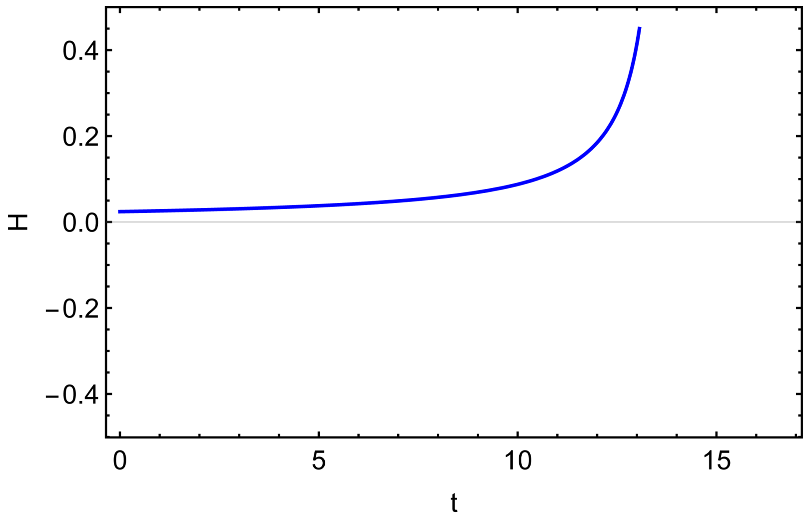

Figure 3.7: Hubble parameter H with t when $k = 13.80497512437811$.

From Fig. 3.6, it is obvious that the graph of the pressure of DE p_d lies in the negative plane during the entire course of evolution, which aligns with the ambiguous property of DE, which accounts for the accelerated expansion. From Fig. 3.7, it is clear that the Hubble's parameter H of the model universe tends to remain almost constant during the early evolution so that the model was in an inflationary epoch experiencing rapid exponential expansion (Kremer et al. 2019). The latest Planck 2018 result (Collaboration et al. 2020), estimates the present age of the universe to be 13.825 ± 0.037 Gyr. Assuming $t = 13.8$ and $k = 13.80497512437811$, from Eq. (3.2.27), the value of Hubble parameter is measured to be $H = 67$, approximately equal to the value $H_0 = 67.36 \pm 0.54$ kms^{-1} Mpc^{-1} of the latest Planck 2018 result (Collaboration et al. 2020).

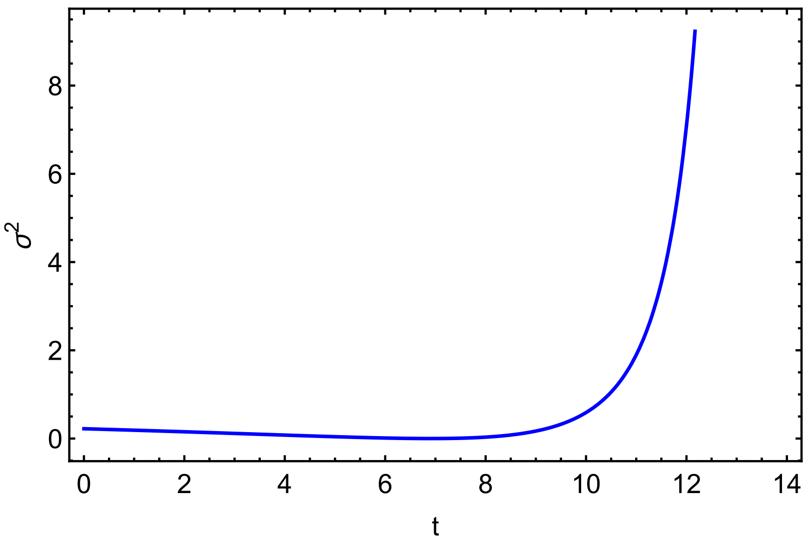

Figure 3.8: Variation of the shear scalar σ^2 with t when $k = 13.80497512437811$.

Fig. 3.8 shows us the variation of σ^2 with cosmic time t. Initially, σ^2 appears to decrease negligibly, and then, it tends to diverge. σ^2 shows us the rate of deformation of the matter flow within the massive cosmos (Ellis & Elst 1999). From Eq. (3.2.29), the anisotropic parameter $A_h = 0$. So, we can sum up that the universe is isotropic and expands with a slow and uniform change of size in the early evolution, whereas the change tends to become faster at late times. This is in agreement with the present observation of the accelerated expansion of the universe.

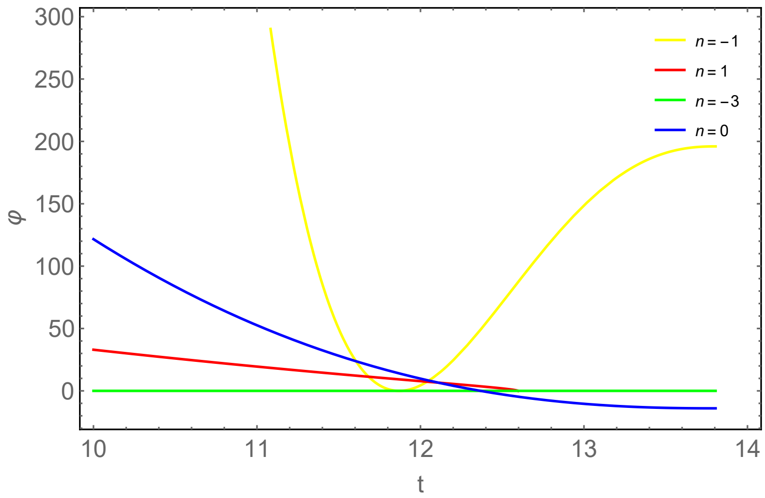

Figure 3.9: Variation of φ with t when $c_1 = c_2 = 1$.

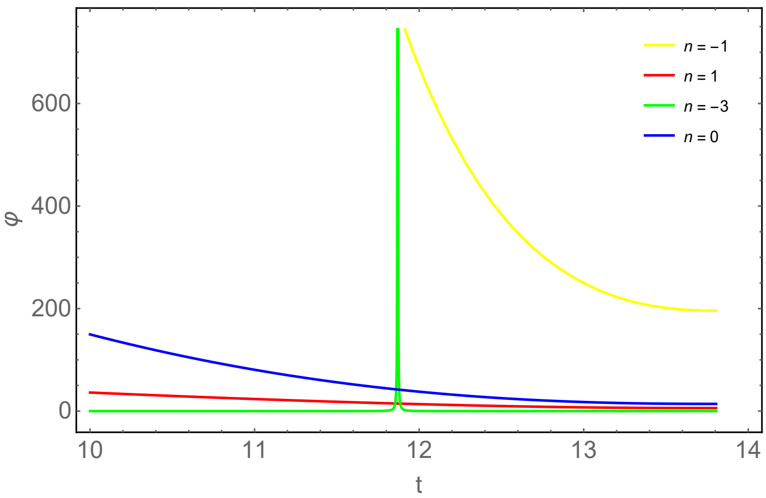

Figure 3.10: Variation of φ with t when $c_1 = -1, c_2 = 1$.

Fig. 3.9 shows the variation the SB scalar field φ with cosmic time t when $c_1 = c_2 = 1$ (both $c_1, c_2 > 0$) whereas Fig. 3.10 shows the variation when $c_1 = -1, c_2 = 1$ ($c_1 < 0, c_2 >$

0). In both cases, the real value of φ can't be obtained for $n = -2$. In Fig. 3.9, we can see the decreasing nature of φ. However, when $n = -1$, it decreases up to a minimum value and increases to attain a constant positive value. It decreases to become negative when $n = 0$. In Fig. 3.10, φ decreases and is positive all through. When $n = -1$, it tends to attain a constant positive value after decreasing for a finite time. When $n = -3$, it attains its maximum and minimum values during the evolution. Hence, in both cases, when $n = -1$, φ tends to attain almost the same large positive constant, which might be the reason for the phantom-like nature of the DE at present. This observation is somewhat similar to that obtained by Naidu et al. (2019), where after both increasing and decreasing, the scalar field tends to attain a positive constant value.

Lastly, from Eq. (3.2.33), the value of the jerk parameter is obtained to be $j(t) = 28$. It can be used as a tool to describe the closeness of models to the standard ΛCDM model. Its value for the standard ΛCDM model is $j(t) = 1$.

3.4 Conclusions

In this chapter, we have investigated an interacting model of HDE and matter in a SS space-time in 5D setting within the framework of SBT. We have obtained an accelerating model where HDE with phantom-like nature dominates the universe. The model doesn't evolve from an initial singularity. To preserve the theoretical foundation of HDE scenario, an extra dimension is employed. In the far future, the DE departs from phantom-like nature to cosmological constant thereby bypassing future singularity and ultimately leading to the de-Sitter phase. The universe is predicted to be isotropic. At $t = 13.8$ Gyr, the approximate present age of the universe, the values of Hubble parameter and DE EoS parameter are measured to be $H = 67$ and $\omega = -1.00011$, which agree with the respective values $H_0 = 67.36 \pm 0.54$ kms^{-1} Mpc^{-1} and $\omega = -1.03 \pm 0.03$ of the latest Planck 2018 result (Collaboration et al. 2020). It is predicted that the model expands with a slow and uniform change of size in the early evolution whereas the change tends to become faster at late times. We observe that when $n = -1$, the SB scalar field φ tends to attain a positive constant value in the course of evolution.

Chapter 4

Vacuum energy in Saez-Ballester theory and stabilization of extra dimensions

The work presented in this chapter is published in Universe 8 (2022) 60, DOI: 10.3390/universe8020060 (IF-2.278)

4.1 Introduction

Since the discovery of dark energy (DE) (Riess et al. 1998; Perlmutter et al. 1999), it has gained a reputation as one of the topics of paramount importance among the cosmological forums. Despite investing tremendous scientific efforts to explore it, its origin, bizarre nature, and future aspects to modern cosmology are still up for grabs. It is characterized by the distinctive feature of possessing a huge negative pressure opposing gravity resulting in the enigmatic phenomenon of the universe expanding at an expedited rate at late times. This cryptic dark entity is considered to be uniformly distributed and varies slowly or nearly unchanged with time (Carroll 2001a, 2001b; Chan 2015b; Peebles & Ratra 2003). Some worth mentioning studies on this mystic dark component that have not escaped our attention in the last few years are briefly presented below.

Recently, Singh & Singh (2021a) study a higher dimensional cosmological model to find the origin of DE. They further predict an $f(R,T)$ gravity model as a DE source (Singh & Singh 2021b). A presentation on the evolution of DE considering recent findings can be seen in the work presented by Wang et al. (2018). Collaboration et al. (2016) investigate the future of this dark entity beyond the bound of cosmological aspects. The estimation of DE density is presented by Dikshit (2019). Moradpour et al. (2014) put forward arguments for the need for DE. Gutierre (2015) analyses the status of the experimental data on DE. A fascinating comparison of the speed of DE with that of a photon can be found in work of Hecht (2013). The atom-interferometry constraints on DE are studied by Hamilton et al. (2015). In the publication of Josset et al. (2017), DE is obtained from the violation of energy conservation. Clery (2017) predicts that galaxies cluster as a result of stirring effect of DE. Lastly, Chan (2015a) claims that particles with imaginary energy density can lead us to the root of the ambiguous dark component.

Cosmologists have witnessed numerous theoretical attempts to obtain hints as to exactly predict the underlying physics of the miraculous expanding phenomenon of the universe at late times. Two well-appreciated methods have been adapted to explain this mystic phenomenon. First, several types of DE are constructed. Second, modifying ETG (Ahmed & Pradhan A 2020b; Clifton et al. 2012). Other than these two, recently, cosmologists and theoretical physicists have been successful in developing other interesting and convincing approaches. In the work of Gorji (2016), the phenomenon is explained by the infrared corrections. Narain & Li (2018) predict that an Ultraviolet Complete Theory leads to the expansion. A fascinating illustration is presented by Berezhiani (2017) where the expedited expansion occurs in the absence of DE.

To figure out the ambiguous nature of DE in as much detail as possible, the equation of state (EoS) parameter ω is studied with utmost importance. The most recent Planck 2018 results (Collaboration et al. 2020), estimates its value to be $\omega = -1.03 \pm 0.03$. The late time expedited expansion of the universe is obtained when $\omega < -\frac{1}{3}$ (Tripathi et al. 2017). $\omega = -1$ corresponds to the natural candidate of DE, the cosmological constant (CC), or in other words, vacuum energy (VE). However, CC or VE comes up short to explain the mystery of the coincidence problem (CP) (Zlatev et al. 1999). After multiple efforts, many other well-appreciated forms of DE are developed (Copeland et al. 2006). One such candidate that has not escaped our notice is the holographic dark energy (HDE). As a result of the holographic principle (Bousso 2002) being applied to DE, HDE is formed. The work of Wang et al. (2017) provides a peek of HDE's fundamental nature and properties. Recent works on some of the different forms of HDE can be seen in the publications by Korunur (2019), Pradhan et al. (2021), Prasanthi & Aditya (2020) and Srivastava et al. (2019). Construction of interacting HDE and dark matter (DM) models in spherically symmetric space-time settings is studied by Reddy D R K et al. (2016, 2016a), Singh & Singh (2019b) and many others. Interacting models can successfully represent modified gravity in the Einstein frame (Cai et al. 2016; Felice & Tsujikawa 2010; He et al. 2011; Kofinas et al. 2016; Zumalacarregui et al. 2013). According to the works of Amendola & Tocchini-Valentini (2001), Cai & Wang (2005), Zimdahl et al. (2001) and Zimdahl & Pavon (2004), we can find that such interacting models are effective in mollifying the CP.

Due to the fascinating natures of the HDE and VE, a spark of interest has been ignited among cosmologists so that they have started to examine HDE paired with VE. Singh & Kumar (2015) predict that their HDE model evolved from ΛCDM in early time and approaches to the same ΛCDM in the late time. They further mention that for a fixed value of a coupling parameter involved, their HDE model remains fixed in the ΛCDM model all through. Sadri et al. (2018) present an accelerating HDE model behaving similarly to the ΛCDM model. An explanation is presented by Dubey & Sharma (2020) in which the HDE model can't be discriminated from ΛCDM in the high-redshift region. Lee et al. (2007) assert that the vacuum entanglement energy is the probable candidate for HDE, where entanglement energy is the disturbed vacuum energy due to the presence of a boundary (Mukohyama et al. 1997). Hu et al. (2015) develop a heterotic DE model where the DE has two parts, the cosmological constant and HDE. A study of an HDE model where $\omega = -1$ is obtained is presented by Myung (2007). Lastly, Mathew et al. (2013) study a model where HDE ends at ΛCDM in the future.

Saez-Ballester Theory (SBT), introduced by Saez and Ballester (Saez & Ballester 1986), can be regarded a viable approach for studying DE and the expanding cosmos. It belongs to the Scalar Tensor Theory (STT) class of gravity theories. In SBT, the metric potentials are associated with a scalar field φ. Scalar fields are regarded to be important in gravitation and cosmology since they may depict phenomena such as DE, DM, and so on (Aditya et al. 2021). They can be considered a probable contributor to the universe's late-time acceleration (Kim 2005). STT is a straightforward extension and generalization of GR (Panotopoulos & Rincon 2018). STT appears to be an ideal contender for DE (Mandal et al. 2018). Linde (1982) and Guth (1981) assert that the inflation at the start of the evolution might have been caused by a scaler field. Pradhan et al. (2013) and Sharma et al. (2019) discuss Bianchi Type-V cosmology in SBT obtaining a transit from decelerating universe to accelerating phase. Currently, SBT and general relativity are held to align with observation.

The higher-dimensional model has become one of the good choices among cosmologists and theorological physicists. The idea of such a model was put forward by Kaluza and Klein (Kaluza 1921; Klein 1926). Aly (2019) and Banik & Bhuyan (2017) claim that such a model can explain the late time expanding phenomenon. In the work of Farajollahi & Amiri (2010), it is mentioned that extra-dimensional theories of gravity might explain the early inflation and late-time acceleration of the universe. There is a remarkable improvement in our knowledge and the logical consistency of physics by the introduction of the fifth dimension (Wesson 2015). A study to validate the existence of the extra dimension is presented by Marciano (1984). There is a chance that the unknown fifth dimension might be related to two the ambiguous and unseen dark components - dark energy and dark matter (Chakraborty & Debnath 2010). According to Zhang (2010), the employment of an extra dimension makes HDE models more complete and consistent. Some recent worth mentioning studies on higher dimension can be seen in the works of Ahmed & Pradhan (2020a), Astefanesei et al. (2020), Ghaffarnejad et al. (2020), Mishra et al. (2019), Montefalcone et al. (2020) and Saha & Ghose (2020).

Taking into consideration the above noteworthy related studies, we consider a minimal DE-DM interaction within the framework of SBT using a 5D spherically symmetric (SS) space-time. In this chapter, we present an in-depth discussion on every cosmological parameter obtained. The definition of shear scalar and its physical significance are provided. We discuss the initial and future singularity of the model universe. Additionally, we calculate the present values of the overall density parameter, deceleration parameter, and the dark energy EoS parameter. We also discuss the conditions to solve the stabilization problem of extra dimensions in general relativity. The chapter is divided into sections. After the introduction, in Sect. 4.2, we present the formulation of the problem with solutions to the parameters. In Sect. 4.3, the solutions are discussed with graphical representations. In Sect. 4.4, we present the explanation on the solution to stabilization problem of extra dimensions in GR. Lastly, to sum up the observations, a concluding note is provided in Sect. 4.5.

4.2 Formulation of problem and solutions

In our universe, the five-dimensional SS metric (Samanta & Dhal 2013) of following the form is considered

$$ds^2 = dt^2 - e^{\mu}\left(dr^2 + r^2 d\Theta^2 + r^2 \sin^2\Theta d\phi^2\right) - e^{\delta} dy^2 \tag{4.2.1}$$

where μ and δ are cosmic scale factors which are functions of time only.

We consider the following Saez-Ballester field equations

$$R_{ij} - \frac{1}{2}g_{ij}R - \omega_{SB}\varphi^n\left(\varphi_{,i}\varphi_{,j} - \frac{1}{2}g_{ij}\varphi_{,k}\varphi^{,k}\right) = -\left(T_{ij} + S_{ij}\right) \tag{4.2.2}$$

where T_{ij} and S_{ij} are the energy momentum tensors for matter and HDE respectively, R and R_{ij} are respectively the Ricci scalar and tensors, whereas the scalar field φ satisfies

$$2\varphi^n\varphi^{;i}_{;i} + n\varphi^{n-1}\varphi_{,k}\varphi^{,k} = 0 \tag{4.2.3}$$

where n is an arbitrary constant.

We define $T_{i\,j}$ and $S_{i\,j}$ as

$$T_{i\,j} = \rho_m u_i u_j \tag{4.2.4}$$

$$S_{i\,j} = (\rho_d + p_d)\, u_i u_j - g_{i\,j} p_d \tag{4.2.5}$$

where ρ_m and ρ_d represent the energy densities of matter and HDE respectively and p_d represents the pressure of the HDE.

Here, the energy is conserved and obviously, we have

$$T^{i\,j}_{;j} + S^{i\,j}_{;j} = 0 \tag{4.2.6}$$

By using the co-moving coordinate system, the surviving field equations are obtained as follows

$$\frac{3}{4}\left(\dot{\mu}^2 + \dot{\mu}\dot{\delta}\right) + \frac{\omega_{SB}}{2}\varphi^n\dot{\varphi}^2 = \rho \tag{4.2.7}$$

$$\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 + \frac{\ddot{\delta}}{2} + \frac{\dot{\delta}^2}{4} + \frac{\dot{\mu}\dot{\delta}}{2} - \frac{\omega_{SB}}{2}\varphi^n\dot{\varphi}^2 = -p_d \tag{4.2.8}$$

$$\frac{3}{2}\left(\ddot{\mu} + \dot{\mu}^2\right) - \frac{\omega_{SB}}{2}\varphi^n\dot{\varphi}^2 = -p_d \tag{4.2.9}$$

and from Eq. (4.2.6), we have

$$\ddot{\varphi} + \dot{\varphi}\left(\frac{3\dot{\mu} + \dot{\delta}}{2}\right) + \frac{n}{2}\dot{\varphi}^2\varphi^{-1} = 0 \tag{4.2.10}$$

where an overhead dot represents differentiation w.r.t. t.

Considering ω as the EoS parameter of the dark energy so that we have

$$p_d = \omega\rho_d \tag{4.2.11}$$

Now, the conservation equation is given by

$$\dot{\rho}_d + (1+\omega)\left(\frac{3\dot{\mu} + \dot{\delta}}{2}\right)\rho_d + \dot{\rho}_m + \rho_m\left(\frac{3\dot{\mu} + \dot{\delta}}{2}\right) = 0 \tag{4.2.12}$$

Due to the minimal interaction of HDE and matter, according to Sarkar (2014a, 2014b), both the components conserve separately thereby obtaining

$$\dot{\rho}_m + \rho_m\left(\frac{3\dot{\mu} + \dot{\delta}}{2}\right) = 0 \tag{4.2.13}$$

$$\dot{\rho}_d + (1+\omega)\,\rho_d \left(\frac{3\dot{\mu}+\dot{\delta}}{2}\right) = 0 \tag{4.2.14}$$

Also, we have

$$\dot{\rho} + (\rho + p) \left(\frac{3\dot{\mu}+\dot{\delta}}{2}\right) = 0 \tag{4.2.15}$$

From Eqs. (4.2.13) and (4.2.14), we have

$$\rho_m = a_0 e^{-\left(\frac{3\mu+\delta}{2}\right)} \tag{4.2.16}$$

$$\rho_d = b_0 e^{-(1+\omega)\left(\frac{3\mu+\delta}{2}\right)} \tag{4.2.17}$$

where a_0 and b_0 are arbitrary constants.

From Eqs. (4.2.8) and (4.2.9), we obtain the expression for cosmic scale factors as

$$\mu = c_1 + \log\left(v\,t - uc_2\right)^{\frac{u}{v}} \tag{4.2.18}$$

$$\delta = kc_1 + \log\left(v\,t - uc_2\right)^{\frac{ku}{v}} \tag{4.2.19}$$

where $c_1,\ c_2,\ u,\ v$ and $k \neq 0$ are arbitrary constants.

From Eqs. (4.2.16)-(4.2.19), the energy densities of matter and DE are respectively obtained as

$$\rho_m = a_0 e^{-\frac{(k+3)c_1}{2}} \left(vt - uc_2\right)^{-\frac{(k+3)u}{2v}} \tag{4.2.20}$$

$$\rho_d = b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}} \left(vt - uc_2\right)^{-\frac{(1+\omega)(k+3)u}{2v}} \tag{4.2.21}$$

Using Eqs. (4.2.18) and (4.2.19) in Eq.(4.2.10), the expression for scalar field is obtain as

$$\varphi = c_2 e^{\frac{2\log\left(e^{\frac{u}{2}(k+3)\left(\frac{t}{v^2 t - uvc_2} - 2c_1\right)} - (n+2)\left(uvc_2 - v^2 t\right)\right) - \frac{(k+3)u\,t}{v^2 - uvc_2}}{n+2}} \tag{4.2.22}$$

From Eqs. (4.2.20) and (4.2.21), the expression for energy density of the model universe is obtained as

$$\rho = a_0 e^{-\frac{(k+3)c_1}{2}} \left(vt - uc_2\right)^{-\frac{(k+3)u}{2v}} + b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}} \left(vt - uc_2\right)^{-\frac{(1+\omega)(k+3)u}{2v}} \tag{4.2.23}$$

Using Eqs. (4.2.18), (4.2.19) and (4.2.23) in Eq. (4.2.15), the expression for pressure of the model universe is obtained as

$$p = -\left(a_0 e^{-\frac{(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(k+3)u}{2v}} + b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(1+\omega)(k+3)u}{2v}}\right) \tag{4.2.24}$$

From Eqs. (4.2.11) and (4.2.21), the pressure of dark energy is obtained as

$$p_d = \omega \; b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(1+\omega)(k+3)u}{2v}} \tag{4.2.25}$$

At any time $t = t_0$, we can assume that $p = p_d$ so that

$$\left(a_0 e^{\frac{\omega(k+3)c_1}{2}} (vt - uc_2)^{\frac{\omega(k+3)u}{2v}} + b_0 (1+\omega)\right) e^{-\frac{(1+\omega)(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(1+\omega)(k+3)u}{2v}} = 0 \tag{4.2.26}$$

The expression for ω will be given by Eq. (4.2.26).

Now, the expressions for the different cosmological parameters are obtained as given below

Spatial volume:

$$V = e^{\frac{(k+3)c_1}{2}} (vt - uc_2)^{\frac{(k+3)u}{2v}} \tag{4.2.27}$$

Scalar expansion:

$$\theta = \frac{(k+3)\,u}{2\,(vt - uc_2)} \tag{4.2.28}$$

Hubble parameter:

$$H = \frac{(k+3)\,u}{8\,(vt - uc_2)} \tag{4.2.29}$$

Deceleration parameter:

$$q = \frac{8v}{(k+3)\,u} - 1 \tag{4.2.30}$$

Shear scalar:

$$\sigma^2 = \frac{1}{72}\left(\frac{16vt^2 - 4\,(3k + 8c_2 + 9)\,uvt + 3\,(3k + 4kc_2 + 12c_2 + 9)\,u^2 + 16uc_2{}^2}{(vt - uc_2)^2}\right) \tag{4.2.31}$$

Anisotropic parameter:

$$A_h = 3\left(\frac{k-1}{k+3}\right)^2 \tag{4.2.32}$$

Dark energy density parameter:

$$\Omega_d = \frac{\rho_d}{3H^2} = \frac{64}{3}\left(\frac{b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}} (vt - uc_2)^{2-\frac{(1+\omega)(k+3)u}{2v}}}{3\,(k+3)^2\,u^2}\right) \tag{4.2.33}$$

Matter density parameter:

$$\Omega_m = \frac{\rho_m}{3H^2} = \frac{64}{3}\left(\frac{a_0 e^{-\frac{(k+3)c_1}{2}}(vt-uc_2)^{2-\frac{(k+3)u}{2v}}}{3\,(k+3)^2\,u^2}\right) \tag{4.2.34}$$

Overall density parameter:

$$\Omega = \frac{64}{3}\left(\frac{\left(a_0 + b_0 e^{-\frac{\omega(k+3)c_1}{2}}(vt-uc_2)^{-\frac{\omega(k+3)u}{2v}}\right) e^{-\frac{(k+3)c_1}{2}}(vt-uc_2)^{2-\frac{(k+3)u}{2v}}}{3\,(k+3)^2\,u^2}\right) \tag{4.2.35}$$

According to Ghaffari et al. (2015), the expression for the state finder diagnostic pair $\{r, s\}$ is given by

$$r = 1 + \frac{3\dot{H}}{H^2} + \frac{\ddot{H}}{H^3} \tag{4.2.36}$$

$$s = \frac{r-1}{3\left(q-\frac{1}{2}\right)} \tag{4.2.37}$$

From Eqs. (4.2.29), (4.2.36) and (4.2.37), we have

$$\{r,\, s\} = \{1,\, 0\} \tag{4.2.38}$$

4.3 Discussion

In this section, for convenience sake, we opt to choose $a_0 = b_0 = c_1 = c_2 = k = 1, u = 2.78$ and $v = \frac{1}{2}$. The discussion on the nature of the parameters with respect to cosmic time t are presented in details with graphs as follows.

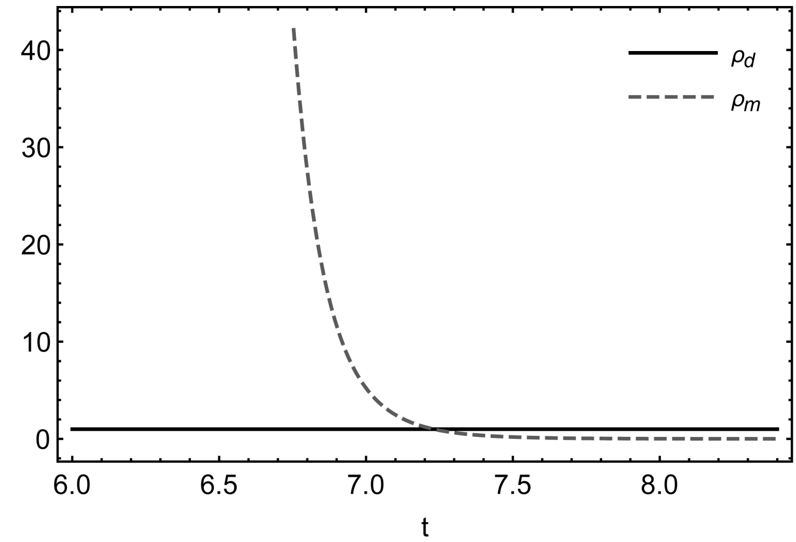

Figure 4.1: Variation of the energy densities of DE ρ_d and DM ρ_m with t when $a_0 = b_0 = c_1 = c_2 = k = 1, u = 2.78, v = \frac{1}{2}$

From Eqs. (4.2.20) and (4.2.21), it is obvious that ρ_d and ρ_m are functions of t. Fig. 4.1 shows that ρ_d is almost consistent throughout whereas ρ_m decreases in the entire course of evolution, which are acceptable scenarios as the ambiguous DE varies slowly or is unchanged with time (Carroll 2001a, 2001b; Chan 2015b; Peebles & Ratra 2003), on the other hand, DM diminishes continuously as a result of the galaxies scattering away from one another during expansion (Carroll 2001b). Moreover, when $t \to \infty$, $\rho_m \to 0$. From these, it would be appropriate to conclude that the universe will be progressively dominated by this cryptic DE. Similar increasing dominant nature of DE can also be seen in the papers of Singh & Singh (2019b), Singh & Samanta (2019) and Caldwell et al. (2003).

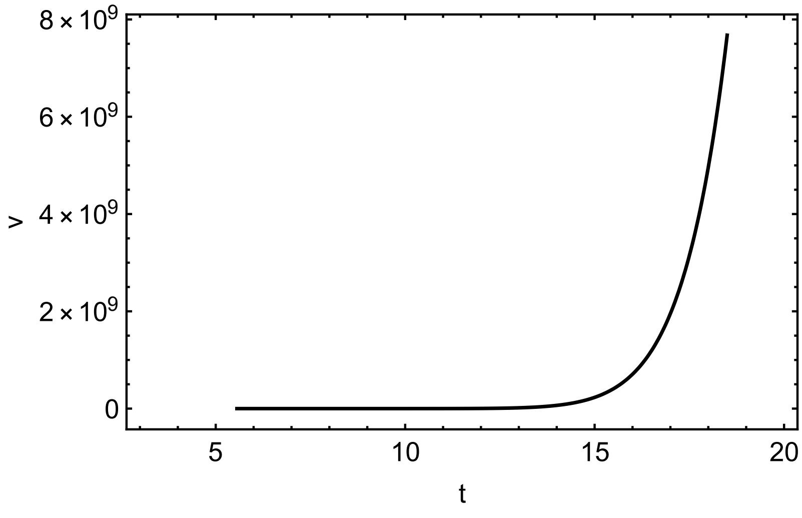

Figure 4.2: Variation of the spatial volume V with t when $c_1 = c_2 = k = 1, u = 2.78, v = \frac{1}{2}$

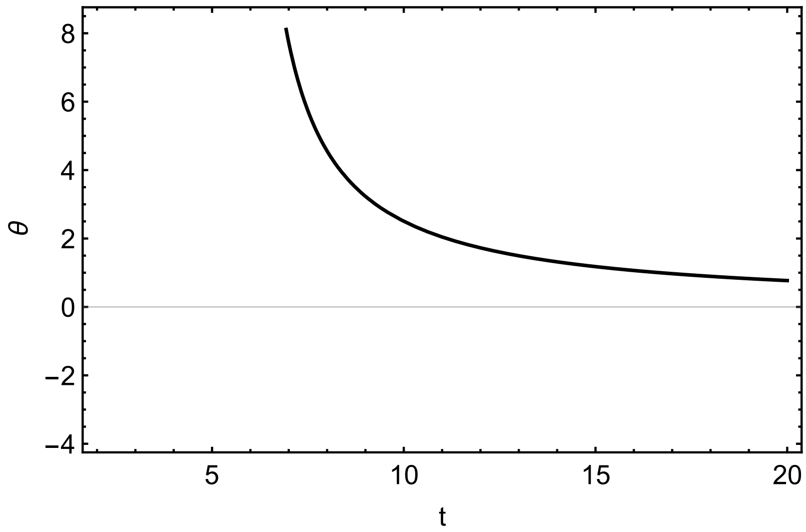

Figure 4.3: Variation of the expansion scalar θ with t when $c_2 = k = 1, u = 2.78, v = \frac{1}{2}$

Fig. 4.2 can be regarded as perfect supporting evidence for the present observation of the spatial expansion of the universe. However, at the initial epoch when $t = 0$, $V = 0$. Also, from Fig. 4.3, we can see that θ initially emerges with a large value, decreases with evolution, and finally, tends to become constant after some finite time which is the indication of the Big-Bang scenario (Mollah et al. 2018). The prediction of a similar scenario with similar cosmological settings can also be seen in a research of Aditya & Reddy (2018). On considering $a_0 = b_0 = c_1 = c_2 = k = 1, u = 2.78$, $v = \frac{1}{2}$ and assuming the present age of the universe to be $t_0 = 13.8$ Gyr which align with the estimated present age by the most recent Plack 2018 results (Collaboration et al. 2020), from Eq. (4.2.26), the value for EoS parameter is measured to be $\omega = -1$. The Planck 2018 results estimates its value to be $\omega = -1.03 \pm 0.03$ (Collaboration et al. 2020). So, the dark energy candidate we are dealing with is the vacuum energy or the cosmological constant. Moreover, from Fig. 4.1, it can be seen that the dark energy density ρ_d remains almost constant throughout evolution, and from Eq. (4.2.27), $v \to \infty$ when $t \to \infty$. So, it would be a pertinent fact that the universe has no end; expanding forever, ultimately, leading to the Big Freeze singularity in the far future. In a thermodynamic sense, the model universe will enter a point of minimum temperature and maximum entropy. It will be almost as though all astrophysical process is being smothered, as the fuel for growth and reproduction gets so diffuse that it can't be used (Skibba 2020). It will be an ending point characterized by increasing isolation, inexorable decay, and an eons-long fade into darkness (Mack 2020).

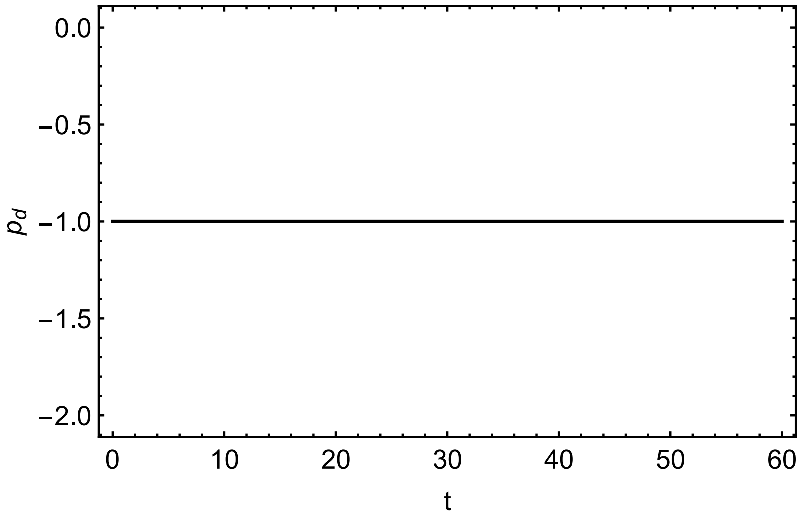

Figure 4.4: Variation of the DE pressure p_d with t when $b_0 = c_1 = c_2 = k = 1, \omega = -1, u = 2.78, v = \frac{1}{2}$

The pressure of DE p_d varies in the negative plane throughout, as seen in Fig. 4.4, which is consistent with the enigmatic feature of DE accountable for the universe's rapid expansion.

u	v	k	q
2.78	$\frac{1}{2}$	1	-0.64
2.78	$\frac{1}{1.6}$	1	-0.55
2.25	$\frac{1}{2}$	1	-0.55
2.25	$\frac{1}{1.6}$	1.9	-0.54

Table 4.1: Values of deceleration parameter q for different values of u, v and k.

From Eq. (4.2.30), the deceleration parametr q depends on u, v and k. In Table 4.1, we present different values of q for different values of u, v and k. Recently, Camarena & Marra (2020) predict its value as $q = -0.55$, whereas Capozziello et al. (2019) estimate the value as $q = -0.644 \pm 0.223$ and $q = -0.6401 \pm 0.187$. With all the values of the constants in Table 4.1, we obtain the EoS paramter of CC. Since q lies in the range $-1 < q < 0$, the accelerating model universe undergoes exponential expansion (Singh & Bishi B K 2017), in agreement with the present cosmology.

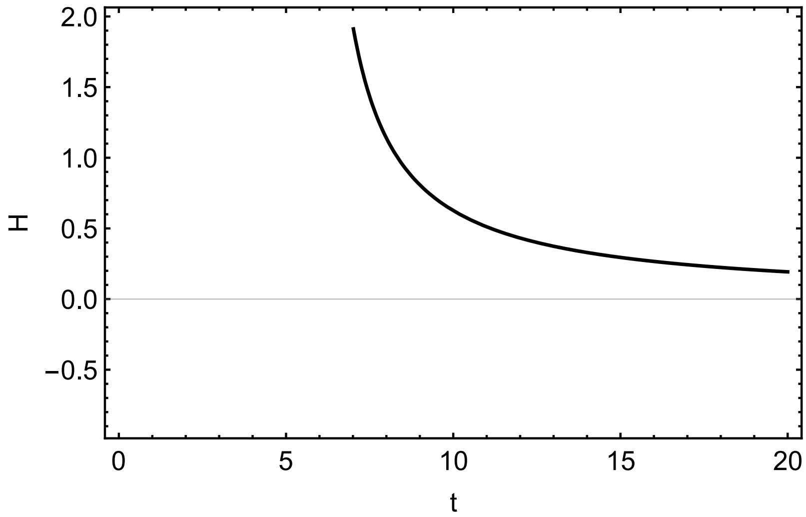

Figure 4.5: Variation of the Hubble parameter H with t when $c_2 = k = 1, u = 2.78, v = \frac{1}{2}$

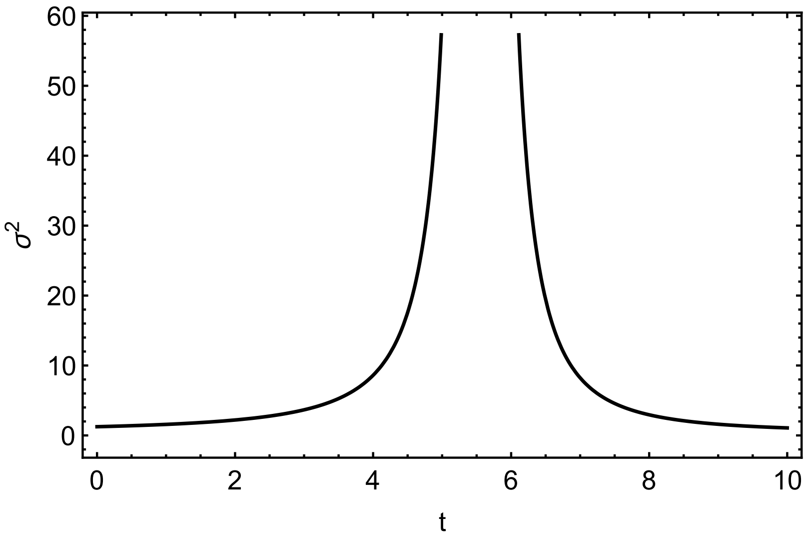

Figure 4.6: Variation of the shear scalar σ^2 with t when $c_2 = k = 1, u = 2.78$ and $v = \frac{1}{2}$

Fig. 4.5 shows the decreasing nature of Hubble parameter H which is within the limit of the present cosmological scenario (Biswas et al. 2019; Mishra & Chand 2020). Shear scalar σ^2 shows us the rate of deformation of the matter flow within the massive cosmos (Ellis & Elst 1999). The evolution of σ^2 can be seen in Fig. 4.6. It evolves with a constant value, then diverges after some finite time, and again converges to become constant. It vanishes for a finite period during evolution. From these, we can summarize that in the initial epoch, the model universe expands with a very slow and uniform change of shape, but after some finite time, the change becomes faster. Then, it again tends to become very slow and uniform after expanding without any deformation for a finite period. From

Eq. (4.2.32), the anisotropic parameter $A_h = 0$ for $k = 1$ so that the constructed model is isotropic.

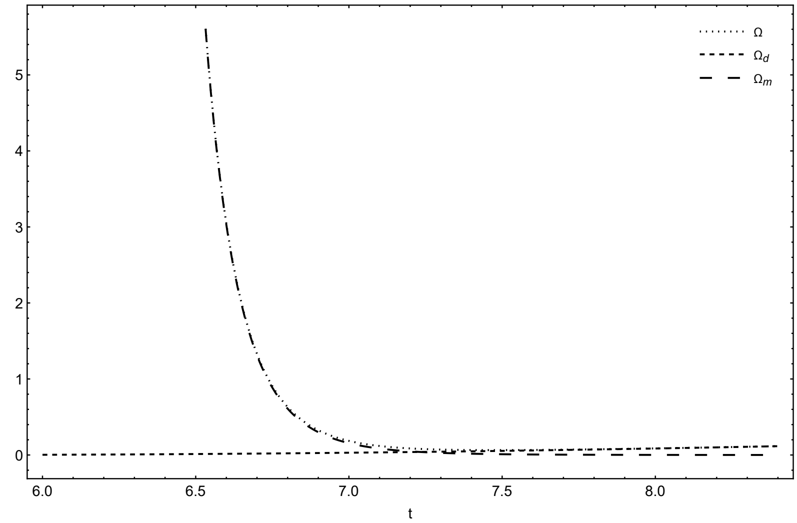

Figure 4.7: Variation of the overall density parameter Ω, DE density parameter Ω_d and DM density parameter Ω_m with t when $a_0 = b_0 = c_1 = c_2 = k = 1, u = 2.78$ and $v = \frac{1}{2}$

Fig. 4.7 shows us the variation of Ω, Ω_d and Ω_m with t. Here, since DE varies slowly or is unchanged with time (Carroll S M 2001a, 2001b; Chan 2015b; Peebles & Ratra 2003), we can see that Ω_d tends to remain constant or increases very slowly. However, Ω_m decreases in the entire course of evolution as a result of the galaxies scattering away from one another leading DM to diminish continuously (Carroll 2001b). Above all, with $k = 1, u = 2.78$ and $v = \frac{1}{2}$, from Eq. (4.2.35), the overall density parameter is obtained to be $\Omega = 0.97(\approx 1)$. For an exactly flat universe, $\Omega = 1$ (Holman 2018; Khodadi et al. 2015; Levin & Freese 1994). Recently, many authors advocate against the belief of an exactly flat universe (Khodadi et al. 2015; Valentino et al. 2020; Javed et al. 2020; Nashed & Hanafy 2014). It will be a right conclusion to say that the universe is close to or nearly flat, but not exactly flat (Khodadi et al. 2015; Nashed & Hanafy 2014; Adler & Overduin 2005). Above all, the most recent Planck 2018 results (Collaboration et al. 2020) obtaining Ω ranging close to unity can be treated as a perfect piece of evidence for a nearly flat universe. Hence, our model obtaining Ω not exactly equal to 1 is justified.

Lastly, from Eq. (4.2.38), we can see that the value of the state finder diagnostic pair $\{r, s\} = \{1, 0\}$ which corresponds to the ΛCDM scenario so that the model universe we are considering is a ΛCDM model. Hence, our interacting HDE model can be considered as an alternate cosmological model to the standard ΛCDM model.

4.4 Stabilization of extra dimensions

The study on the stabilization of extra dimensions can be considered as a phenomenological necessity in higher-dimensional models. The discussion on stabilization is mostly confined to particle physics, supersymmetry, supergravity, string theory, and braneworld models. We require a stabilization mechanism to prevent modification of gravity to an experimentally undesirable manner (Kribs 2006). The stabilization also makes sure the visible 4D universe with a long lifetime (Ketov 2019). Another benefit of stabilization is that we can ignore any unwanted outcomes of quantum gravity at Planck length distances (Hamed et al. 2002). One of the most classic solutions for stabilization is the Goldberger-Wise mechanism (Goldberger & Wise 1999), where stabilization is achieved in the presence of an additional scalar field. Carroll et al. (2002) claim that stabilization can be achieved by introducing a potential of the dilaton field. Chung & Freese (1999) present a study of an isotropic 3-brane model where stabilization is achieved with the only value of the EoS $\omega(t) = -\frac{2}{3}$. Another observation of stabilization in an isotropic perfect fluid model in 5D with the value of EoS $\omega > -\frac{1}{3}$ is presented by Arapoglu et al. (2018). Bronnikov & Rubinn (2006) show that the issue of stabilization can be overcome in a theory of gravity involving high-order curvature invariants. Sundrum (2005) obtains stabilization by quantum corrections from massive matter. In the investigation of Kainulainen & Sunhede (2006), we can find the investigation of a class of dilatonic STT where stabilization is achieved by quantum corrections to the effective 4D Ricci scalar. Mazumdar (1999) presents an argument calming that stabilization is attained as soon as inflation ends, on the contrary, Ferrer & Rasanen (2007) assert that inflation ends if stabilization is attained. According to Chirkov & Pavluchenko (2021), to achieve a realistic theoretical model, we should assume that the visible three dimensions are expanding isotropically, whereas the extra dimensions are contracting (or contracted for a period during the evolution). Similarly, Rasouli & Moniz (2017) predict that the extra dimension contracts with the cosmic time. According to Moraes & Correa (2019), the hidden extra dimension is related to scalar fields. Bruck & Longden (2019) also represent the size of the extra dimensions in terms of a scalar field. Hamed et al. (2001) investigate 4D gauge theories that dynamically generate a 5D, where stabilization is no longer needed. In his works (Tosa 1984, 1985), Tosa studies the Kaluza-Klein cosmology for a torus space with a cosmological constant and matter. He predicts that the number of the extra dimensions should be more than 1, and the extra dimensions should be of small size. However, during recent years, many authors have successfully predicted models with just one extra dimension, where stabilization is obtained (Das et al. 2018; Dudas & Quiros 2005; Egorov & Volobuev 2017; Kanti et al. 2002; Ponton & Poppitz 2001; Wongjun 2015). Additionally, we can also witness large extra dimensions the works by Gong et al. (2008), Wu et al. (2008) and Wang (2002), and infinite-volume extra dimensions in the fourth paragraph of this section.

In our work, we have discussed a 5D SS cosmological model in general relativity (GR) with the cosmological constant (CC), or in other words, vacuum energy (VE) as the DE candidate. In GR, generally, we cannot find conditions for stabilization, and all dimensions want to be dynamical (Bruck & Longden 2019). In an accelerating model with CC, stabilization cannot be obtained (Rador 2007). Therefore, in a trial to solve the stabilization problem in GR, we consider two options. The first one is the Casimir energy and the second is the infinite-volume extra dimension, which is discussed below.

Casimir energy is a DE candidate with the ability to drive the late-time accelerated expansion and stabilize the extra dimensions automatically (Wongjun 2015; Greene & Levin 2007). Casimir energy is VE emerging from imposing boundary conditions on the quantum fluctuations of fields and the EoS's of both Casimir energy and CC are of the same form (Wongjun 2015). Further, Wongjun (2015) interpretes Casimir energy as CC. Additionally, Roberts (2000) equates VE with Casimir energy. Ichinose (2012) also identifies Casimir energy with CC. If the CC is to be created from the Casimir energy, then there will be only one extra dimension (Dupays et al. 2013). Coincidently, in our spherically symmetric cosmological model with the CC as the DE candidate, there is only one extra dimension.

The study on extra dimensions has been widely considered in brane world models (Dick 2001; Freese & Lewis 2002; Hogan 2001; Ichiki et al. 2002; Langlois 2003; Shiromizu et al. 2000; Zhu & Fujimoto 2002, 2003, 2004), one of which is the DGP model (Dvali et al. 2000), which presents an accelerating 5D scenario with an infinite-volume extra dimension. This infinite-volume extra dimension drives the expedited expansion of the universe at late times (Alcaniz 2006). Kumar & Suresh (2005) and Satheeshkumar & Suresh (2011) assert that with an infinite-volume extra dimension, one doesn't need stabilization. They further claim that the infinite-volume scenario can explain us the late time cosmology and the acceleration of the universe driven by DE, which are one of the core components of GR. According to Dvali & Michael (2003), infinite-volume extra dimensions might result to the emergence of DE. Hence, it would be appropriate to conclude that the extra dimension in our study on 5D spherically symmetric cosmological model is of infinite-volume.

One of the most classic solutions for stabilization is the Goldberger–Wise (GW) mechanism (Goldberger & Wise 1999). We can witness the application of the GW mechanism in the field of string theory, M-Theory, and Randall and Sundrum (RS) model in the noteworthy works by Wang (2010), Wu et al. (2009), Wang & Santos (2008, 2010), Devin et al. (2009) and Wang et al. (2008). In these works, the authors consider a 5D static metric with a 4D Poincare symmetry. To obtain stability, they introduce the proper distance and a massive scalar field and show that the effective radion potential has a minimum. Since the Casimir energy (force) provides a natural alternative to the GW mechanism (Garriga & Pomarol 2003), the stabilization mechanism applied by the aforementioned noteworthy works might have some sort of relationship with the Casimir energy stabilization approach which we have predicted above. Above all, one may consider it as an advantage above the GW mechanism that the introduction of an ad hoc classical interaction between the branes is not needed in the Casimir energy approach of stabilization (Garriga & Pomarol 2003). We may note the work by Garriga et al. (2001) predicting that the Casimir force will not lead to stabilization to the right value unless a tuning of parameters. Fortunately, the work by Garriga & Pomarol (2003) shows that this conclusion of Garriga et al. (2001) is not general, and proves that Casimir energy (force) provides a natural alternative to the GW mechanism in the RS model. There might be more advantages or relationships of our predicted stabilization approaches with the GW mechanism, which we would like to find out in our future works.

We have presented two conditions for the stabilization of extra dimensions. Probably, our work might be the first to predict such conditions in GR. Nevertheless, these two conditions are toy models which require further in-depth analysis considering different cosmological aspects. We need more investigation on the reliability of considering, within GR, the identification of Casimir energy with cosmological constant, or in other words, vacuum energy. We also need to verify all the possible outcomes of assuming the extra dimension is of infinite volume in a higher-dimensional vacuum energy model within GR.

4.5 Conclusions

In this chapter, we have analysed a cosmological model in SS space-time in a 5D setting with minimally interacting matter and HDE in SBT. We predict that the expanding isotropic universe will be progressively DE dominated. The pressure of DE is negative all through. We estimate few values of the deceleration parameter and the values are found very close to the recently predicted values. The Hubble's parameter H decreases which agree with the present cosmological scenario. In the initial epoch, the model universe expands with a very slow and uniform change of shape, but after some finite time, the change becomes faster. Then, it again tends to become very slow and uniform after expanding without any deformation for a finite period. The value of the DE EoS parameter is measured to be $\omega = -1$ indicating that the DE we are dealing with is the VE or CC. The value of the overall density parameter is obtained as $\Omega = 0.97(\approx 1)$, which is not exactly equal to 1, since the universe is close to or nearly flat, but not exactly flat. We observe that the model universe starts with the Big-Bang and ends at the Big Freeze singularity. The value of the state finder diagnostic pair obtained corresponds to the ΛCDM model so that our interacting HDE model can be considered as an alternate cosmological model to the standard ΛCDM model. Lastly, we present two conditions to solve the stabilization problem of extra dimension in GR, the first one is the identification of Casimir energy with CC, or in other words, VE and the second is assuming the extra dimension is of infinite volume.

Chapter 5

A higher dimensional cosmological model for the search of dark energy source

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5.1 Introduction

The enigmatic dark energy (DE) has gained the reputation of being one of the most discussed topics of paramount importance in cosmology since its profound discovery (Riess et al. 1998; Perlmutter et al. 1999). Its property with huge negative pressure with repulsive gravitation causing the universe to expand at an expedited rate still remains a mystery. It is believed to uniformly permeate throughout the space and vary slowly or almost consistent with time (Chan 2015b; Carroll 2001a, 2001b; Peebles & Ratra 2003). Cosmologists around the world have invested tremendous scientific efforts with a strong focus to hunt its origin and are still scrabbling for a perfect answer. Different authors have put forward their own versions of the answer with convincing evidences in support. Some worth mentioning studies on the search of the root of DE which have not escaped our attention in the past few years are briefly discussed below.

Singh & Kar (2019) assert that emergent D-instanton might lead us to the root of DE. Huterer & Shafer (2018) analyse the twenty years old history of DE and the current status. Wang et al. (2018) investigate the evolution of the DE using a non-parametric Bayesian approach in the light of the latest observation. According to Capolupo (2018), vacuum condensate can indicate us the origin of DE. A study on neutrino mixing as the origin of DE is presented by Capolupo et al. (2007). The explanation for DE with pure quantum mechanical method is presented by Dikshit (2019). According to Alexander et al. (2010), DE emerges due to condensation of fermions formed during the early evolution. The explanation of a physical mechanism as a source of DE can be seen in the work of Gontijo (2012). A cosmological model involving an antineutrino star is proposed by Neiser (2020) in an attempt to find the origin of DE. Josset et al. (2017) obtain DE from the violation of energy conservation. A unified dark fluid is obtained as a source of DE by Josset et al. (2017). Lastly, we can witness a claim by Tripathy et al. (2015) that the presence of particle with imaginary energy density can lead us to the source of DE.

From literatures and observations (Agrawal et al. 2018; Araujo 2005; Carroll 2001a; Law 2020; Ray et al. 2013; Singh & Singh 2019b; Straumann 2007; Wu & Yu 2005), it is obvious that the massive universe is dominated by the mystic DE with negative pressure and positive energy density. This qualifies DE a completely irony of nature as the dominating component is also the least explored. So, there a lot more hidden physics behind this dark entity yet to be discovered. Contradicting to the condition of positive energy density, it is surprising that many authors has come up with the notion of negative energy density (NED) with convincing and fascinating arguments in support. Carroll (2001a) predicts a condition in which NED is possible only if the DE is in the form of vacuum energy. Besides defying the energy conditions of GR, NED also disobeys the second law of thermodynamics (Hawking & Ellis 1973). However, the condition should be solely obeyed on a large scale or on a mean calculation, thereby neglecting the probable violation on a small scale or for a short duration, in relativity (Epstein 1965; Fewster 2012; Ford & Roman 1996; Graham & Olum 2003; Helfer 1998a, 1998b; Pfenning & Ford 1998; Roman 1986; Visser & Barcelo 2000). Hence, in the initial epoch, if there were circumstance of defiance for a short duration measured against the present age which is estimated to be 13.825 ± 0.037 Gyr by the latest Planck 2018 result (Collaboration et al. 2020), it will remain as an important part in the course of evolution. According to Nemiroff et al. (2015), under certain conditions, a repelling negative gravitational pressure can be seen with NED. It further mention of a repelling negative phantom energy with NED. Energy density assuming negative value with equation of state parameter (EoS) $\omega < -1$ is predicted by Macorra & German (2004). Fay (2014) asserts that the universe evolves by inflation when the coupled fluid has NED in the initial epoch. Wong et al. (2019) discuss negative vacuum energy density in Rainbow Gravity. According to Parker & Fulling (1973), the introduction of quantized matter field with NED to energy momentum tensor might by pass cosmological singularity. Huang (1990) investigates models which evolved with NED in the infinite past. Ijjas & Steinhardt (2019) discuss NED asserting that their models evolve with a bounce. The authors continued that there might be bounces in the future too. Lastly, an accelerating universe with NED is studied by Sawicki & Vikman (2013).

The sessions of the Prussian Academy of Science during the four Thursdays of the month of November 1915 can be marked as the most memorable moments in the life of the great Einstein. On 4^{th}, 11^{th}, 18^{th} and 25^{th} of the month, he presented four of his notable communications (Einstein 1915a, 1915b, 1915c, 1915d) at the sessions, which led to the foundation of GR. Since then, a number of authors have been exploring gravitation in different settings. In Weyl's work of 1918 (Weyl 1918), we can witness the first trial to extend GR with the aim of bringing together gravitation and electromagnetism geometrically. Similar to that of Weyl, Lyra's modification (Lyra 1951) by proposing a gauge function into the structureless manifold provides one of the well appreciated alternate or modified theories of gravitation. The static model with finite density in Lyra's modified Riemannian geometry is similar to the static Einstein model (Singh & Singh 1991). The scalar-tensor treatment based on Lyra's geometry yields the same effects, as GR, under certain limits (Yadav 2020). Lyra's work are further extended by other well known authors (Halford 1970, 1972; Sen 1957, 1960; Sen & Dunn 1971). Cosmologists choose to opt alternate or modified theories of gravitation in order to precisely understand the underlying mechanism of the late time expedited expansion of the universe. Many other authors too have succeed in developing fascinating and worth appreciating modified theories which have served the purpose of explaining the expanding paradigm in a quite convincing way (Barker 1978; Bekenstein 2004; Brans & Dicke 1961; Chamseddine & Mukhanov 2013; Nojiri & Odintsov 2003, 2014; Nordtvedt 1970; Saez & Ballester 1986; Sotiriou & Faraoni 2010).

During the past few years, there has been an increasing interest among cosmologists to study the ambiguous DE paired with Lyra Manifold (LM). Recently, Bhardwaj & Rana (2020) investigate the existence of Lyra's cosmology with interaction of normal matter and DE. In the work of Hova (2013), we can witness a DE model in a LM which proves that the expansion paradigm can be illustrated in the absence of a negative pressure energy component. It is further mentioned that DE is naturally of geometrical origin. The investigation of a two component DE model in LM can be seen in the publication of Khurshudyan et al. (2014). Ram et al. (2020) predict a cosmological model of anisotropic DE paired with LM in consonant with the present observation. The study of of a magnetized DE model in Lyra setting is presented by Pawar et al. (2014). A discussion on the effect of DE on model with linear varying deceleration parameter in LM can be found in the work of Patra et al. (2019). Katore & Hatkar (2015) present the isotropization of DE distribution in LM. A Kaluza-Klein DE model is studied in LM thereby obtaining an exponentially expanding universe by Aditya et al. (2019). From the research of Singh & Sharma (2014a), we can find a DE model in LM where constant deceleration parameter is assumed. Brans–Dicke scalar field as a DE candidate in LM is illustrated by Zia & Maurya (2018). A DE model with quadratic equation of state is presented in the framework of LM by Mollah et al. (2018). Lastly, Yadav & Bhardwaj (2018) search for the existence of Lyra's cosmology with minimal interaction between DE and normal matter and obtain that the time varying displacement $\beta(t)$ co-relates with the nature of cosmological constant $\Lambda(t)$.

The chance of space-time having more than 4D has captivated many authors. This has ignited a spark of interest among cosmologists and theorological physicists so that, in the past few decades, there has been a trend among authors opting to choose higher dimensional space-time to study cosmology. Higher dimensional model was introduced by Kaluza (1921) and Klein (1926) in an attempt to unify gravity with electromagnetism. Higher dimensional model can be regarded as a tool to illustrate the late time expedited expanding paradigm (Banik & Bhuyan 2017). Investigation of higher dimensional space-time can be regarded as a task of paramount importance as the universe might have come across a higher dimensional era during the initial epoch (Singh et al. 2004). From the investigations of Alvax & Gavela (1983) and Guth (1981), it can be found that extra dimensions generate huge amount of entropy which gives possible solution to flatness and horizon problem. Marciano (1984) asserts that the detection of a time varying fundamental constants can possibly show us the proof for extra dimensions. Since we are living in a 4D space-time, the hidden extra dimension in 5D is highly likely to be associated with the invisible DM and DE (Chakraborty & Debnath 2010). Astefanesei et al. (2020), Bahrehbakhsh (2018), Demirel (2019), Ghaffarnejad et al. (2020), Montefalcone et al. (2020), Oli (2014), Saha & Ghose (2020), Samanta et al. (2014), Singh & Desikan (1997), Shinkai & Torii (2015) and Singh & Singh (2019b) are some of the authors who have worked on higher dimensional space-time during the last few years.

Keeping in mind the above notable works by different authors, we have analysed a spherically symmetric (SS) metric in 5D setting within the framework of LM, with the aim of predicting a possible source of DE. Here, we observe the field equations with due consideration of reasonable cosmological assumptions within the limit of the present cosmological scenario. The chapter is structured into sections. In Sect. 5.2, in addition to obtaining the solutions of the field equations, the cosmological parameters are also solved. In Sect. 5.3, the physical and kinematical aspects of our model are discussed with graphs. Considering everything, a closing remark is presented in Sect. 5.4.

5.2 Formulation of problem with solutions

The five-dimensional SS metric (Samanta & Dhal 2013) is given by

$$ds^2 = dt^2 - e^{\mu}\left(dr^2 + r^2 d\Theta^2 + r^2 \sin^2\Theta d\phi^2\right) - e^{\delta} dy^2 \tag{5.2.1}$$

where $\mu = \mu(t)$ and $\delta = \delta(t)$ are cosmic scale factors.

The modified Eintein's field equations in Lyra geometry appear in the form

$$R_{ij} - \frac{1}{2}g_{ij}R + \frac{3}{2}\varphi_i\varphi_j - \frac{3}{4}g_{ij}\varphi_k\varphi^k = -T_{ij} \tag{5.2.2}$$

where φ_i is the displacement vector and other symbols have their usual meaning as in Riemannian geometry. The displacement vector φ_i takes the time dependent form

$$\varphi_i = (\beta(t),\ 0,\ 0,\ 0,\ 0) \tag{5.2.3}$$

The assumption that φ_i is time independent i.e. constant is vague as there is no specific mathematical or physical explanation showing that a constant displacement vector contributes to the late time acceleration of the universe (Yadav 2020). Above all, assuming displacement vector field as a constant is just for convenience sake without any scientific reason (Singh & Desikan 1997).

The energy momentum tensor T_{ij}, considered as a perfect fluid, in the co-moving coordinates is given by

$$T_{ij} = (\rho + p)\, u_i u_j - p g_{ij} \tag{5.2.4}$$

where ρ and p respectively represent the energy density and isotropic pressure of the matter source. The five velocity vector u^i satisfies

$$u^i u_i = 1,\ u^i u_j = 0 \tag{5.2.5}$$

Now, the surviving field equations are obtained as follows

$$\frac{3}{4}\left(\dot{\mu}^2 + \dot{\mu}\dot{\delta} - \beta^2\right) = \rho \tag{5.2.6}$$

$$\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 + \frac{\ddot{\delta}}{2} + \frac{\dot{\delta}^2}{4} + \frac{\dot{\mu}\dot{\delta}}{2} + \frac{3}{4}\beta^2 = -p \tag{5.2.7}$$

$$\frac{3}{2}\left(\ddot{\mu} + \dot{\mu}^2\right) + \frac{3}{4}\beta^2 = -p \tag{5.2.8}$$

where an overhead dot represents differentiation w.r.t. t.

From continuity equation, we have

$$\dot{\rho} + \frac{3}{2}\dot{\beta}\beta + 3H\left(\rho + p + \frac{3}{2}\beta^2\right) = 0 \tag{5.2.9}$$

According to Bahrehbakhsh (2018), assuming that β and ρ are independent without any interaction, Eq. (5.2.9) can be separately written as

$$\dot{\rho} + 3H\left(\rho + p\right) = 0 \tag{5.2.10}$$

$$\dot{\beta}\beta + 3H\beta^2 = 0 \tag{5.2.11}$$

where H is the Hubble parameter.

From Eqs. (5.2.7) and (5.2.8), the expression for cosmic scale factors are obtained as

$$\mu = l - \log\left(k - t\right)^{\frac{2}{3}} \tag{5.2.12}$$

$$\delta = m - \log(k - t)^{\frac{2}{3}} \tag{5.2.13}$$

where $l,\ m,\ k$ are arbitrary constants.

Now, we obtain the expression for the cosmological parameters as follows.

Spatial volume:

$$V = e^{\frac{3\mu+\delta}{2}} = e^{\frac{3l+m}{2}}\left(k - t\right)^{-\frac{4}{3}} \tag{5.2.14}$$

Scale factor:

$$a(t) = V^{\frac{1}{4}} = e^{\frac{3l+m}{8}}\left(k - t\right)^{-\frac{1}{3}} \tag{5.2.15}$$

Scalar expansion:

$$\theta = u^i_{;j} = \frac{3\dot{\mu}}{2} + \frac{\dot{\delta}}{2} = \frac{4}{3}\left(k - t\right)^{-1} \tag{5.2.16}$$

Hubble parameter:

$$H = \frac{\Theta}{4} = \frac{1}{3}\left(k - t\right)^{-1} \tag{5.2.17}$$

Deceleration parameter:

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 = -4 \tag{5.2.18}$$

With $\Delta H_i = H_i - H, (i = 1,2,3,4)$ representing the directional Hubble's parameters, anisotropic parameter A_h is defined as

$$A_h = \frac{1}{4}\sum_{i=1}^{4}\left(\frac{\Delta H_i}{H}\right)^2 = 0 \tag{5.2.19}$$

Shear Scalar:

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{2}\sum_{i=1}^{4}\left(H_i^2 - 4H\right) = \frac{2}{9}\left(1 - \frac{1}{k-t}\right)^2 \tag{5.2.20}$$

From Eqs. (5.2.11) and (5.2.17), the expression for displacement vector is obtained as

$$\beta = d\,(t-k) \tag{5.2.21}$$

where d is an arbitrary constant.

From Eqs. (5.2.6), (5.2.12), (5.2.13) and (5.2.21), the expression for energy density is obtained as

$$\rho = \frac{3}{4}\left(\frac{8 - \left(3d\,(k-t)^2\right)^2}{9\,(k-t)^2}\right) \tag{5.2.22}$$

From Eqs. (5.2.8), (5.2.12) and (5.2.13), the expression of pressure is obtained as

$$p = -\frac{8\left(1 + (k-t)^2\right) + \left(3d\,(k-t)^3\right)^2}{12(k-t)^4} \tag{5.2.23}$$

5.3 Discussion

For convenience sake and to obtain realistic results, specific values of the constants are chosen i.e., $l = m = 1, d = -1, k = 3.45497$ and the variations of some of the physical parameters with cosmic time t are provided as figures in this section. A scale of 1 Unit = 4 Gyr is taken along the time axis of each graph so that the point $t = 3.45$ corresponds to 13.8 Gyr which align with 13.825 ± 0.037 Gyr, the present age of the universe estimated by the latest Planck 2018 result (Collaboration et al. 2020).

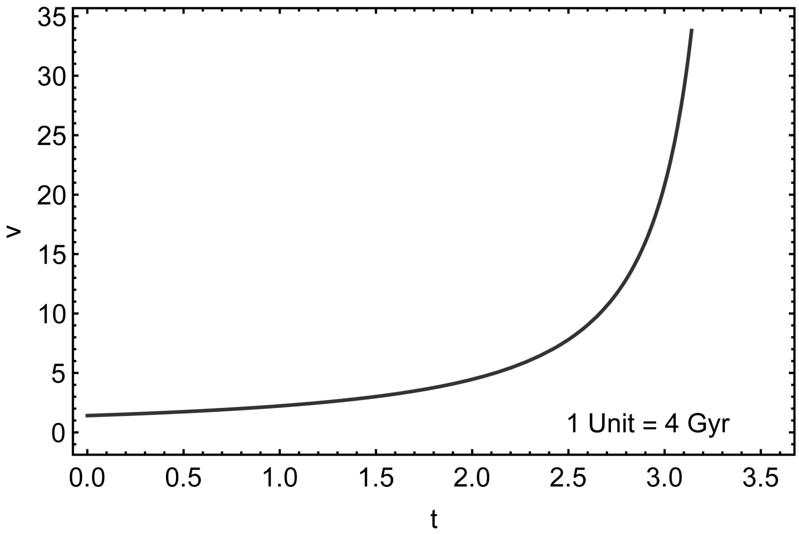

Figure 5.1: Variation of spatial volume V with time t when $l = m = 1, k = 3.45497$ showing its increasing nature throughout the evolution.

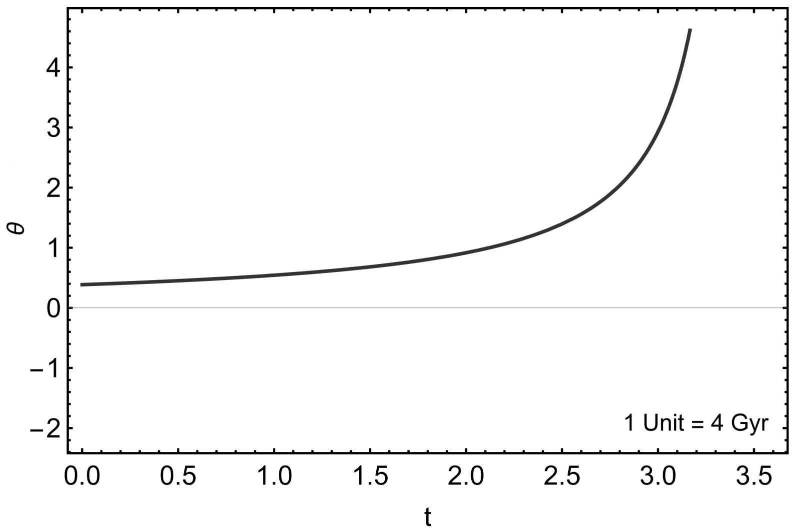

Figure 5.2: Variation of scalar expansion θ with time t when $l = m = 1, k = 3.45497$ showing its increasing nature throughout the evolution.

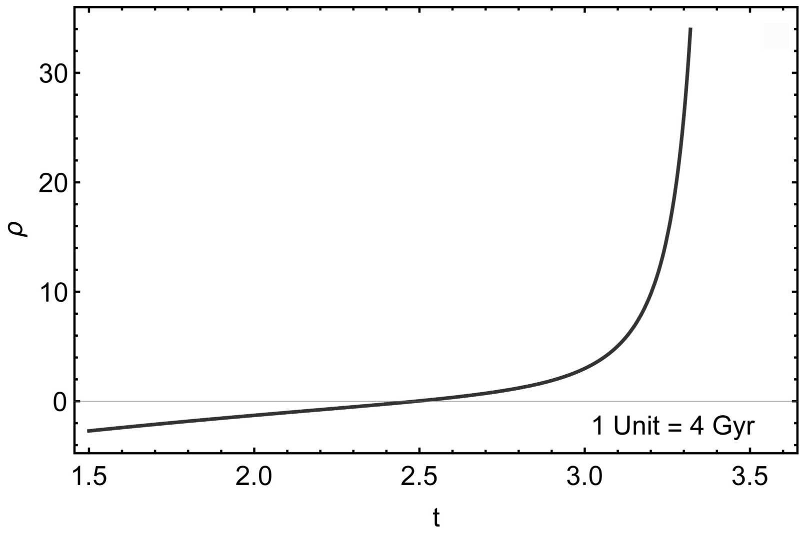

Figure 5.3: Variation of energy density ρ with time t when $l = m = 1, k = 3.45497$ showing its transition from being negative to positive during evolution.

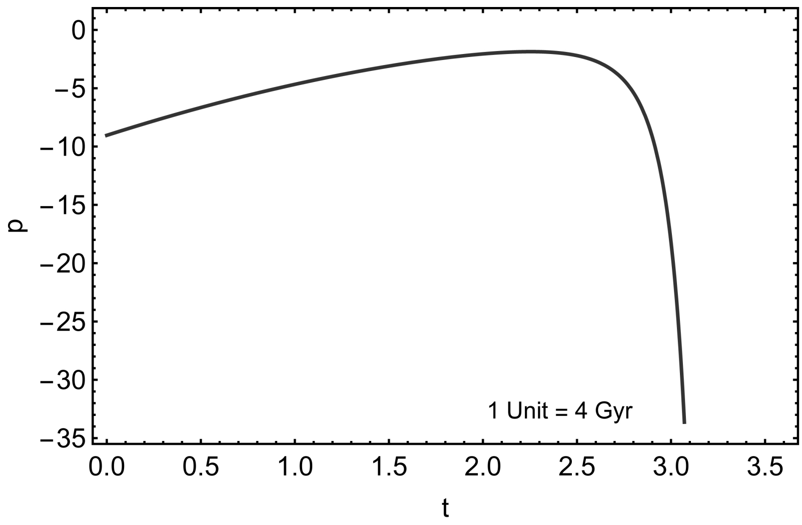

Figure 5.4: Variation of pressure p with time t when $l = m = 1, k = 3.45497$ showing its negative nature all through.

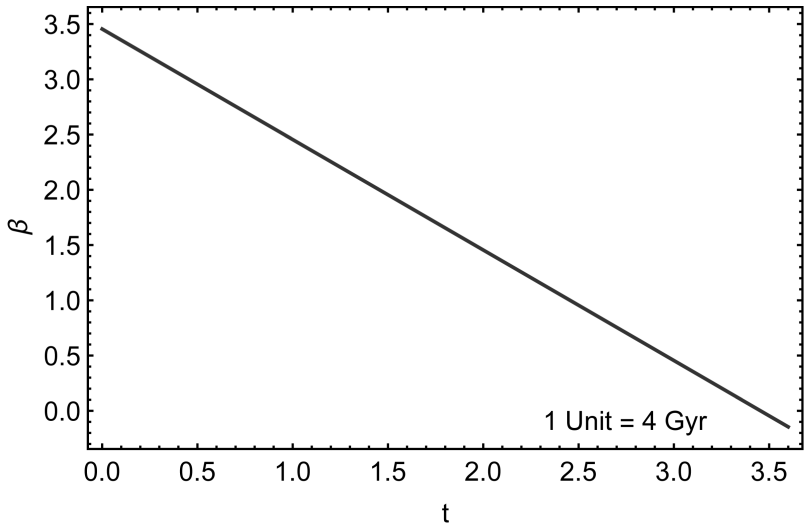

Figure 5.5: Variation of displacement vector β with time t when $d = -1, k = 3.45497$ showing its decreasing nature.

Figs. 5.1 and 5.2 can be regarded as the perfect evidences of the present spacial expansion of the universe at an expedited rate. From Figs. 5.3 and 5.4, we can witness a transition of the energy density ρ of the model universe from being negative to positive during the course of evolution whereas the pressure p of the model is negative all through. In short, the universe expands at an expedited rate with ρ and p both negative. This negative p can be regarded as the indication of the presence of DE. In this scenario, we can predict that DE in the form of vacuum energy (VE) or cosmological constant (CC) is dominating the model, as mentioned in the work of Carroll (2001a), NED is possible only if the DE is in the form of VE. When $t \to \infty$, both V and $\theta \to 0$ showing that, in the far future, the expanding phenomenon will cease, the universe will be dominated by gravity, resulting to collapse and ultimately ending at the big crunch singularity. This may be supported by the fact that DE density may decrease faster than matter leading DE to vanish at $t \to \infty$ (Peebles & Ratra 2003). Additionally, when $t \to \infty$, ρ again starts to become negative. In this condition, due to the presence of NED, according to Ijjas & Steinhardt (2019), we can assume that the model universe represents an oscillating model, each cycle evolving with a big bang and ending at a big crunch, undergoing a series of bounces. Additionally, from Fig. 5.5, it is clear that the displacement vector β is a decreasing function of time. Here, we can assert that β acts as the time dependent cosmological constant (Agarwal et al. 2011; Halford 1970; Perlmutter et al. 1999; Riess et al. 1998). Hence, it is fascinating to observe that LM itself can be regarded as a DE model.

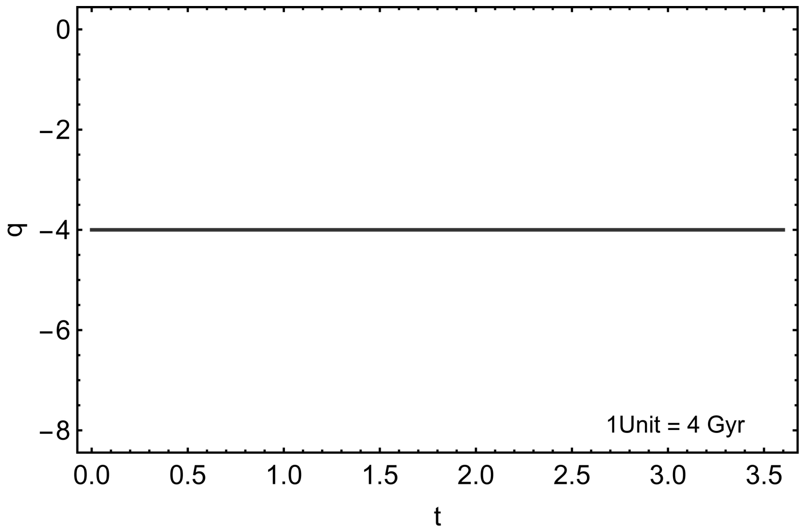

Figure 5.6: Variation of deceleration parameter q with time t when $l = m = 1, k = 3.45497$.

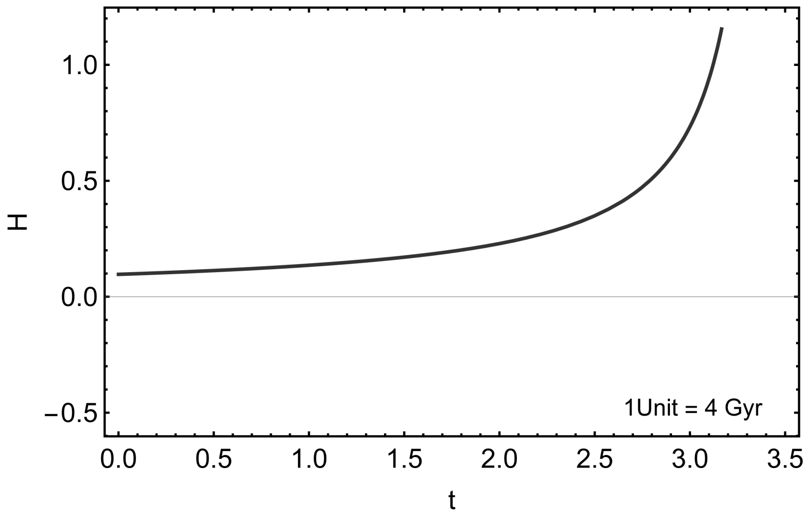

Figure 5.7: Variation of Hubble parameter H with time t when $k = 3.45497$.

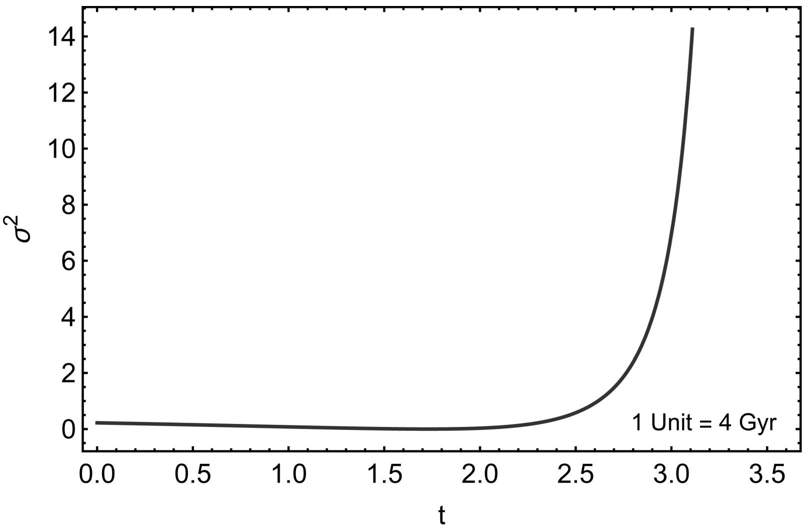

Figure 5.8: Variation of shear scalar σ^2 with time t when $l = m = 1, k = 3.45497$.

Accelerated expansion can be attained when $-1 < q < 0$ whereas $q < -1$ causes super-exponential expansion (Singh & Bishi 2017). Fig. 5.6 shows that the deceleration parameter q is a negative constant -4 indicating that the model universe undergoes super-exponential expansion. It may be noted that in a higher dimensional theory with CC, super-exponential inflation (expansion) can be attained if H increases with t (Polock 1988; Shaft & Wetterich 1985; Wenerich 1985). In our case, H is increasing as shown in Fig. 5.7. Shear scalar σ^2 provides us the rate of deformation of the matter flow within the massive cosmos (Ellis & Elst 1999). From Fig. 5.8, we can see that σ^2 evolves almost constantly, then diverges after some finite time. From Eq. (5.2.19), the anisotropic parameter $A_h = 0$. From these, we can sum up that initially, the isotropic universe expands with a slow and uniform change of shape, but after some finite time, the change becomes faster.

Lastly, with a scale of 1 Unit = 4 Gyr, the point $t = 3.45$ corresponds to 13.8 Gyr which align with 13.825 ± 0.037 Gyr, the present age of the universe estimated by the latest Planck 2018 result (Collaboration et al. 2020). At the point $t = 3.45$ and assuming $k = 3.45497$, from Eq. (5.2.17), the numeric value of the Hubble parameter is measured to be $H = 67.0691$ which is very close to $H_0 = 67.36 \pm 0.54$ kms^{-1} Mpc^{-1}, the value estimated by the latest Planck 2018 result (Collaboration et al. 2020).

5.4 Conclusions

In this chapter, with due consideration of reasonable cosmological assumptions within the limit of the present cosmological scenario, we have analysed a SS metric in 5D setting within the framework of LM. The model universe is predicted to be a DE model, dominated by VE or CC. The displacement vector also acts as the time dependent DE. The model represents an oscillating model, each cycle evolving with a big bang and ending at a big crunch, undergoing a series of bounces. Our universe undergoes super-exponential expansion. It may be noted that in a higher dimensional theory with CC, super-exponential inflation (expansion) can be attained if H increases with t (Polock 1988; Shaft & Wetterich 1985; Wenerich 1985). In our case, H is increasing. Initially, the isotropic universe expands with a slow and uniform change of shape, but after some finite time, the change becomes faster. Then, the change slows down and tends to become uniform after expanding without any deformation of the matter flow for a finite time period. Lastly, the Hubble parameter is measured to be $H = 67.0691$ which is very close to $H_0 = 67.36 \pm 0.54$ kms^{-1} Mpc^{-1}, the value estimated by the latest Planck 2018 result (Collaboration et al. 2020). We have constructed a model in LM appearing as a DE model; nonetheless, the work we have put forward is just a toy model. The model needs further deep study considering all the observational findings, which will be our upcoming work.

Chapter 6

$f(R,T)$ gravity model behaving as a dark energy source

The work presented in this chapter is published in New Astronomy 84 (2021) 101542, DOI: 10.1016/j.newast.2020.101542 (IF-1.325)

6.1 Introduction

The ambiguous dark energy (DE) has been regarded as one of the most tantalizing topics in cosmology since its profound discovery in 1998 (Riess 1998; Perlmutter 1999). It is considered to be the reason behind the late time expanding universe at an expedited rate due to its huge negative pressure with repulsive gravitation. It is uniformly permeated throughout the space and vary slowly or almost consistent with time (Carroll 2001a, 2001b; Chan 2015b; Peebles & Ratra 2003). Cosmologists all over the map have conducted a series of studies with the aim of hunting its origin and are still scrabbling for a perfect answer. Some worth mentioning such studies that have not escaped our notice in the recent years are briefly discussed below.

Singh & Kar (2019) assert that emergent D-instanton might indicate us a hint to the root of DE. A cosmological model associated with an antineutrino star is constructed by Neiser (2020) in order to search the origin of DE. Dikshit (2019) presents an explanation for DE with pure quantum mechanical method. Huterer & Shafer (2018) investigate the twenty years old history of DE and the current status. The authors in Wang et al. (2018) study the evolution of the DE using a non-parametric Bayesian approach in the light of the latest observation. Capolupo (2018) claims that vacuum condensate can provide us the origin of DE. According to Josset et al. (2017), DE is originated from the violation of energy conservation. A unified dark fluid is obtained as a source of DE by Tripathy et al. (2015). The presence of particle with imaginary energy density can lead us to the source of DE (Chan 2015a). The explanation of a physical mechanism as a source of DE is presented by Gontijo (2012). Lastly, according to the work of Alexander et al. (2010), DE evolves as a result of the condensation of fermions formed during the early evolution.

It is an obvious fact that the universe is dominated by the cryptic DE with negative pressure and positive energy density (Araujo 2005; Agrawal et al. 2018; Carroll 2001a; Law 2020; Ray et al. 2013; Singh & Singh 2019b; Straumann 2007; Wu & Yu 2005). This qualifies DE a completely irony of nature as the dominating component is also the least explored. As against the positive energy density condition, it is fascinating to see many authors introducing the concept of possibility of negative energy density (NED) with convincing arguments in support. Ijjas & Steinhardt (2019) discuss NED where models evolve with a bounce. The authors continued that there might be bounces in the future too. The discussion of negative vacuum energy (VE) density in Rainbow Gravity can be seen in the work of Wong et al. (2019). According to Nemiroff et al. (2015), under certain conditions, a repelling negative gravitational pressure with NED. Further, we can find a repelling negative phantom energy with NED. Fay (2014) claims that the universe evolves by inflation when the coupled fluid has NED in the initial epoch. An accelerating universe with NED is studied by Sawicki & Vikman (2013). Macorra & German (2004) present an explanation of energy density with negative value with equation of state parameter (EoS) $\omega < -1$. Carroll (2001a) predicts that NED is possible only if the DE is in the form of VE. Huang (1990) investigates models which evolved with NED in the infinite past. According to Parker & Fulling (1973), the introduction of quantized matter field with NED to energy momentum tensor might by pass cosmological singularity. Besides defying the energy conditions of GR, NED also disobeys the second law of thermodynamics (Hawking & Ellis 1973). However, the condition should be solely obeyed on a large scale or on a mean calculation, thereby neglecting the probable violation on a small scale or for a short duration, in relativity (Epstein et al. 1965; Fewster 2012; Ford & Roman 1996; Graham & Olum 2003; Helfer 1998a, 1998b; Pfenning & Ford 1998; Roman 1986; Visser & Barcelo 2000). Hence, in the initial epoch, if there were circumstance of defiance for a short duration measured against the present age of 13.830 ± 0.037 Gyr estimated by the latest Planck 2018 result (Collaboration et al. 2020), it will remain as an important part in the course of evolution.

In the present cosmology, authors prefer to opt alternate or modified theories of gravitation in order to precisely understand the underlying mechanism of the late time expedited expansion of the universe. One such well appreciated modified theory is the $f(R,T)$ gravity introduced by Harko et al. (2011) in which the gravitational Lagrangian is represented by an arbitrary function of the Ricci scalar R and the trace T of the energy-momentum tensor. In the past few years, this theory has captivated many cosmologists and theoretical physicists as it presents natural gravitational substitutes to DE (Chirde & Shekh 2019). Recently, Myrzakulov (2020) studies the theory and predicts the conditions to obtain expanding universe in the absence of any dark component. Sahoo et al. (2020) investigate a mixture of barotropic fluid and DE in $f(R,T)$ gravity where the model evolves from the Einstein static era and approaches ΛCDM. Pawar et al. (2019) study a modified holographic Ricci DE model in the theory obtaining a singularity free model. Zia et al. (2018) investigate $f(R,T)$ gravity discussing future singularities in DE dominated universe. In the research of Srivastava & Singh (2018), we can find a discussion of new holographic DE model in $f(R,T)$ gravity thereby obtaining ΛCDM in the late times. Fayaz et al. (2016) examine ghost DE model within the theory, predicting model behaving as phantom or quintessence like nature. The investigation of cosmological models within the theory without DE is observed in the work of Sun & Huang (2016). Mishra et al. (2016b) and Singh & Kumar (2016) study the relation of the theory with DE. Houndjo and Piattella (2012) present a reconstruction of the theory from holographic DE. The study cosmological model in $f(R,T)$ gravity obtaining DE induced cosmic acceleration is presented by Mishra et al. (2016a). Zubair et al. (2016) discuss Bianchi space-time within the theory with time-dependent deceleration parameter. Ahmed et al. (2016) investigate model in which the cosmological constant is considered as a function of T. Rao & Rao (2015) discuss a higher dimensional anisotropic DE model within the theory obtaining the EoS parameter $\omega = -1$. Jamil et al. (2012) construct models within the theory asserting that dust fluid leads to ΛCDM. Houndjo (2012) predicts a model in $f(R,T)$ gravity that transit from matter dominated to accelerating phase. From these worth appreciating studies, it won't be a wrong guess to sum up that there must be some sort of hidden correspondence between the pair of DE and $f(R,T)$ gravity. Consequently, in this work, we will try to find out if $f(R,T)$ itself behaves as a DE source.

The possibility of space-time possessing with more than 4D has fascinated many authors. In the recent years, there has been a trend of preferring higher dimensional space-time to study cosmology. Higher dimensional model, in GR, was introduced by Kaluza (1921) and Klein (1926) in an effort to unify gravity with electromagnetism. Higher dimensional model can be regarded as a tool to illustrate the late time expedited expanding paradigm (Banik & Bhuyan 2017). Investigation of higher dimensional space-time can be regarded as a task of paramount importance as the universe might have come across a higher dimensional era during the initial epoch (Singh et al. 2004). Marciano (1984) asserts that the detection of a time varying fundamental constants can possibly show us the proof for extra dimensions. According to Alvarez & Gavela (1983) and Guth (1981), extra dimensions generate huge amount of entropy which gives possible solution to flatness and horizon problem. Since we are living in a 4D space-time, the hidden extra dimension in 5D is highly likely to be associated with the invisible DM and DE (Chakraborty & Debnath

2010).

Keeping in mind the above notable works by different authors, we have analysed a spherically symmetric (SS) metric in 5D setting within the framework of $f(R,T)$ gravity with focus to predict a possible source of DE. Here, we observe the field equations with due consideration of reasonable cosmological assumptions within the limit of the present cosmological scenario. The chapter is divided into sections. After introduction, in Sect. 6.2, the field equations of $f(R,T)$ gravity theory are discussed. In Sect. 6.3, in addition to obtaining the solutions of the field equations, the cosmological parameters are also solved. In Sect. 6.4, the physical and kinematical aspects of our model are discussed with graphs. Considering everything, a closing remark is presented in Sect. 6.5.

6.2 The field equations of $f(R,T)$ gravity theory

The action of $f(R,T)$ gravity theory is given by

$$S = \int \left(\frac{1}{16\pi} f(R,T) + \mathcal{L}_m \right) \sqrt{-g} d^4x \tag{6.2.1}$$

where $g \equiv det(g_{ij})$, f is an arbitrary function of the Ricci scalar $R = R(g)$ and the trace $T = g^{ij}T_{ij}$ of the energy-momentum tensor of matter T_{ij} defined by Koivisto (2006) as

$$T_{ij} = -\frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta\, g^{ij}} \tag{6.2.2}$$

Here, the matter Lagrangian density $\mathcal{L}_m$ is assumed to rely solely on g_{ij} so that we obtain

$$T_{ij} = g_{ij}\mathcal{L}_m - 2\frac{\partial \mathcal{L}_m}{\partial g^{ij}} \tag{6.2.3}$$

The action S is varied w.r.t. the metric tensor g^{ij} and hence, the field equations of $f(R,T)$ gravity is given by

$$f_R\left(R,T\right)R_{ij} - \frac{1}{2} f\left(R,T\right) g_{ij} + \left(g_{ij}\Box - \nabla_i\nabla_j\right) f_R\left(R,T\right) = 8\pi T_{ij} - f_T\left(R,T\right)T_{ij} - f_T\left(R,T\right)\theta_{ij} \tag{6.2.4}$$

where

$$\theta_{ij} = -2\,T_{ij} + g_{ij}\mathcal{L}_m - 2g^{lk}\frac{\partial^2 \mathcal{L}_m}{\partial g^{ij}\partial g^{lk}} \tag{6.2.5}$$

Here, the subscripts appearing in f represent the partial derivative w.r.t. R or T and $\Box \equiv \nabla^i\nabla_i, \nabla_i$ being the covariant derivative.

With ρ and p respectively representing the energy density and pressure such that the five velocity u^i satisfies $u^i u_i = 1$ and $u^i \nabla_j u_i = 0$, we opt to use the perfect fluid energy-momentum tensor of the form

$$T_{ij} = (p + \rho)\, u_i u_j - p g_{ij} \tag{6.2.6}$$

We assume that $\mathcal{L}_m = -p$ so that Eq. (6.2.5) is reduced to

$$\theta_{ij} = -2T_{ij} - p g_{ij} \tag{6.2.7}$$

In general, the field equations of $f(R,T)$ gravity also rely on the physical aspect of the matter field and consequently, there exists three classes of field equations as follows

$$f(R,T) = \begin{cases} R + 2f(T) \\ f_1(R) + f_2(T) \\ f_1(R) + f_2(R) f_3(T) \end{cases} \tag{6.2.8}$$

Our study will be dealing with the class $f(R,T) = R + 2f(T)$, where $f(T)$ represents an arbitrary function so that the field equations of the modified theory is be reduced to

$$R_{ij} - \frac{1}{2} R g_{ij} = 8\pi\, T_{ij} + 2f'(T)\, T_{ij} + \left\{ 2p\, f'(T) + f(T) \right\} g_{ij} \tag{6.2.9}$$

where the prime indicates differentiation w.r.t. T and we assume that $f(T) = \lambda T$, where λ is an arbitrary constant.

6.3 Formulation of the problem and solutions

The five-dimensional SS metric is given by (Samanta & Dhal 2013)

$$ds^2 = dt^2 - e^{\mu} \left(dr^2 + r^2 d\Theta^2 + r^2 \sin^2 \Theta d\phi^2 \right) - e^{\delta} dy^2 \tag{6.3.1}$$

where $\mu = \mu(t)$ and $\delta = \delta(t)$ are cosmic scale factors.

Now, using co-moving co-ordinates, the surviving field equations are obtained as follows

$$-\frac{3}{4} \left(\dot{\mu}^2 + \dot{\mu}\dot{\delta} \right) = (8\pi + 3\lambda)\, \rho - 2p\lambda \tag{6.3.2}$$

$$\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 + \frac{\ddot{\delta}}{2} + \frac{\dot{\delta}^2}{4} + \frac{\dot{\mu}\dot{\delta}}{2} = (8\pi + 4\lambda)\, p - \lambda\rho \tag{6.3.3}$$

$$\frac{3}{2}\left(\ddot{\mu}+\dot{\mu}^2\right)=(8\pi+4\lambda)\,p-\lambda\rho \tag{6.3.4}$$

where an overhead dot indicates differentiation w.r.t. t.

From Eqs. (6.3.12) and (6.3.13), the expressions for the cosmic scale factors are obtained as

$$\mu=a-3\log\left(2\left(k-3t\right)\right) \tag{6.3.5}$$

$$\delta=b-3\log\left(2\left(k-3t\right)\right) \tag{6.3.6}$$

where a, b, k are arbitrary constants.

Now, the expressions for spatial volume v, scalar expansion θ, Hubble parameter H, deceleration parameter q, shear scalar σ^2 and anisotropic parameter A_h are obtained as follows.

$$V=e^{\frac{3a+b}{2}}\left(2\left(k-3t\right)\right)^{-6} \tag{6.3.7}$$

$$\theta=18\left(k-3t\right)^{-1} \tag{6.3.8}$$

$$H=\frac{9}{2}\left(k-3t\right)^{-1} \tag{6.3.9}$$

$$q=-1.7 \tag{6.3.10}$$

$$\sigma^2=\left(\frac{27-2\left(k-3t\right)}{18\left(k-3t\right)}\right)^2 \tag{6.3.11}$$

$$A_h=0 \tag{6.3.12}$$

From Eqs. (6.3.11) and (6.3.13), the expressions for the pressure p and the energy density ρ of the model universe are respectively obtained as

$$p=\frac{243\lambda-324\left(8\pi+3\lambda\right)}{4\left(k-3t\right)^2\left(-5\lambda^2-32\pi^2-28\pi\lambda\right)} \tag{6.3.13}$$

$$\rho=\frac{(8\pi+4\lambda)}{\lambda}\left(\frac{243\lambda-324\left(8\pi+3\lambda\right)}{4\left(k-3t\right)^2\left(-5\lambda^2-32\pi^2-28\pi\lambda\right)}\right)-\frac{162}{\lambda\left(k-3t\right)^2} \tag{6.3.14}$$

The expression for the scalar curvature R is obtained as

$$R = \frac{513}{(k-3t)^2} \tag{6.3.15}$$

6.4 Discussions

For convenience sake and to obtain realistic results, specific values of the arbitrary constants involved are chosen i.e., $a = b = 1, k = 15$ and $\lambda = -5.06911$ and -12.5856. The graphs of the cosmological parameters w.r.t. cosmic time t are presented with the detailed discussion in view of the latest observations.

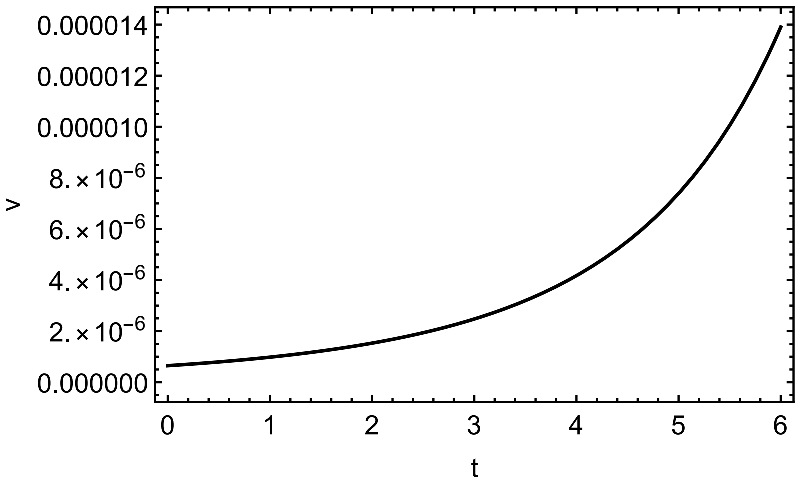

Figure 6.1: Variation of the spatial volume V with t when $a = b = 1, k = 15$.

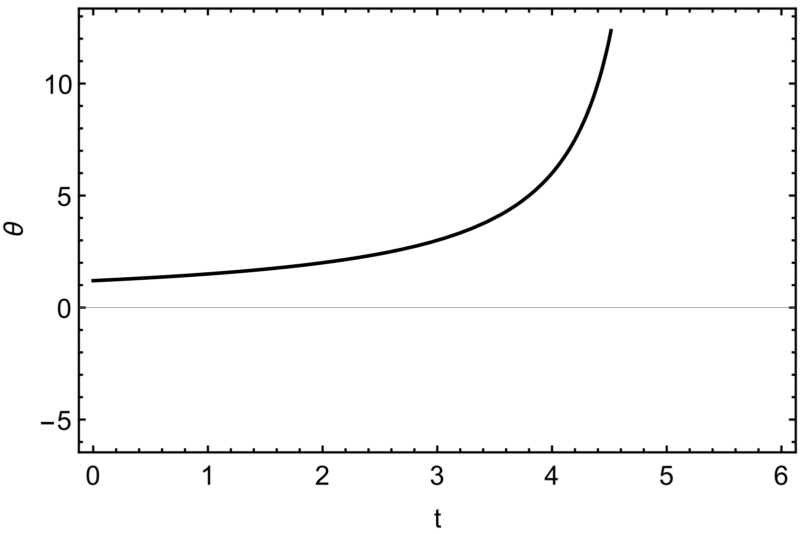

Figure 6.2: Variation of the expansion scalar θ with t when $a = b = 1, k = 15$.

Figs. 6.1 and 6.2 can be regarded as the perfect evidences for the present spatial expansion at an expedited rate. When $t \to 0$, V and other related parameters are constants ($\neq 0$), implying that the model universe doesn't evolve from an initial singularity.

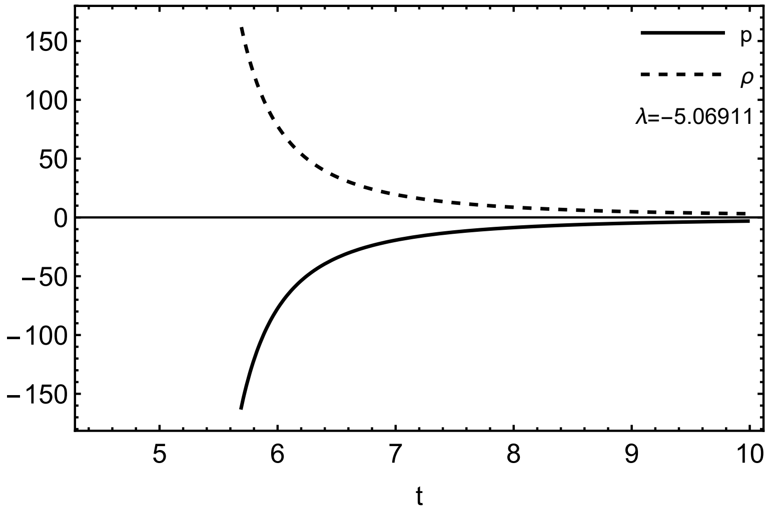

Figure 6.3: Variation of the pressure p and energy density ρ with t when $\lambda = -5.06911$.

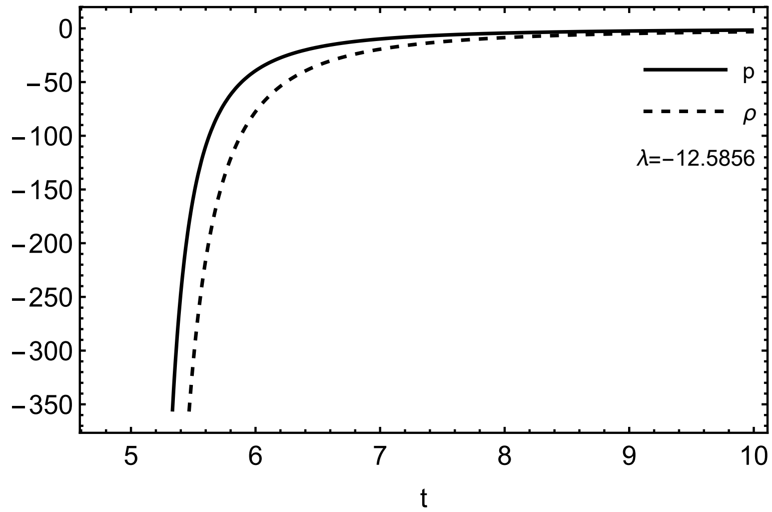

Figure 6.4: Variation of the pressure p and energy density ρ with t when $\lambda = -12.5856$.

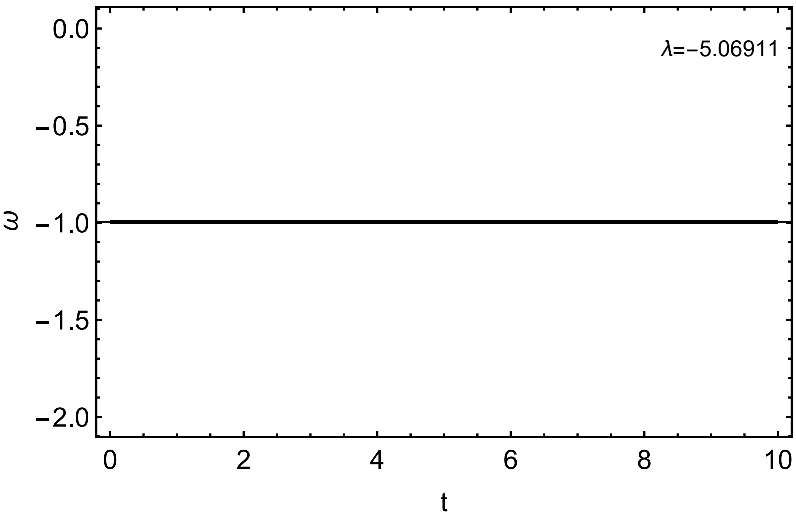

Figure 6.5: Variation of the DE EoS ω with t when $a = b = 1, k = 15, \lambda = -5.06911$.

Fig. 6.3 shows the variation of the pressure p and energy density ρ when $a = b = 1, k = 15, \lambda = -5.06911$. From the graph, it is obvious that the model is experiencing accelerated expansion with negative p and positive ρ. Here, the model evolves with a large ρ and it converges to become constant at late times. This phenomenon is a clear indication of the presence of DE as the present cosmology believes that the late time accelerating universe is due to the dominant and slowly varying or constant DE with negative pressure and positive energy density (Araujo 2005; Agrawal et al. 2018; Carroll 2001a; Law 2020; Ray et al. 2013; Singh & Singh 2019b; Straumann 2007; Wu & Yu 2005). In order to predict the nature, the graph of $\omega = \frac{p}{\rho}$ which is the DE EoS parameter is plotted in Fig. 6.5 which shows that $\omega = -1$. Hence, we can sum up that the $f(R, T)$ gravity model we have constructed turns out to be a DE model, DE in the form of VE or the CC. Fig. 6.4 shows the variation of the pressure p and energy density ρ when $a = b = 1, k = 15, \lambda = -12.5856$. In this case, the model undergoes expansion at an expedited rate with p and ρ both negative. This negative p can be regarded as the indication of the presence of DE. In this scenario too, we can predict that DE in the form of VE is dominating the model, as predicted by Carroll (2001a), NED is possible only if the DE is in the form of VE. Hence, in both the cases, it is fascinating to see that the constructed $f(R, T)$ gravity theory model behaves as a DE (vacuum energy) model. We have not considered the case when $\lambda > 0$ as it yields positive pressure which is not reliable in the present scenario.

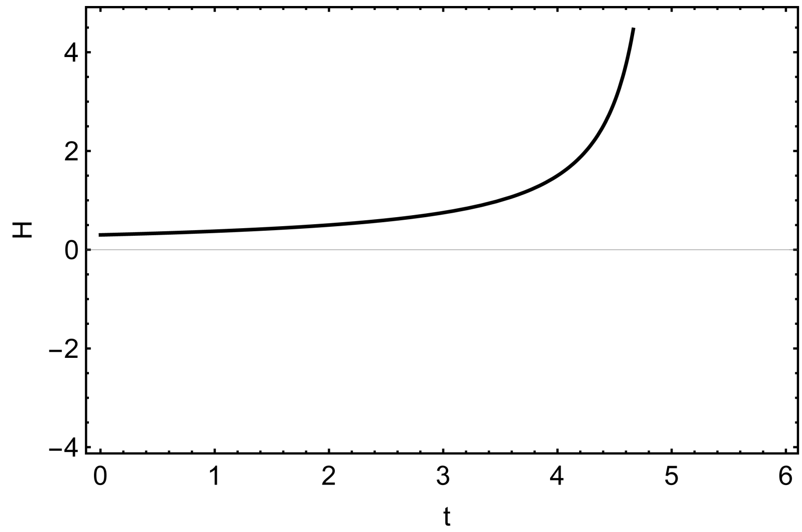

Figure 6.6: Variation of the Hubble parameter H with t when $k = 15$.

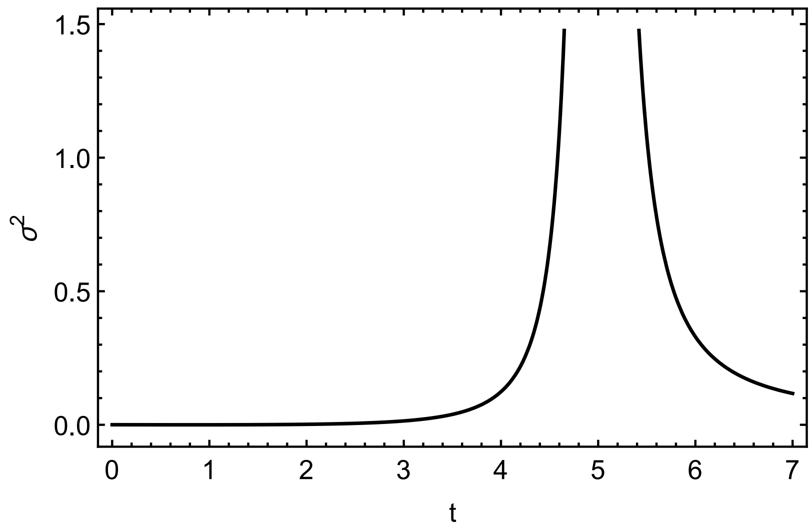

Figure 6.7: Variation of the scalar scalar σ^2 with t when $k = 15$.

Fig. 6.6 shows that the Hubble parameter H appears to remain almost constant in the early evolution so that our universe was in the inflationary epoch experiencing rapid exponential expansion (Crevecoeur 2016). Accelerated expansion can be attained when $-1 < q < 0$ whereas $q < -1$ causes super-exponential expansion (Singh & Bishi 2017). Eq. (6.3.10) shows that $q = -1.7$ indicating that the model universe undergoes super-exponential expansion. It may be noted that in a higher dimensional theory with CC or VE, super-exponential inflation (expansion) can be attained if H increases with t (Polock 1988; Shaft & Wetterich 1985; Wenerich 1985). In our case, H increases after the initial epoch as shown in Fig. 6.6. Shear scalar σ^2 provides us the rate of deformation of the matter flow within the massive cosmos (Ellis & Elst 1999). From Fig. 6.7, we can see that

σ^2 evolves constantly, then diverges after some finite time and again converges to become constant after vanishing for a finite period. From Eq. (6.3.21), the anisotropic parameter $A_h = 0$. From these, we can sum up that initially, the isotropic universe expands with a slow and uniform change of shape, but after some finite time, the change becomes faster. Then, the change slows down and tends to become uniform after expanding without any deformation of the matter flow for a finite time period.

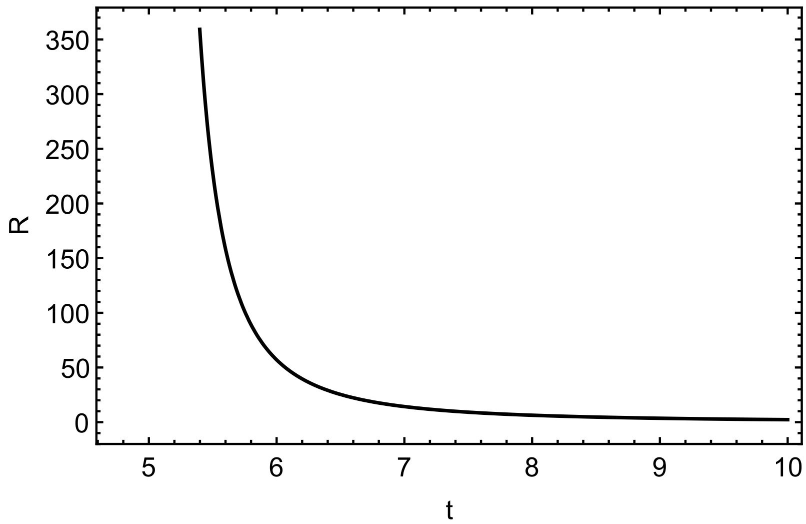

Figure 6.8: Variation of the scalar curvature R with t when $k = 15$.

Fig. 6.8 shows the decreasing nature of the scalar curvature R with cosmic time t. Similar observation can also be seen in the recent studies (Pavlovic & Sossich 2017; Pashitskii & Pentegov 2016). It tends to become constant in the future. At $t = 13.8$ Gyr which align with 13.830 ± 0.037 Gyr, the approximate present age of the universe estimated by the latest Planck 2018 result (Collaboration et al. 2020), the scalar curvature is obtained to approach a constant $R = 0.72$. $R = 0$ corresponds to an exactly flat expanding universe (Bevelacqua 2006; Gueorguiev & Maeder 2020; Kleban & Senatore 2016). However, in the recent years, arguments against the notion of exactly flat universe have been put forwarded by many authors (Javed et al. 2020; Khodadi et al. 2015; Nashed & Hanafy 2014; Valentino et al. 2020). In the present scenario, the universe is assumed to be close to or nearly flat, but not exactly flat (Adler & Overduin 2005; Levin & Freese 1994; Nashed & Hanafy 2014). Additionally, the latest Planck 2018 results (Collaboration et al. 2020) estimating the value of overall density parameter Ω ranging close to unity can also regarded as an evidence for nearly flat universe, as for an exactly flat universe, $\Omega = 1$ (Holman 2018; Khodadi et al. 2015; Levin & Freese 1994). Hence, our model obtaining a small and constant $R = 0.73$ is justified. Kim et al. (2002) and Tiwari (2016), in their studies, assert that R is constant for de-Sitter phase. So, the reason for R approaching a constant can be regarded as an indication for the model approaching the de-Sitter phase dominated by VE or CC in the finite time future avoiding singularity. According to Falls et al. (2018), accelerated expansion will lead R to approach a nearly constant value so that the universe behaves in the same manner as a de-Sitter universe in the future. Many other authors have also asserted that the expanding universe will end at the de-Sitter phase dominated by VE, avoiding singularity (Basilakos et al. 2018; Carneiro 2006; Dymnikova 2019; Dyson et al. 2002; Krauss and Starkman 2000; Markkanen 2018; Nojiri and Odintsov (2004); Sakharov 1966; Starobinsky 2000; Zilioti et al. 2018).

6.5 Conclusions

In this chapter, within the framework of $f(R,T)$ gravity theory, we have analysed a SS space-time in 5D setting. The variation of the pressure p and energy density ρ with cosmic time t are analysed when $\lambda = -5.06911$ and -12.5856. In both the cases, it is fascinating to see that our $f(R,T)$ gravity theory model behaves as a DE (vacuum energy) model. The model is isotropic and free from an initial singularity. The model expands with a slow and uniform change of shape, but after some finite time, the change becomes faster. Then, the change slows down and tends to become uniform after expanding without any deformation of the matter flow for a finite time period. The scalar curvature R is decreasing with time which is consistent with the recent studies. The model is predicted to approach the de-Sitter phase dominated by VE or CC in the finite time future avoiding singularity. We have constructed a model where $f(R,T)$ gravity theory itself behaves as a DE (VE) model; nonetheless, the work we have put forward is just a toy model. The model needs further deep study considering all the observational findings, which will be our upcoming work.

Chapter 7

Scale covariant theory as a dark energy model

The work presented in this chapter is under review at International Journal of Geometric Methods in Modern Physics (IF-1.874)

7.1 Introduction

Since its profound discovery in 1998, dark energy (DE) (Riess et al. 1998; Perlmutter et al. 1999) has become a topic of paramount importance in the field cosmology. This ambiguous dark entity is the leading factor in the present expansion of the universe at an increasing rate. In fact, DE has earned the reputation of being one of the most mysterious components of the universe. Cosmologists all over the map have put tremendous scientific efforts into figuring out its root and enigmatic nature, but are still in the quest for the right answers. Few of such noteworthy recent efforts are mentioned below.

An $f(R,T)$ gravity model is proposed as a DE source in the study of Singh & Singh (2021b). They further study a 5D cosmological model to find a source of DE (Singh & Singh 2021a). A cosmological model is proposed by Neiser (2020) so as to find the root of DE. Paul & Sengupta (2020) discuss the generalized phenomenological models of DE. Capolupo (2018) predicts that vacuum condensate may lead the way to the source. Wang et al. (2018) present the evolution of this dark entity considering the recent findings. Collaboration *et al.* (2016) present the future of DE beyond the bound of cosmological aspects. Chan (2015a) claims that particles with imaginary energy density might give us a clue to the origin of the dark component. Lastly, a physical mechanism is presented as one of the origins by Gontijo (2012).

Cosmologists all over the globe have performed numerous theoretical and practical attempts to obtain hints as to exactly predict the hidden physics behind the late time expansion. Two well appreciated theoretical approaches have been brought to light to serve the purpose of illustrating the phenomenon. First, viable candidates of DE are developed. Second, modifying ETG. Aside from these two, several other cosmologists have subsequently proposed additional intriguing theories that explain the expanding phenomena satisfactorily. Narain & Li (2018) assert that an Ultraviolet Complete Theory causes the expanding phenomenon. Berezhiani (2017) explains the expansion by dark matter-baryon interactions in the absence of DE. Lastly, Gorji (2016) explains the mystic phenomenon by the infra red corrections.

To classify DE to specific categories, the equation of state (EoS) parameter ω is considered as a good choice. $\omega = -1$ represents the cosmological constant or vacuum energy. Phantom energy has $\omega < -1$ whereas the range $-1 < \omega < \frac{-1}{3}$ signifies quintessence. The latest Planck 2018 result (Collaboration et al. 2020) predicts its possible bound to be $\omega = -1.03 \pm 0.03$, which is an indication that the form of DE in the present universe is highly likely to be phantom energy.

Over the years, many cosmologists have successfully introduced many well appreciated optimized modifications of ETG which align with the current cosmological trends quite convincingly (Weyl 1918; Nojiri and Odintsov 2014; Harko et al. 2011; Chiba et al. 2007; Brans & Dicke 1961; Scheibe 1952). Cosmologists and theoretical physicists prefer to opt such modified theories to study the late time accelerating phenomenon as Einstein's General Relativity doesn't provide the accurate explanation of gravity (Sbisa 2014). One of such modifications which has not escaped our attention is the scale covariant theory (SCT) introduced by Canuto et al. (1977a) and Canuto et al. (1977b). They developed the theory by applying scale transformation in order to calculate space-time distances Canuto et al. (1977a). According to them, the generalized Einstein's field equations (EFE) are invariant under scale transformation and they successfully investigated many astrophysical tests with the theory Canuto et al. (1977b). In SCT, the EFE are valid in gravitational units, on the other hand, atomic units are used for physical quantities. The metric tensors associated with these two unit systems are connected by the scale transformation $\overline{g_{ij}} = \varphi^2(x^k)g_{ij}$, where the bar and unbar respectively represent the gravitational units and atomic quantities. φ is a gauge function satisfying $0 < \varphi < \infty$, without possessing any wave equation. Using this transformation, Canuto et al. (1977a, 1977b) transform the usual EFE into

$$R_{ij} - \frac{1}{2}g_{ij}R + f_{ij}(\varphi) = -8\pi G(\varphi)T_{ij} + \Lambda(\varphi)\, g_{ij} \tag{7.1.1}$$

such that

$$\varphi^2 f_{ij} = 2\varphi\varphi_{i;j} - 4\varphi_{,i}\varphi_{,j} - g_{ij}\left(\varphi\varphi^{,k}_{;k} - \varphi^{,k}\varphi_{,k}\right) \tag{7.1.2}$$

where all the symbols have their usual meanings.

According to Katore et al. (2014), SCT is one of the best alternatives to ETG. This theory permits the variation of the gravitational constant G (Wesson 1980; Will 1984). The ambiguous DE and the mysterious expanding phenomenon have been successfully studied by many authors within the framework of SCT. In the recent study by Singh et al. (2020), it is asserted that SCT might be one of the probable contributors to the late time accelerated expanding phenomenon. Zeyauddin et al. (2020) present a cosmological model that decelerates during the initial phase and accelerates during the present evolution. Ram et al. (2015) present a forever expanding DE dominated universe which tends to de-sitter universe in the future. Naidu et al. (2015) present an DE model with early inflation and late time acceleration. Katore et al. (2014) investigates three Bianchi type space-times involving magnetized anisotropic DE. Singh & Sharma (2014b) investigate a Bianchi type-II space-time with variable ω. Zeyauddin & Saha (2013) study an endlessly expanding and shearing model with an initial singularity. Reddy et al. (2012) construct an expanding DE model, which doesn't evolve from a singularity in the initial epoch. In the present scenario, SCT paired with DE is considered to align with cosmological observations.

Cosmological models based on higher dimension have become a preferred choice among many authors. The concept of higher dimension in cosmology was put forward by Kaluza (1921) and (Klein 1926). Banik & Bhuyan (2017) assert that models based on higher dimension can be considered as means to explain the expanding phenomenon of the universe. An explanation in support of the extra dimension can be seen in the work of Marciano (1984). Most probably, the unknown fifth dimension might correspond to the two ambiguous and unseen dark entities - DE and DM (Chakraborty & Debnath 2010). Many well known authors have put forward noteworthy discussions on higher dimension during the past few decades (Astefanesei et al. 2020; Ghaffarnejad et al. 2020; Montefalcone et al. 2020; Saha & Ghose 2020; Demirel 2019; Bahrehbakhsh 2018; Shinkai & Torii 2015; Samanta et al. 2014; Oli 2014; Singh & Desikan 1997).

Taking into consideration the above noteworthy related studies, we try to find out if SCT itself can behave as a DE model, within the framework of a 5D spherically symmetric (SS) space-time. In this chapter, we present an in-depth discussion on every cosmological parameter obtained. We estimate the variation of gravitational constant G. After the introduction, in Sect. 7.2, we present the formulation of the problem with solutions to the parameters. In Sect. 7.3, the solutions are discussed with graphical representations. Lastly, to sum up the observations, a concluding note is provided in Sect. 7.4.

7.2 Problem formulation with solutions

We consider a SS metric in 5D of following the form (Samanta & Dhal 2013)

$$ds^2 = dt^2 - e^{\mu}\left(dr^2 + r^2 d\Theta^2 + r^2 \sin^2\Theta d\phi^2\right) - e^{\delta} dy^2 \tag{7.2.1}$$

where $\mu = \mu(t)$ and $\delta = \delta(t)$ are cosmic scale factors.

The energy-momentum tensor is given by

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij} \tag{7.2.2}$$

where p and ρ represent pressure and energy density, whereas u^i satisfies $u^i u_i = 1$, in co-moving co-ordinate system.

Now, the surviving field equations from Eqs. (7.1.1) and (7.1.2) for the spherically symmetric metric are obtained as

$$\frac{3}{4}\left(\dot{\mu}^2 + \dot{\mu}\dot{\delta}\right) - \frac{\dot{\varphi}}{\varphi} + 3\left(\frac{\dot{\varphi}}{\varphi}\right)^2 + \frac{\dot{\varphi}}{\varphi}\left(\frac{3\dot{\mu} + \dot{\delta}}{2}\right) = 8\pi G(\varphi)\rho \tag{7.2.3}$$

$$\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 + \frac{\ddot{\delta}}{2} + \frac{\dot{\delta}^2}{4} + \frac{\dot{\mu}\dot{\delta}}{2} + \frac{\ddot{\varphi}}{\varphi} + \frac{\dot{\varphi}}{\varphi}\left(\frac{\dot{\mu} + \dot{\delta}}{2}\right) - \left(\frac{\dot{\varphi}}{\varphi}\right)^2 = -8\pi G(\varphi)p \tag{7.2.4}$$

$$\frac{3}{2}\left(\ddot{\mu} + \dot{\mu}^2\right) + \frac{\ddot{\varphi}}{\varphi} - \left(\frac{\dot{\varphi}}{\varphi}\right)^2 + \frac{\dot{\varphi}}{\varphi}\left(\frac{3\dot{\mu} - \dot{\delta}}{2}\right) = -8\pi G(\varphi)p \tag{7.2.5}$$

where a superscribed dot represents derivative w.r.t. t.

From Eqs. (7.2.6) and (7.2.7), we have

$$\mu = x - \frac{2}{3}\log(z - t) \tag{7.2.6}$$

$$\delta = y - \frac{2}{3}\log(z - t) \tag{7.2.7}$$

where x, y and z are arbitrary constants.

The expressions for the parameters are obtained as follows.

Spatial volume:

$$V = e^{\frac{3\mu+\delta}{2}} = e^{\frac{3x+y}{2}} (z-t)^{-\frac{4}{3}} \tag{7.2.8}$$

Scale factor:

$$a(t) = V^{\frac{1}{4}} = e^{\frac{3x+y}{8}} (z-t)^{-\frac{1}{3}} \tag{7.2.9}$$

Scalar expansion:

$$\theta = u^i_{;j} = \frac{3\dot{\mu}}{2} + \frac{\dot{\delta}}{2} = \frac{4}{3} (z-t)^{-1} \tag{7.2.10}$$

Hubble parameter:

$$H = \frac{\theta}{4} = \frac{1}{3} (z-t)^{-1} \tag{7.2.11}$$

The most recent Planck 2018 results (Collaboration et al. 2020) estimates the value of the Hubble parameter to be $H = 67.4 \pm 0.5 \text{kms}^{-1}\ \text{Mpc}^{-1}$.

With $\Delta H_i = H_i - H, (i = 1, 2, 3, 4)$ representing the directional Hubble parameters, the anisotropic parameter A_h is given by

$$A_h = \frac{1}{4} \sum_{i=1}^{4} \left(\frac{\Delta H_i}{H} \right)^2 = 0 \tag{7.2.12}$$

Shear Scalar:

$$\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij} = \frac{1}{2} \sum_{i=1}^{4} \left(H_i^2 - 4H \right) = \frac{2}{9} \left(1 - \frac{1}{z-t} \right)^2 \tag{7.2.13}$$

In our study, φ is time dependent and we consider the well appreciated relation $\varphi = m\,(a(t))^n$ (Zeyauddin et al. 2020; Zeyauddin & Saha 2013; Singh & Sharma 2014b), where m and n are arbitrary constants so that from Eq. (7.2.11), we have

$$\varphi = m e^{\frac{n(3x+y)}{8}} (z-t)^{-\frac{n}{3}} \tag{7.2.14}$$

Using Eqs. (7.2.8), (7.2.9) and (7.2.16) in Eqs. (7.2.5) and (7.2.7), we have

$$\rho = \frac{6 + 3n^2 + n\,(4 - 3(z-t))}{72\pi G(\varphi)(z-t)^2} \tag{7.2.15}$$

$$p = -\frac{5(3+n)}{72\pi G(\varphi)(z-t)^2} \tag{7.2.16}$$

Dirac (1937) asserts that G decreases with cosmic time t. The study on the accelerating universe with decreasing G can be seen in the studies of Hova (2020) and Tiwari et al. (2010) whereas increasing G is presented in the studies of Oli (2014), Massa (1995) and Levit (1980). Models with variable G are also investigated by many other authos (Sahni & Shtanov 2014; Kordi 2009; Srivastava 2008; Debnath & Paul 2006; Grigorian & Saharian

1990; Narlikar 1983). Beesham (1986) assumes G to be $G \propto t^{\alpha}$, where α is a constant, Sistero (1991) considers $G \propto (a(t))^{\alpha}$ whereas Ram et al. (2009) assert $G = \varepsilon t$ where ε is a proportionality constant. In our study, we consider G in the following form as suggested by Dirac (1937).

$$G = ct^{-1} \tag{7.2.17}$$

where c is a proportionality constant.

Using Eq. (7.2.19) in Eqs. (7.2.17) and (7.2.18), we obtain the expressions for ρ and p as functions of t only as follows

$$\rho = \frac{6 + 3n^2 + n\,(4 - 3(z - t))}{72\pi ct^{-1}(z - t)^2} \tag{7.2.18}$$

$$p = -\frac{5(3 + n)}{72\pi ct^{-1}(z - t)^2} \tag{7.2.19}$$

7.3 Discussion

In this section, for simplicity purposes and reasonable outcomes, we opt to choose $c = m = x = y = 1, n = 0.2$ and $z = 13.8049$ and the parameters are plotted with respect to the cosmic time t.

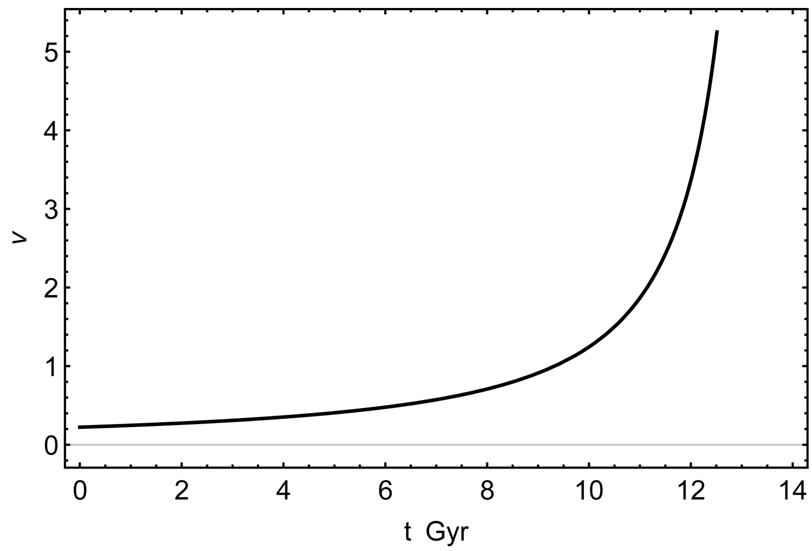

Figure 7.1: Volume V increasing with t when $x = y = 1, z = 13.8049$.

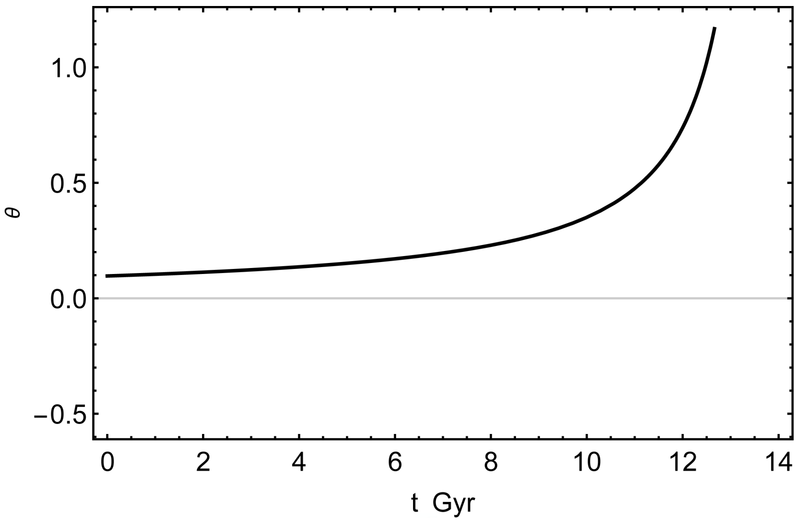

Figure 7.2: Expansion scalar θ increasing with t when $z = 13.8049$.

It can be seen from Figs. 7.1 and 7.2 respectively that V and θ diverge with t, which favour the expansion of the universe at an increasing rate. Above all, V is constant ($\neq 0$) at $t = 0$. Hence, it won't be wrong to conclude that the model doesn't evolve from a singularity.

From Fig. 7.3, it can be seen that throughout the evolution, the graph of p lies within the negative plane. So, the model undergoes accelerated spatial expansion with negative pressure. This is the indication that our SCT model is in fact a DE dominated model. From Fig. 7.4, we can witness the increasing nature of ρ during the course of evolution. To precisely understand the nature of the DE, in Fig. 7.5, we have plotted the graph of $\omega = \frac{p}{\rho}$ which is the DE EoS parameter showing that $\omega < -1$. From these, we can conclude that the model is a phantom energy dominated model which agrees with the present observation. It may be noted that phantom energy is also characterised by the increasing positive energy density with cosmic time (Ram et al. 2009; Baushev 2010; Caldwell et al. 2003). Above all, in Fig. 7.5, we can observe that during the course of evolution, ω tends very close to -1 in the future which agrees with observations of Amirhashchi (2017) and Carroll et al. (2003) asserting that the value of ω in phantom energy model should reduce to -1 in the far future so that the dominating DE will be transformed to VE or the CC. This will make sure the model universe bypass the future finite time big rip singularity thereby, ultimately, leading to the de-Sitter phase. The concept of de-Sitter phase avoiding future singularity is also presented by many other authors (Dymnikova 2019; Sakharov 1966; Gliner 1966).

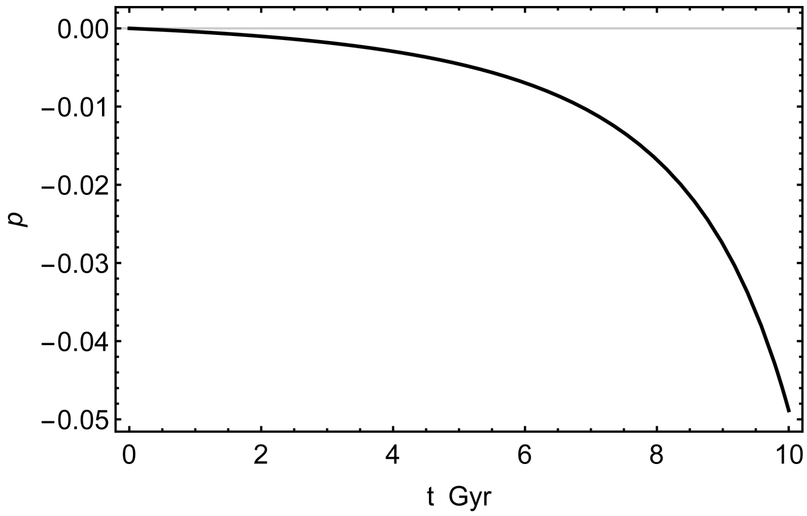

Figure 7.3: Pressure p ranging in the negative plane when $c = 1, n = 0.2, z = 13.8049$.

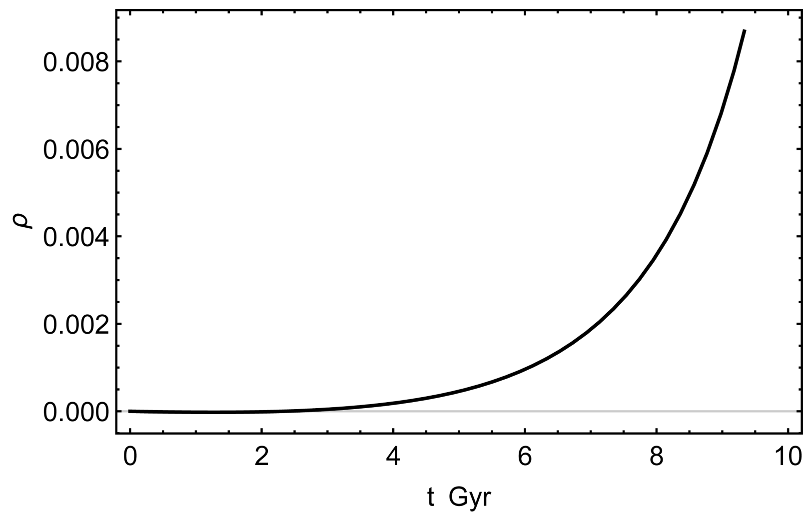

Figure 7.4: Energy density ρ increasing with t when $c = 1, n = 0.2, z = 13.8049$.

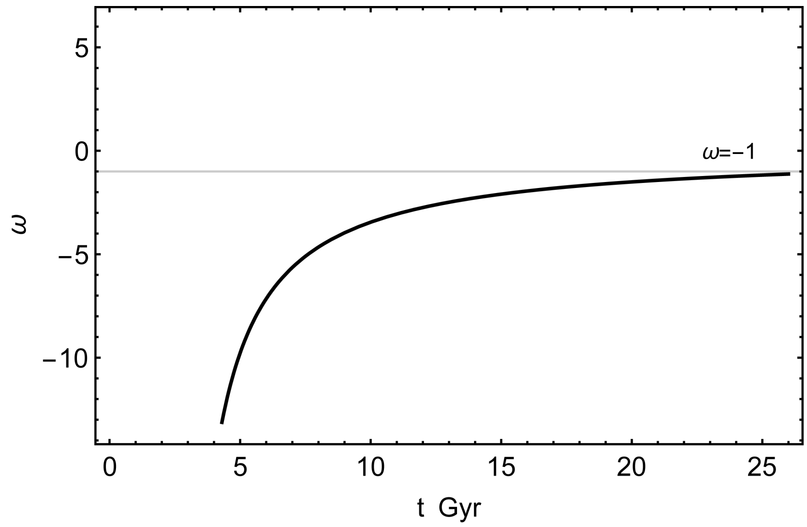

Figure 7.5: EoS parameter ω with t when $n = 0.2$ showing that it tends very close to -1 in the future.

Fig. 7.6 shows that σ^2 decreases slowly in the early state of evolution and then, it starts to decrease with a greater extent and finally, it tends to diverge. σ^2 estimates the rate of distortion of the matter flow (Ellis & Elst 1999). So, we can conclude that the model expands with a steady and consistent change of structure in the early evolution and then, the change become more steady and consistent and finally, the change tends to become faster at late times. From Fig. 7.7, we can see the decreasing nature of G which is supported by the observation of Dirac (1937). Similar plots of decreasing G with accelerating universe can also be seen in the investigation of Hossain et al. (2017) and Tiwari et al. (2010). From Eq. (7.2.14), $A_h = 0$ so that the constructed model in isotropic throughout.

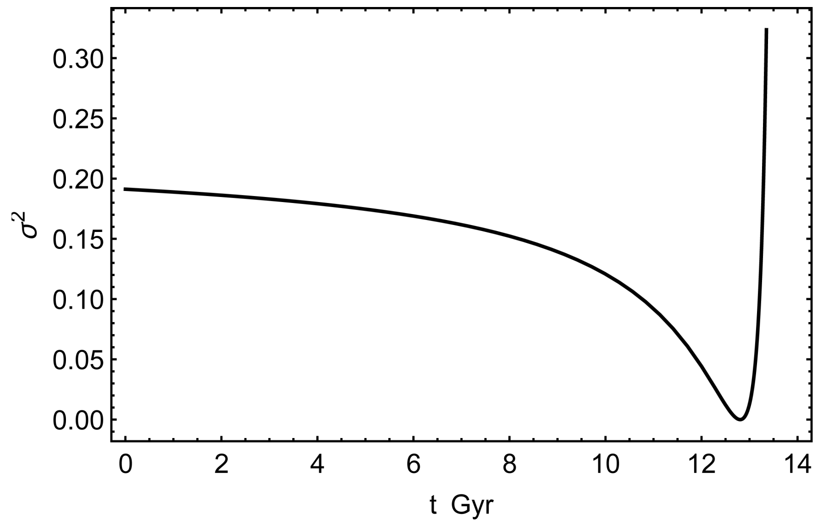

Figure 7.6: Shear scalar σ^2 with t when $z = 13.8049$.

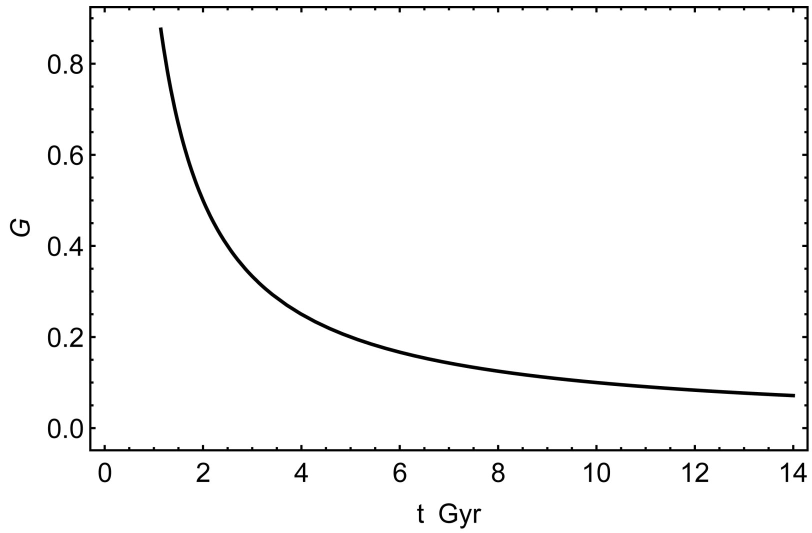

Figure 7.7: Gravitational constant G decreasing with t when $c = 1$.

$\dot{G}/G$ **in yr**$^{-1}$	**Reference**
-7×10^{-11}	Kordi (2009)
-7.4×10^{-11}, 10^{-14}—10^{-11}	Steinhardt & Wesley (2010)
$-10^{-11} \leq \frac{\dot{G}}{G} \leq 0$	Gaztanaga et al. (2001)
$-2.3 \times 10^{-11} < \frac{\dot{G}}{G} < +0.3 \times 10^{-11}$	Loren-Aguilar et al. (2003)
-5×10^{-11}, -1×10^{-11}, -1×10^{-12}	Althaus (2011)
$-2.50 \times 10^{-10} \leq \lvert\frac{\dot{G}}{G}\rvert < 0$	Benvenuto et al. (2004)
$\lvert\frac{\dot{G}}{G}\rvert \leq 4.1 \times 10^{-10}$	Biesiada & Malec (2004)
$\lvert\frac{\dot{G}}{G}\rvert = 2 \times 10^{-10}$	Sahoo & Singh (2003)
$0.7^{+3.8}_{-4.3} \times 10^{-12}$	Alvey et al. (2020)
$\lvert\frac{\dot{G}}{G}\rvert \leq 3 \times 10^{-12}, 1.5 \times 10^{-12}$	Zhao et al. (2018)
$(-0.7 \pm 3.8) \times 10^{-13}$	Hofmann et al. (2010)
-2.8×10^{-13}	Li (2018)
$(7.1 \pm 7.6) \times 10^{-14}$	Hofmann & Muller (2018)

Table 7.1: Estimated values of variation of G during the past few decades.

Since the prediction of the decreasing nature of the gravitational constant G with time (Dirac 1937), a number of authors have put forward different values of the variation of G i.e., $\frac{\dot{G}}{G}$ with convincing arguments and evidences in support, some of which are presented in Table 7.1. Banerjee & Pavon (2001) predict that $\frac{\dot{G}}{G}$ is safely below 4×10^{-10} yr^{-1}. G changes with a fraction of $\frac{\dot{G}}{G}$ per year (Kordi 2009). In our study, with the current age of the universe to be 13.8 Gyr, i.e. 13.8×10^{9} years (Collaboration et al. 2020), the variation of G is measured to be $\frac{\dot{G}}{G} = -7.2 \times 10^{-11}$ yr^{-1}. Additionally, at t=13.8 Gyr, from Eq. (7.2.13), we obtain $H = 68$ which align with the result of the most recent Planck 2018 results (Collaboration et al. 2020).

7.4 Conclusions

In this chapter, we have studied SCT within the framework of a SS space-time in 5D. For simplicity purposes and reasonable outcomes, we opt to choose $c = m = x = y = 1, n = 0.2$ and $z = 13.8049$. The universe is isotropic. The model behaves as a phantom energy dominated model, which doesn't evolve from a singularity and tends to the de-Sitter phase avoiding finite time future singularity (big rip). During the early evolution, the universe expands with a steady and consistent change of structure and then, the change become more steady and consistent and finally, the change tends to become faster at late times. The value of G is predicted to be decreasing with a variation of $\frac{\dot{G}}{G} = -7.2 \times 10^{-11}$ yr^{-1}. At t=13.8 Gyr, we obtain $H = 68$ which align with the result of the most recent Planck 2018 results. In the study, an SCT model, which acts as a phantom energy model is presented. In other words, the SCT model acts as a DE source. However, this constructed model is a toy model which requires more in depth analysis taking into account all the latest cosmological findings, which we are planning to work on.

Chapter 8

Dark energy on higher dimensional spherically symmetric Brans-Dicke universe

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8.1 Introduction

Topics on the accelerated expansion of the universe have attracted wide attention from many theoretical physicists and cosmologists around the world energizing them for further investigations and many clear and convincing evidence have been produced in support. This accelerated expansion is explained by the so-called dark energy (DE) (Riess et al. 1998; Perlmutter et al. 1999), a completely mysterious form of energy with an exotic property of negative pressure which generates a negative gravity that causes the acceleration by emitting a strong repulsive force resulting in an anti-gravity effect. This uniformly distributed mystical component dominating the universe is slowly varying with time and space (Carroll 2001a, 2001b; Peebles & Ratra 2003). Since its discovery, it has become one of the most discussed topics among the cosmological society and great scientific efforts have been invested in order to explore its bizarre nature, properties, future characteristics and applications to modern cosmology. Frampton & Takahashi (2003) obtain that the universe might be DE dominant or free from DE in future time. Steinhardt et al. (2003) studies the quintessential introduction to DE. Sahoo & Mishra (2014b) investigate wet dark fluid, a DE candidate. Sahoo & Mishra (2014a) further study an axially symmetric cosmological model in the presence of anisotropic DE in which the solutions obtained could give us an appropriate description of the evolution of the universe. DE Survey Collaboration (Collaboration et al. 2016) describes the future prospect and discovery potential of the Dark Energy Survey (DES) beyond cosmological studies. Singh et al. (2017a) examine if DE could neutralize the global warming. Singh et al. (2017b) put forward interesting explanations to show that Lyra's manifold could be the hidden source of DE. Nair & Jhingan

(2013) examine whether DE is evolving or not. Abbott et al. (2018) provide us the first public data release of the DES. Risaliti & Lusso (2018) observe that the DE density is increasing with time. According to Cooper (2018), there are cosmologists who doubt if DE is behind the increasing expansion of the universe and the author analysed the arguments. Lastly, Calder & Lahav (2008) hunt down the origin of DE as far back as Newton and Hooke and presented a comprehensive summary of 90 years old history of the cosmological constant.

To understand this dark component as precisely as possible to obtain hints as to exactly predict its nature and properties, cosmologists have opted for analysing the equation of state (EoS) parameter ωwhich is the ratio of the pressure to the density of the DE. In recent years, different authors have calculated different viable limits on the value of ω with strong evidence in support. Knop et al. (2003) calculate two different ranges -1.61< ω <-0.78 and -1.67 < ω < -0.62 on two different situations. Melchiorri (2003) measures a bound of -1.38< <-0.82, whereas according to the most recent Planck 2018 results (Collaboration et al. 2020), the value of ω is measured to be ω = -1.03±0.03.

We can describe the accelerated expansion of the universe by two approaches: (i) DE approach in which different viable candidates of DE are developed (ii) Modified theories of gravitation approach in which ETG is modified to many optimized forms. Besides these approaches, many authors have put forward other possible ways to explain the late time acceleration of the universe. It is shown that the acceleration of the universe is the result of the back reaction of cosmological perturbations, rather than the effect of a negative pressure DE fluid or a modification of general relativity (Kolb et al. 2006). Gorji (2016) addresses the late-time cosmic acceleration the infrared corrections. An interesting explanation of cosmic acceleration using only dark matter (DM) and ordinary matter can be seen in the study of Berezhiani et al. (2017). An approach is also suggested by Narain & Li (2018) where the late time cosmic acceleration is obtained from an Ultraviolet Complete Theory.

The natural candidate for DE is the cosmological constant or the vacuum energy (VE) with $\omega = -1$. But, VE fails to illustrate many riddles of physics, one of which worth mentioning is the coincidence problem (Zlatev et al. 1999) in which the similar densities, at the present epoch, of the differently evolved DE and DM remains a mystery. Therefore, many other viable candidates of DE have been introduced (Copeland et al. 2006). Cosmologist started to construct models which involve the interaction of these two dark components to explain the small value of Λ (Wetterich 1995, 1988). Afterwards, these constructed models were found applicable to mollify the coincidence problem (Amendola & Valentini

2001; Zimdahl et al. 2001; Zimdahl & Pavon 2004; Cai & Wang 2005). During the last decade, evidence have been put forward which confirm that modified gravity can be presented in terms of interaction of these two dark components in the Einstein frame (Felice & Tsujikawa 2010; He et al. 2011; Zumalacarregui et al. 2013; Kofinas et al. 2016; Cai et al. 2016). This can enable us to broaden the gravitational theory beyond the breadth of general relativity if we can figure out the specific interaction term. Recently, great scientific efforts have been utilized to study the DE-DM interaction, for both theoretical and observational point of view, in the holographic dark energy (HDE) setting (Sadjadi 2007; Sadjadi & Vadood 2008; Setare & Vagenas 2009; Chimento et al. 2013; Kiran et al. 2014; Adhav et al. 2014a, 2014b; Umadevi & Ramesh 2015; Reddy et al. 2016a; Raju et al. 2016). HDE, a consequence of the application of the holographic principle (Wang et al. 2017) to the repulsive dark entity, was introduced by Gerard 't Hooft (Hooft 2009). Interacting models involving this dark holographic entity and matter in spherically symmetric (SS) space-time were studied in (Raju et al. 2016; Reddy et al. 2016). The mysterious nature of these two dark components have arisen many fundamental questions indicating that there are many new physics yet to be uncovered.

In the past few decades, many modified theories of gravitation challenging Einstein's theory have been put forward and these theories succeeded to fit the present cosmological trends in a quite satisfactory way, a handful of which that have not escaped our notice are Weyl's theory (Weyl 1918), Lyra geometry (Scheibe 1952), Brans-Dicke theory (Brans & Dicke 1961), $f(R)$ gravity (Chiba al. 2007), $f(R,T)$ gravity (Harko et al. 2011), Mimetic $F(R)$ gravity (Nojiri & Odintsov 2014) etc. Brans-Dicke theory (BDT) of gravitation has become one of the favourite choices among many cosmological audiences and enormous efforts have been employed to study its modern cosmological aspects (Miyazaki 2000; Kim 2005; El-Nabulsi 2007, 2010; Hrycyna & Szydlowski 2013b; Rani et al. 2018; Cruz & Peracaula 2018; Sadri & Vakili 2018; Brando et al. 2018). In this theory, a metric tensor g_{ij} is introduced along with a scalar filed φ which represents the space-time varying gravitational constant. In Einstein theory, gravity is explained by the lone entity - the space-time metric tensor or, in simple word, geometry. Whereas, in this modified theory, all matters are the reason for the gravitational behaviour of φ, so that, in this logic, it can be treated as a modification from purely geometric to geometric-scalar nature and thus, becoming a part of the family of scalar-tensor theory.

BDT can be of good choice to study DE and the expansion of the universe. It can be considered as the most natural choice of the scalar-tensor generalization of general relativity due to its easiness and is less stringent than general relativity. Above all, the scalar field and the theory itself are of classical origin and can be considered as viable candidates to contribute in the late time evolution of the universe (Kim 2005). BDT or its modified versions are also the possible agents generating the present cosmic acceleration (Banerjee & Pavon 2001a; Brunier et al. 2004). It has also been shown that the theory can potentially generate sufficient acceleration in the matter dominated era (Banerjee & Pavon 2001b). In most of the studies in the BDT setting, it can be seen that the accelerated expansion of the universe needs a very small value of ω, in the order of unity (Das & Mamon 2014) and to be negative. It is shown that if the Brans-Dicke scalar field interacts with the DM, a generalized BDT may cause the acceleration of the universe even with a high value of ω (Das & Banerjee 2006). Interestingly, Joyce et al. (2016) show that the theory is essentially equivalent to a DE model. At present, both BDT and general relativity are generally held to be in agreement with observation.

Sadjadi (2007) studies a spatially homogeneous and anisotropic Bianchi type-V universe filled with minimally interacting fields of HDE and matter obtaining a universe which decelerate initially and accelerate in infinite time. Sadjadi & Vadood (2008) and Setare & Vagenas (2009) examine interacting models in Bianchi type-I and Bianchi type-V universe respectively showing that for suitable choice of interaction between matter and DE, there is no coincidence problem. Chimento et al. (2013) find an interacting models between the two dark components in BDT setting and the authors obtained a model that exhibits early inflation and late time acceleration. Kiran et al. (2014) present an five dimensional interaction model in BDT obtaining an anisotropic universe. In the paper, the authors further mentioned that their universe will become isotropic in finite time due to cosmic re-collapse. Adhav et al. (2014) obtain an interacting model in a 5D spherically universe where the model experiences a transition from decelerated to accelerated phase due to cosmic re-collapse. Reddy et al. (2016a) study DE and matter using a relation between metric potentials and an equation of state representing disordered orientation obtaining the flat ΛCDM model as a particular case.

Inspired by the above studies, in this chapter, the minimal interaction model of the two dark entities has been presented with a 5D SS space-time in BDT of gravitation. Here, we consider some reasonable assumptions in agreement with the present cosmological observations. With particular choices of the constants involved, the values of the overall density parameter and the Hubble's parameter are obtained to be very close to the latest observational values. We obtain a model universe which will be increasing DE dominated. We also obtain that the model universe will face the big crunch singularity in the far future. The chapter has been structured into sections. In Sect. 8.2, the formulation of the problem is presented along with the solutions of the field equations. Related cosmological parameters are also solved in this section. In Sect. 8.3, the graphs of the parameters are plotted and the physical and kinematical aspects of our model in comparison with the present observational findings are discussed. Considering everything, a concluding note is provided in Sect. 8.4.

8.2 Formulation of problem with solutions

For our universe, we consider the 5D spherically symmetric metric (Samanta & Dhal 2013))

$$ds^2 = dt^2 - e^{\alpha}\left(dr^2 + r^2 d\Theta^2 + r^2 \sin^2\Theta d\phi^2\right) - e^{\beta} dy^2 \tag{8.2.1}$$

where μ and δ are cosmic scale factor which are functions of time only.

Here, BD field equations take the form

$$R_{ij} - \frac{1}{2} g_{ij} R + \omega_{_{BD}} \varphi^{-2}\left(\varphi_{,i}\varphi_{,j} - \frac{1}{2} g_{ij}\varphi_{,k}\varphi^{,k}\right) + \varphi^{-1}\left(\varphi_{i;j} - g_{ij}\varphi^{k}_{;k}\right) = -8\pi\varphi^{-1}\left(T_{ij} + S_{ij}\right) \tag{8.2.2}$$

where φ is the BD scalar field and T_{ij} and S_{ij} are respectively the energy momentum tensors for matter and HDE, whereas R is the Ricci scalar and R_{ij} is the Ricci tensor.

In our study we define T_{ij} and S_{ij} as follows

$$T_{ij} = \rho_m u_i u_j \tag{8.2.3}$$

$$S_{ij} = \left(\rho_d + p_d\right) u_i u_j - g_{ij} p_d \tag{8.2.4}$$

where ρ_m is the energy density of matter whereas ρ_d and p_d are respectively the energy density and the pressure of the HDE.

The wave equation satisfied by the scalar field is written as

$$\varphi^{k}_{;k} = 8\pi\left(3 + 2\omega_{_{BD}}\right)^{-1}(T + S) \tag{8.2.5}$$

The energy conservation equation in its obvious form is given by

$$T^{ij}_{;j} + S^{ij}_{;j} = 0 \tag{8.2.6}$$

We consider the co-moving co-ordinate system so that the flow vector satisfies the relation

$$g_{\mu\upsilon}u^{\mu}u^{\upsilon}=1 \tag{8.2.7}$$

We obtain the field equations as follows

$$\frac{3}{4}\left(\dot{\mu}^2+\dot{\mu}\dot{\delta}\right)-\frac{\omega_{BD}}{2}\frac{\dot{\varphi}^2}{\varphi^2}+\frac{\dot{\varphi}}{\varphi}\left(\frac{3\dot{\mu}+\dot{\delta}}{2}\right)=8\pi\varphi^{-1}\left(\rho_m+\rho_d\right) \tag{8.2.8}$$

$$\ddot{\mu}+\frac{3}{4}\dot{\mu}^2+\frac{\ddot{\delta}}{2}+\frac{\dot{\delta}^2}{4}+\frac{\dot{\mu}\dot{\delta}}{2}+\frac{\omega_{BD}}{2}\frac{\dot{\varphi}^2}{\varphi^2}+\frac{\ddot{\varphi}}{\varphi}+\frac{\dot{\varphi}}{\varphi}\left(\dot{\mu}+\frac{\dot{\delta}}{2}\right)=-8\pi\varphi^{-1}p_d \tag{8.2.9}$$

$$\frac{3}{2}\left(\ddot{\mu}+\dot{\mu}^2\right)+\frac{\omega_{BD}}{2}\frac{\dot{\varphi}^2}{\varphi^2}+\frac{\ddot{\varphi}}{\varphi}+\frac{3}{2}\frac{\dot{\varphi}}{\varphi}\dot{\mu}=-8\pi\varphi^{-1}p_d \tag{8.2.10}$$

And Eq. (8.2.6) gives

$$\ddot{\varphi}+\dot{\varphi}\left(\frac{3\dot{\mu}+\dot{\delta}}{2}\right)=8\pi\left(3+2\omega_{BD}\right)^{-1}\left(\rho_m+\rho_d-4p_d\right) \tag{8.2.11}$$

where an overhead dot represents differentiation with respect to time t.

Taking ω as the equation of state (EoS) parameter of HDE, we have

$$p_d=\omega\rho_d \tag{8.2.12}$$

Then, the conservation equation takes the form

$$\dot{\rho}_d+(1+\omega)\left(\frac{3\dot{\mu}+\dot{\delta}}{2}\right)\rho_d+\dot{\rho}_m+\rho_m\left(\frac{3\dot{\mu}+\dot{\delta}}{2}\right)=0 \tag{8.2.13}$$

Since the HDE and matter are interacting minimally, both the components will conserve separately. Thus, we can write (Sarkar 2014a, 2014b)

$$\dot{\rho}_m+\rho_m\left(\frac{3\dot{\mu}+\dot{\delta}}{2}\right)=0 \tag{8.2.14}$$

$$\dot{\rho}_d+(1+\omega)\,\rho_d\left(\frac{3\dot{\mu}+\dot{\delta}}{2}\right)=0 \tag{8.2.15}$$

Also, we have

$$\dot{\rho}+(\rho+p)\left(\frac{3\dot{\mu}+\dot{\delta}}{2}\right)=0 \tag{8.2.16}$$

Now, from Eqs. (8.2.9) and (8.2.10), we have

$$\frac{1}{2}\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 - \frac{\ddot{\delta}}{2} - \frac{\dot{\delta}^2}{4} - \frac{\dot{\mu}\dot{\delta}}{2} + \frac{1}{2}\frac{\dot{\varphi}}{\varphi}\left(\dot{\mu} - \dot{\delta}\right) = 0 \tag{8.2.17}$$

From Eq. (8.2.14), we have

$$\rho_m = a_0 e^{-\left(\frac{3\mu+\delta}{2}\right)} \tag{8.2.18}$$

Similarly,

$$\rho_d = b_0 e^{-(1+\omega)\left(\frac{3\mu+\delta}{2}\right)} \tag{8.2.19}$$

where a_0 and b_0 are arbitrary constants.

From Eqs. (8.2.9) and (8.2.10), we get

$$\mu = a_1 - \log\left(c_1 - t\right)^{\frac{2}{3}} \tag{8.2.20}$$

$$\delta = b_1 - \log\left(c_1 - t\right)^{\frac{2}{3}} \tag{8.2.21}$$

where a_1 and b_1 are arbitrary constants.

Thus, from Eqs. (8.2.18)-(8.2.21), we obtain

$$\rho_m = a_0 e^{-\frac{1}{2}(3a_1+b_1)}\left(c_1 - t\right)^{\frac{4}{3}} \tag{8.2.22}$$

$$\rho_d = b_0 e^{-\frac{1}{2}(3a_1+b_1)(1+\omega)}\left(c_1 - t\right)^{\frac{4}{3}(1+\omega)} \tag{8.2.23}$$

Now, using Eqs. (8.2.12), (8.2.20)-(8.2.23) in Eq. (8.2.11), we obtain the expression of the scalar field φ as

$$\varphi = M_0\left(c_1 - t\right)^{\frac{10}{3}} + N_0\left(c_1 - t\right)^{\frac{10}{3}+\frac{4}{3}\omega} \tag{8.2.24}$$

where

$$M_0 = \frac{36}{65}\pi\left(3 + 2\omega\right)^{-1} a_0 e^{-\frac{1}{2}(3a_1+b_1)} \tag{8.2.25}$$

$$N_0 = 8\pi b_0 (1-4\omega)(3+2\omega)^{-1}\left(\frac{10}{3}+\frac{4}{3}\omega\right)^{-1}\left(\frac{11}{3}+\frac{4}{3}\omega\right)^{-1} e^{-\frac{1}{2}(1+\omega)(3a_1+b_1)} \tag{8.2.26}$$

From Eqs. (8.2.22) and (8.2.23), we obtain the expression for the energy density ρ as

$$\rho = \rho_m + \rho_d = a_0 e^{-\frac{1}{2}(3a_1+b_1)}(c_1-t)^{\frac{4}{3}} + b_0 e^{-\frac{1}{2}(1+\omega)(3a_1+b_1)}(c_1-t)^{\frac{4}{3}(1+\omega)} \tag{8.2.27}$$

Using Eqs. (8.2.20), (8.2.21) and (8.2.27) in Eq. (8.2.16), we obtain the expression for the pressure as

$$p = \frac{1}{3}a_0 e^{-\frac{1}{2}(3a_1+b_1)}(c_1-t)^{\frac{4}{3}} + \left(\frac{1}{3}+\frac{4}{3}\omega\right) b_0 e^{-\frac{1}{2}(1+\omega)(3a_1+b_1)}(c_1-t)^{\frac{4}{3}\omega+\frac{4}{3}} \tag{8.2.28}$$

From Eqs. (8.2.12) and (8.2.23), we the pressure of the DE is given by

$$p_d = \omega b_0 e^{-\frac{1}{2}(3a_1+b_1)(1+\omega)}(c_1-t)^{\frac{4}{3}(1+\omega)} \tag{8.2.29}$$

Now, at any time $t = t_0$, we can take

$$p = p_d \tag{8.2.30}$$

Therefore, from Eqs. (8.2.28), (8.2.29) and (8.2.30), we get

$$\left(a_0 e^k + b_0 (1+\omega) e^{k(1+\omega)} (c_1-t_0)^{\frac{4\omega}{3}}\right)(c_1-t_0)^{\frac{1}{3}} = 0 \tag{8.2.31}$$

where $k = -\frac{1}{2}(3a_1+b_1)$

Eq. (8.2.31) will give us the expression for the EoS parameter ω.

Now, we obtain the values of the different cosmological parameters as follows.

Spatial volume:

$$V = e^{\frac{3a_1+b_1}{2}}(c_1-t)^{-\frac{4}{3}} \tag{8.2.32}$$

Scalar expansion:

$$\theta = \frac{4}{3}(c_1-t)^{-1} \tag{8.2.33}$$

Hubble parameter:

$$H = \frac{1}{3}\left(c_1 - t\right)^{-1} \tag{8.2.34}$$

Shear scalar:

$$\sigma^2 = \frac{2}{9}\left(1 - \frac{1}{c_1 - t}\right)^2 \tag{8.2.35}$$

Anisotropic parameter:

$$A_h = 0 \tag{8.2.36}$$

DE density parameter:

$$\Omega_d = \frac{\rho_d}{3H^2} = 3b_0 e^{-\frac{1}{2}(3a_1+b_1)(1+\omega)}\left(c_1 - t\right)^{\frac{2}{3}(5+2\omega)} \tag{8.2.37}$$

Matter density parameter:

$$\Omega_m = \frac{\rho_m}{3H^2} = 3a_0 e^{-\frac{1}{2}(3a_1+b_1)}\left(c_1 - t\right)^{\frac{10}{3}} \tag{8.2.38}$$

Overall density parameter:

$$\Omega = \Omega_d + \Omega_m = 3\left(a_0 e^{-\frac{1}{2}(3a_1+b_1)} + b_0\left(c_1 - t\right)^{\frac{4\omega}{3}} e^{-\frac{1}{2}(3a_1+b_1)(1+\omega)}\right)\left(c_1 - t\right)^{\frac{10}{3}} \tag{8.2.39}$$

8.3 Discussion

For different values of the constants involved, we will obtain different graphs. So, we opt to take particular values of the constants i.e., a_0=b_0=a_1=b_1=1, c_1=14.301443981790266 and plot the graphs of some of the parameters showing their variations with time as shown in the figures of this section.

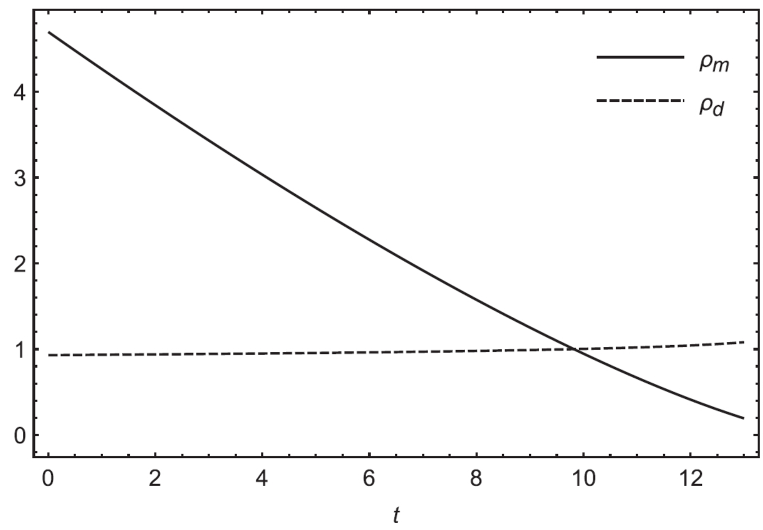

Figure 8.1: Variation of energy density of DE ρ_d and matter ρ_m with time t when a_0=b_0=a_1=b_1=1, c_1=14.301443981790266 showing that ρ_m decreases throughout evolution whereas ρ_d tends to increase very slowly or is nearly unchanged.

From Eqs. (8.2.22) and (8.2.23), it is obvious that energy densities of matter ρ_m and DE ρ_d are functions of cosmic time. To examine their nature, we plot their graphs showing their variations with cosmic time t as shown in Fig. 8.1. Here, it can be seen that ρ_m decreases throughout the evolution, as with the expansion of the universe, the galaxies get farther away from each other so that the matter density continues to diminish (Carroll 2001b). But, ρ_d tends to increase very slowly or is nearly unchanged. This may be a result of this anti-gravity dark component varying slowly with time and space (Carroll 2001a, 2001b; Peebles & Ratra 2003). So, our model universe will be increasingly dominated by dark energy in the far future.

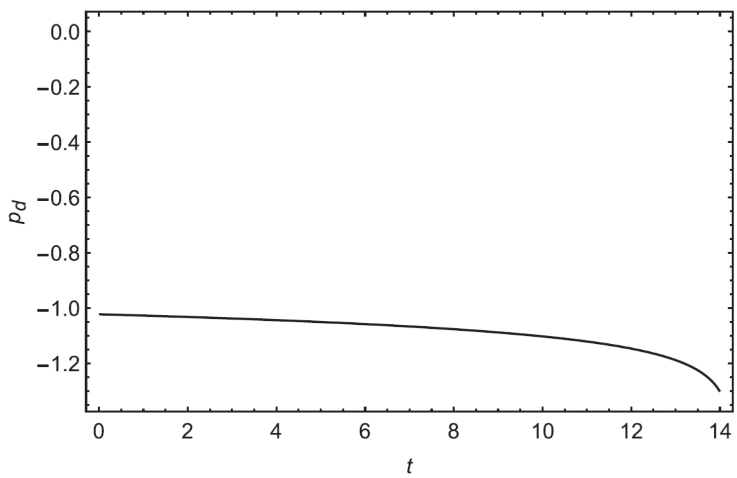

Figure 8.2: Variation of pressure of DE p_d with time t when b_0=a_1=b_1=1, c_1=14.301443981790266 showing it varies in the negative plane throughout evolution.

From Fig. 8.2, it can be clearly seen that the pressure of DE varies in the negative region throughout the evolution which is in agreement with the exotic property of dark energy that causes the universe to expand.

Figs. 8.3 and 8.4 respectively show that spatial volume V and the scalar expansion θ increases with t showing the accelerating spatial expansion of the universe. From Eq. (8.2.32), the spatial volume V of the universe is constant ($V \neq 0$) at time t=0. Also, other related parameters are also constant at time t=0. These show that our universe is free from initial singularity. But, when $t \to \infty$, both V and $\theta \to 0$ which indicates that after an infinite period of time, there will be a phase transition in which the expansion of the universe will cease. This may be supported by the fact that dark energy which causes the expansion of the universe varies slowly with time and space (Carroll 2001a, 2001b; Peebles & Ratra 2003). Also, the energy density of dark energy may decreases faster than that of matter leading to the disappearance of dark energy at $t \to \infty$ (Peebles & Ratra 2003). Then, our model universe will expand up to a finite degree; the expansion will tend to decrease. So, in the far future, this would lead our universe to be dominated by gravity causing it to shrink; finally collapsing resulting to the big crunch singularity.

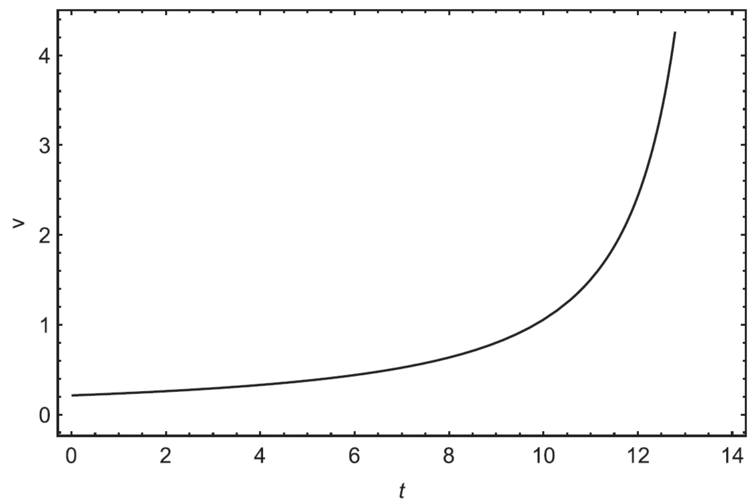

Figure 8.3: Variation of spatial volume V with time t when a_1=b_1=1, c_1=14.301443981790266 showing that it increases with time.

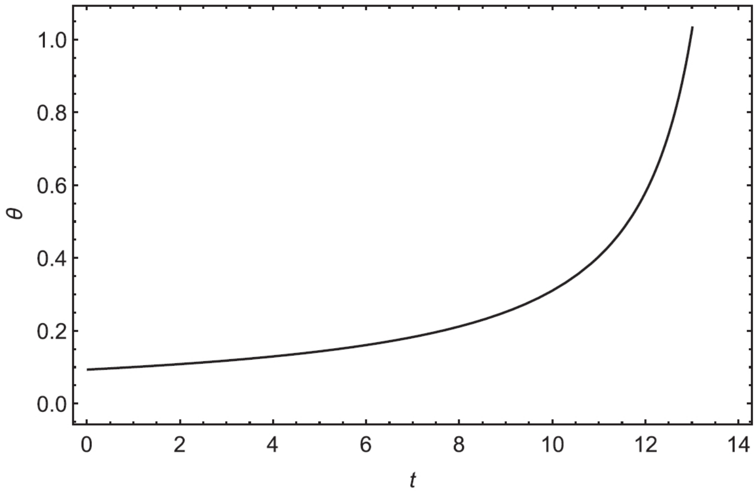

Figure 8.4: Variation of scalar expansion θ with time t when c_1=14.301443981790266 showing that it increases with time.

Fig. 8.5 shows that shear scalar (σ^2) tends to remain constant in the initial stage. Then, it start to converge and finally, diverges with the increase of cosmic time. Shear scalar provides us the rate of distortion of the matter flow of the large scale structure of cosmology (Ellis & Elst 1999). Hence, the model universe expands with a slow and uniform change of shape in the initial stage. Then, the change become more slower and finally, the change becomes faster. From Eq. (8.2.36), it is clear that anisotropic parameter A_h=0 all the time which indicates that our model universe is isotropic throughout the evolution.

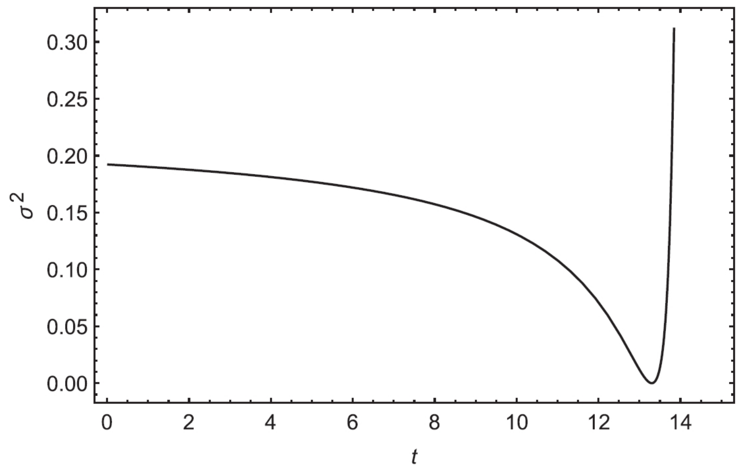

Figure 8.5: Variation of shear scalar σ^2 with time t when c_1=14.301443981790266.

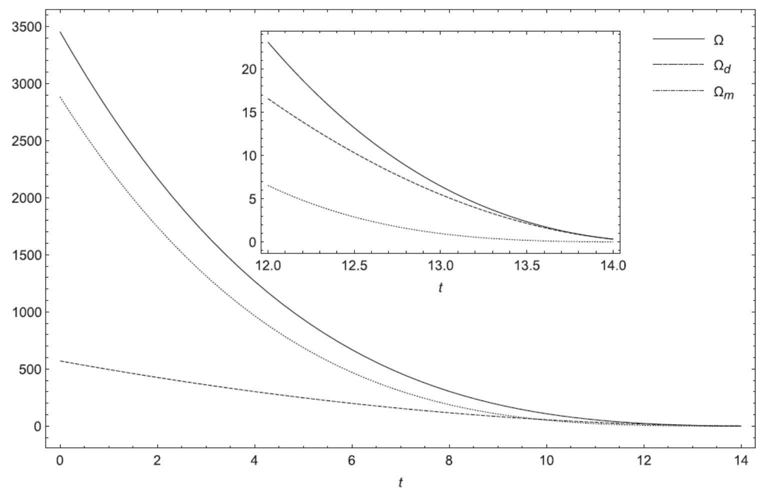

Figure 8.6: Variation of overall density parameter Ω, DE density parameter Ω_d and matter density parameter Ω_m with time t when $a_0{=}b_0{=}a_1{=}b_1{=}1$, $c_1{=}14.301443981790266$ showing that Ω and Ω_d decrease and tend to become constant whereas Ω_m decreases with a greater extent.

The variations of Ω, Ω_d and Ω_m with cosmic time t are shown in Fig. 8.6. Here, Ω and Ω_d are decreasing with the increase of cosmic time t and tend to become constant whereas Ω_m decreases but with a greater extent which might be supported by the fact that the matter density is diminishing with the accelerated expansion of the universe (Carroll 2001b). Here, it may be predicted that our model universe will become increasingly DE dominated in the far future. Moreover, on assuming that $a_0{=}b_0{=}a_1{=}b_1{=}1$, $c_1{=}14.301443981790266$ and taking EoS parameter $\omega{=}-1.047$ which is in agreement with the latest observational value of ω (Knop et al. 2003; Melchiorri 2003; Collaboration et al. 2020), we find that the expression for EoS given by Eq. (8.2.31) is satisfied by time $t_0{=}13.8$ which is age of the universe at the present epoch. Also, under these assumptions, Eq. (8.2.39) gives us the value of the overall density parameter $\Omega{=}0.905988(\approx 1)$ at $t{=}13.8$ which is consistent with the present cosmological belief. Above all, at $t{=}13.8$, Eq. (8.2.34) gives us the value of Hubble parameter $H{=}68$ which is very close to $H_0 = 67.36 \pm 0.54\text{kms}^{-1}\text{Mpc}^{-1}$, the value of Hubble parameter by the most recent Planck 2018 results (Collaboration et al. 2020).

8.4 Conclusions

In this chapter, we have studied a 5D SS space-time accompanied by minimally interacting fields - DM and DE components in BDT. It is predicted that our model universe will be increasingly dominated by DE. It is observed that the model universe is isotropic throughout the evolution. Our model universe is free from initial singularity but may face the big crunch singularity in the far future. With reasonable assumptions of the values of the constants and $\omega{=}-1.047$ which is consistent with the value of ω of the most recent Planck

2018 results (Collaboration et al. 2020), we obtain the value of overall density parameter Ω=0.905988($\approx$1) which agrees with the present cosmological observation. Above all, at t=13.8, we obtain the value of Hubble parameter H=68 which is very close to $H_0 = 67.36 \pm 0.54 \text{kms}^{-1}\text{Mpc}^{-1}$, the value predicted by the most recent Planck 2018 results (Collaboration et al. 2020).

Chapter 9

Conclusions

This chapter summarizes the works presented in the thesis and conveys the relevance of the research. The chapter also explains what new knowledge has been brought to light and suggests future research ideas on the subject.

9.1 Summary

We have examined the dynamics of DE universe models in a 5D context while taking into account a SS space-time. We have anticipated some unforeseen outcomes as well as some that match the observational data. Our work, however, does have certain limitations, which is a general characteristic of a research work. We need further in-depth study and research to overcome the limitations. The works provided in the thesis are briefly summarised below, highlighting the main findings.

In **Chapter 1**, we have explained the motivation for our study and the fundamental concepts of the foundation and formulation of GR cosmological problems in a concise manner. We've also included an overview of the various ideas, space-time, and gravitational theories covered in the thesis. Above all, we have reviewed 108 articles to get a better understanding of similar works from the past and present. The review assisted us in identifying the knowledge gap as well as potential research ideas. We have summarised the review's main findings, as well as the implications and ideas for further research.

In **Chapter 2**, we have presented the research method and process employed to acquire knowledge and data to answer the research problem. The logic behind the approach used to acquire and analyse the results of the study is discussed in this chapter.

In **Chapter 3**, within the framework of SBT, we have studied an interacting model of HDE and matter in an SS space-time in a 5D context. We have established an accelerating universe dominated by phantom energy. The extra dimension maintains the theoretical foundation of the HDE scenario. The DE transits from phantom-like nature to CC, avoiding future singularity and heading to the de-Sitter phase. The Hubble parameter and DE

EoS parameter are measured to be $H = 67$ and $\omega = -1.00011$ at $t = 13.8$ Gyr. The SB scalar field φ tends to reach a positive constant value in the course of evolution when $n = -1$.

In **Chapter 4**, with minimally interacting matter and HDE in SBT, we have investigated a cosmological model in SS space-time in a 5D scenario. We have predicted that the universe will become increasingly DE dominated. We have estimated a few values of the deceleration parameter and found that they are quite close to the recently estimated values. The Hubble parameter H decreases, which is consistent with the current cosmic scenario. The DE EoS parameter has a value of $\omega = -1$, suggesting that the DE we are working with is the VE or CC. The total density parameter is predicted to be $\Omega = 0.97(\approx 1)$. The model universe begins with the Big Bang and ends with the Big Freeze singularity. Our interacting HDE model may be regarded as an alternative cosmological model to the standard ΛCDM model. Finally, we present two criteria for solving the extra dimension stability problem in GR: the first is to identify Casimir energy with the CC, or VE, and the second is to assume the extra dimension is of infinite volume. Probably, our work is the first to predict conditions to stabilize extra dimension in GR.

In **Chapter 5**, we have examined an SS metric in a 5D setting within the framework of LM. The model universe is predicted to be a DE model, with VE or CC dominating. The displacement vector serves as a time-dependent DE as well. The model is an oscillating model, with each cycle beginning with a big bang and ending with a big crunch, undergoing a series of bounces. The Hubble parameter is calculated to be $H = 67.0691$. We have created a model in LM that appeared to be a DE model; nevertheless, the work we have presented is only a toy model. The model requires further investigation, taking into account all of the observational data, which will be the focus of our forthcoming work.

In **Chapter 6**, we have investigated a SS space-time in a 5D scenario employing the $f(R,T)$ gravity framework. When $\lambda = -5.06911$ and -12.5856, the variation of pressure p and energy density ρ with cosmic time t have been investigated. It's intriguing to observe how our $f(R,T)$ model acts as a DE (vacuum energy) model in both situations. The scalar curvature R decreases over time, which is in line with previous research. In the finite time future, the model is projected to reach the de-Sitter phase. We have developed a model in which the $f(R,T)$ gravity theory itself acts like a DE (VE) model; nevertheless, the work we have presented is only a toy model. The model requires additional investigation, taking into account all of the observational data, which will be the matter of our forthcoming work.

In **Chapter 7**, we have investigated SCT in the context of a 5D SS space-time. The isotropic model acts like a phantom energy dominated model, that does not evolve from a singularity and instead tends to the de-Sitter phase, avoiding finite time future singularity (big rip). We have estimated that the value of G decreases with a variation of $\frac{\dot{G}}{G} = -7.2 \times 10^{-11}$ yr^{-1}. We have predicted $H = 68$ at t=13.8 Gyr. We have developed an SCT model that behaves as a phantom energy model, in other words that acts as a DE source. However, this developed model is a toy that requires a more in-depth study that takes into account all of the most recent cosmological facts, which we intend to do in the future.

In **Chapter 8**, in BDT framework, we have investigated a 5D SS space-time with minimally interacting DM and DE. DE is expected to become increasingly dominant. Our isotropic model universe does not start with an initial singularity, but it may encounter a big crunch singularity in the future. The values of the cosmological parameters obtained in this chapters are $\omega = -1.047$, $\Omega = 0.905988 \approx 1$ and $H = 68$, which agree with the current observational data.

9.2 Summary of key findings

Following are some of the key findings of our research.

- As DE sources, many authors have proposed various theories. Here, in our work we have shown that the modified theories of gravity, i.e. Lyra manifold, scale covariant theory, and $f(R,T)$ gravity can act as DE sources.
- In each of our constructed model, except Chapter 6, we have estimated some of the values of the cosmological parameters like Hubble parameter, DE EoS parameter and overall density parameter. Above all, we have also estimated the variation of the gravitational constant in Chapter 7.
- Generally, we can't find criteria stabilizing extra dimensions in GR. However, in Chapter 4, we have predicted two conditions for stabilizing extra dimensions in GR. Probably, our work is the first to predict conditions to stabilize extra dimension in GR.
- Because the dominant DE component is of phantom type, the big rip singularity is the most likely scenario. However, we have presented phantom energy models avoiding the cosmic doomsday, and ending at the de Sitter phase in Chapters 3 and 7. Above all, during research (Chapter 3), we come to know that in a HDE model,

big rip singularity is not permitted, as big rip violates the theoretical foundation of HDE scenario.

- Energy density should be positive to obtain a reliable cosmological model. However, we have presented models in Chapters 5 and 6 involving negative energy density, with arguments in support.

9.3 Ideas for further research

Following are some of the ideas for further research.

- We have shown that that the modified theories of gravity, i.e. Lyra manifold, scale covariant theory, and $f(R,T)$ gravity can act as DE source. Nevertheless, the models we have presented are just toy models. They require additional investigation, taking into account all of the observational data, which will be the matter of our forthcoming work. We may also see if there are any other modified theories that can be employed as DE sources.

- We have proposed two criteria for stabilising extra dimensions in GR in Chapter 4, perhaps for the first in GR. More research is needed to determine the reliability of all potential outcomes of the two criteria in terms of different cosmological factors in GR.

- In Chapter 3, we have mentioned that in a HDE model, big rip singularity is not permitted, as big rip violates the theoretical foundation of HDE scenario. This issue can be solved with the employment of an extra dimension. We can do further study to see if there is another way to solve this issue.

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Vacuum Energy in Saez-Ballester Theory and Stabilization of Extra Dimensions

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Abstract: In this work, we study a spherically symmetric metric in 5D within the framework of Saez-Ballester Theory, where minimal dark energy-matter interaction occurs. We predict that the expanding isotropic universe will be progressively DE dominated. We estimate few values of the deceleration parameter, very close to the recently predicted values. We obtain the value of the DE EoS parameter as $\omega = -1$. Additionally, we measure the value of the overall density parameter as $\Omega = 0.97(\approx 1)$, in line with the notion of a close to or nearly (not exactly) flat universe. We predict that the model universe starts with the Big-Bang and ends at the Big Freeze singularity. In general, we cannot find conditions for stabilization of extra dimensions in general relativity, and all dimensions want to be dynamical. Here, we present two possible conditions to solve this stabilization problem in general relativity.

Keywords: general relativity; Saez-Ballester Theory; vacuum energy; spherically symmetric; singularity; extra dimensions

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1. Introduction

Since the discovery of dark energy (DE) [1,2] in 1998, it has gained a reputation as one of the topics of paramount importance among the cosmological forums. Despite investing tremendous scientific efforts to explore it, its origin, bizarre nature, and future aspects to modern cosmology are still up for grabs. It is characterized by the distinctive feature of possessing a huge negative pressure opposing gravity resulting in the enigmatic phenomenon of the universe expanding at an expedited rate at late times. This cryptic dark entity is considered to be uniformly distributed and varies slowly or nearly unchanged with time [3–6]. Some worth mentioning studies on this mystic dark component that have not escaped our attention in the last few years are briefly presented below.

Recently, in [7], the authors study a higher dimensional cosmological model to find the origin of DE. They further predict an $f(R,T)$ gravity model as a DE source [8]. A presentation on the evolution of DE considering recent findings can be seen in [9]. In [10], the authors investigate the future of this dark entity beyond the bound of cosmological aspects. In [11], the estimation of DE density is presented. In [12], the authors put forward arguments for the need for DE. Gutierre [13] analyses the status of the experimental data on DE. A fascinating comparison of the speed of DE with that of a photon can be found in [14]. The atom-interferometry constraints on DE are studied in [15]. In [16], DE is obtained from the violation of energy conservation. The prediction of clustering galaxy as a result of stirring effect of DE can be seen in [17]. Lastly, in [18], the author claims that particles with imaginary energy density can lead us to the root of the ambiguous dark component.

Cosmologists have witnessed numerous theoretical attempts to obtain hints as to exactly predict the underlying physics of the miraculous expanding phenomenon of the universe at late times. Two well-appreciated methods have been adapted to explain this mystic phenomenon. Firstly, different possible forms of DE are developed. Secondly, modifying the Einstein theory of gravitation [19,20]. Other than these two, recently, cosmologists and theoretical physicists have been successful in developing other interesting and convincing approaches. In [21], the phenomenon is explained by the infrared corrections. Narain [22] predicts that an Ultraviolet Complete Theory leads to the expansion. A fascinating illustration can be seen in [23] where the expedited expansion occurs in the absence of DE.

To figure out the ambiguous nature of DE in as much detail as possible, the equation of state (EoS) parameter ω is studied with utmost importance. The most recent Planck 2018 results [24], estimates its value to be $\omega = -1.03 \pm 0.03$. The late time expedited expansion of the universe is obtained when $\omega < -\frac{1}{3}$ [25]. $\omega = -1$ corresponds to the natural candidate of DE, the cosmological constant (CC), or in other words, vacuum energy (VE). However, CC or VE comes up short to explain the mystery of the coincidence problem (CP) [26]. After multiple efforts, many other well-appreciated forms of DE are developed [27]. One such candidate that has not escaped our notice is the holographic dark energy (HDE), an outcome of the introduction of the holographic principle (HP) [28] to DE. Accordingly, all the physical quantities inside the universe including the energy density of DE can be illustrated by some quantities on the boundary of the universe [29]. Recent works on some of the different forms of HDE can be seen in [30–33]. Construction of interacting HDE and dark matter (DM) models in spherically symmetric space-time settings can be observed in [34–36]. Interacting models can successfully represent modified gravity in the Einstein frame [37–41]. In [42–45], it also is shown that such interacting models are effective in mollifying the CP.

Due to the fascinating natures of the HDE and VE, a spark of interest has been ignited among cosmologists so that they have started to examine HDE paired with VE. In [46], the authors predict that their HDE model evolved from ΛCDM in early time and approaches to the same ΛCDM in the late time. They further mention that for a fixed value of a coupling parameter involved, their HDE model remains fixed in the ΛCDM model all through. In [47], an accelerating HDE model behaving similarly to the ΛCDM model is presented. An explanation can be seen in [48] in which the HDE model cannot be discriminated from ΛCDM in the high-redshift region. In [49], it is asserted that the vacuum entanglement energy is the probable candidate for HDE, where entanglement energy is the disturbed vacuum energy due to the presence of a boundary [50]. Hu et al. [51] develop a heterotic DE model where the DE has two parts, the cosmological constant and HDE. A study of an HDE model where $\omega = -1$ is obtained can be found in [52]. Lastly, a model can be seen in [53] where HDE ends at ΛCDM in the future.

Saez-Ballester Theory (SBT), introduced by Saez and Ballester [54], can be considered to be the right option to study DE and the accelerating universe. It is a member of the family of Scalar Tensor Theory (STT) of gravitation. In SBT, the metric potentials are coupled with a scalar field φ. Scalar fields are considered to play key roles in gravitation and cosmology as they can illustrate prodigies like DE, DM, etc. [55]. They can be regarded as a possible contributing factor in the late time acceleration of the universe [56]. STT is of direct generalization and extension of general relativity [57]. STT can be considered as a perfect candidate for DE [58]. In [59,60], it is asserted that a scaler field might be responsible for the inflation at the initial epoch. The authors in [61,62] discuss Bianchi Type-V cosmology in SBT obtaining a transit from decelerating universe to accelerating phase. Currently, SBT and general relativity are held to align with observation.

The higher-dimensional model has become one of the good choices among cosmologists and theorological physicists. The idea of such a model was put forward by Kaluza and Klein [63,64]. The authors in [65,66] claim that such a model can explain the late time expanding phenomenon. In [67], it is mentioned that extra-dimensional theories of gravity might explain the early inflation and late-time acceleration of the universe. There is a remarkable improvement in our knowledge and the logical consistency of physics by the introduction of the fifth dimension [68]. A study to validate the existence of the extra dimension is presented by Marciano [69]. There is a chance that the unknown fifth dimension might be related to two the ambiguous and unseen dark components—dark energy and dark matter [70]. According to [71], the employment of an extra dimension makes HDE models more complete and consistent. Some recent worth mentioning studies on higher dimension can be seen in [72–77].

Taking into consideration the above noteworthy related studies, we consider a minimal DE-DM interaction within the framework of SBT using a 5D spherically symmetric space-time. In this work, we present an in-depth discussion on every cosmological parameter obtained. The definition of shear scalar and its physical significance are provided. We discuss the initial and future singularity of the model universe. Additionally, we calculate the present values of the overall density parameter, deceleration parameter, and the dark energy EoS parameter. We also discuss the conditions to solve the stabilization problem of extra dimensions in general relativity. The paper is divided into sections. After the introduction, in Section 2, we present the formulation of the problem with solutions to the parameters. In Section 3, the solutions are discussed with graphical representations. In Section 4, we present the explanation of the solution to the stabilization problem of extra dimensions in GR. Lastly, to sum up the observations, a concluding note is provided in Section 5.

2. Formulation of Problem and Solutions

In our universe, the five-dimensional spherically symmetric metric [78] of following the form is considered

$$ds^2 = dt^2 - e^{\alpha}\left(dr^2 + r^2 d\Theta^2 + r^2 \sin^2\Theta d\phi^2\right) - e^{\beta} dy^2, \tag{1}$$

where α and β are cosmic scale factors which are functions of time only.

We consider the following Saez-Ballester field equations

$$R_{ij} - \frac{1}{2} g_{ij} R - \lambda \varphi^n \left(\varphi_{,i}\varphi_{,j} - \frac{1}{2} g_{ij} \varphi_{,k}\varphi^{,k}\right) = -(T_{ij} + S_{ij}), \tag{2}$$

where T_{ij} and S_{ij} are the energy momentum tensors for matter and HDE, respectively, R and R_{ij} are, respectively, the Ricci scalar and tensors, whereas the scalar field φ satisfies

$$2\varphi^n \varphi^{,i}_{;i} + n\varphi^{n-1}\varphi_{,k}\varphi^{,k} = 0, \tag{3}$$

where n is an arbitrary constant.

We define T_{ij} and S_{ij} as

$$T_{ij} = \rho_m u_i u_j, \tag{4}$$

$$S_{ij} = (\rho_d + p_d) u_i u_j - g_{ij} p_d, \tag{5}$$

where ρ_m and ρ_d represent the energy densities of matter and HDE, respectively, and p_d represents the pressure of the HDE.

Here, the energy is conserved and obviously, we have

$$T^{ij}_{;j} + S^{ij}_{;j} = 0. \tag{6}$$

By using the co-moving coordinate system, the surviving field equations are obtained as follows

$$\frac{3}{4}\left(\dot{\alpha}^2 + \dot{\alpha}\dot{\beta}\right) + \frac{\lambda}{2}\varphi^n\dot{\varphi}^2 = \rho, \tag{7}$$

$$\ddot{\alpha} + \frac{3}{4}\dot{\alpha}^2 + \frac{\ddot{\beta}}{2} + \frac{\dot{\beta}^2}{4} + \frac{\dot{\alpha}\dot{\beta}}{2} - \frac{\lambda}{2}\varphi^n\dot{\varphi}^2 = -p_d, \tag{8}$$

$$\frac{3}{4}\left(\ddot{\alpha} + \dot{\alpha}^2\right) - \frac{\lambda}{2}\varphi^n\dot{\varphi}^2 = -p_d, \tag{9}$$

and from Equation (6), we have

$$\ddot{\varphi} + \dot{\varphi}\left(\frac{3\dot{\alpha} + \dot{\beta}}{2}\right) + \frac{n}{2}\dot{\varphi}^2\varphi^{-1} = 0, \tag{10}$$

where an overhead dot represents differentiation w.r.t. t.

Considering ω as the EoS parameter of the dark energy so that we have

$$p_d = \omega\rho_d. \tag{11}$$

Now, the conservation equation is given by

$$\dot{\rho}_d + (1+\omega)\left(\frac{3\dot{\alpha} + \dot{\beta}}{2}\right)\rho_d + \dot{\rho}_m + \rho_m\left(\frac{3\dot{\alpha} + \dot{\beta}}{2}\right) = 0. \tag{12}$$

Due to the minimal interaction of HDE and matter, by [79,80], both the components conserve separately thereby obtaining

$$\dot{\rho}_m + \rho_m\left(\frac{3\dot{\alpha} + \dot{\beta}}{2}\right) = 0. \tag{13}$$

$$\dot{\rho}_d + (1+\omega)\rho_d\left(\frac{3\dot{\alpha} + \dot{\beta}}{2}\right) = 0. \tag{14}$$

Furthermore, we have

$$\dot{\rho} + (\rho + p)\left(\frac{3\dot{\alpha} + \dot{\beta}}{2}\right) = 0. \tag{15}$$

From Equations (13) and (14), we have

$$\rho_m = a_0 e^{-\left(\frac{3\alpha+\beta}{2}\right)}, \tag{16}$$

$$\rho_d = b_0 e^{-(1+\omega)\left(\frac{3\alpha+\beta}{2}\right)}, \tag{17}$$

where a_0 and b_0 are arbitrary constants.

From Equations (8) and (9), we obtain the expression for cosmic scale factors as

$$\alpha = c_1 + \log(v\,t - uc_2)^{\frac{u}{v}}, \tag{18}$$

$$\beta = kc_1 + \log(v\,t - uc_2)^{\frac{ku}{v}}, \tag{19}$$

where $c_1,\ c_2,\ u,\ v$ and $k \neq 0$ are arbitrary constants.

From Equations (16)–(19), the energy densities of matter and DE are, respectively, obtained as

$$\rho_m = a_0 e^{-\frac{(k+3)c_1}{2}}(vt - uc_2)^{-\frac{(k+3)u}{2v}}. \tag{20}$$

$$\rho_d = b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}}(vt - uc_2)^{-\frac{(1+\omega)(k+3)u}{2v}}. \tag{21}$$

Using Equations (18) and (19) in Equation (10), the expression for scalar field is obtain as

$$\varphi = c_2 e^{\frac{2\log\left(e^{\frac{u}{2}(k+3)\left(\frac{t}{v^2t-uvc_2}-2c_1\right)}_{-(n+2)\left(uvc_2-v^2t\right)}\right)-\frac{(k+3)u\,t}{v^2-uvc_2}}{n+2}}. \tag{22}$$

From Equations (20) and (21), the expression for energy density of the model universe is obtained as

$$\rho = a_0 e^{-\frac{(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(k+3)u}{2v}} + b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(1+\omega)(k+3)u}{2v}}. \tag{23}$$

Using Equations (18), (19) and (23) in Equation (15), the expression for pressure of the model universe is obtained as

$$p = -\left(a_0 e^{-\frac{(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(k+3)u}{2v}} + b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(1+\omega)(k+3)u}{2v}} \right). \tag{24}$$

From Equations (11) and (21), the pressure of dark energy is obtained as

$$p_d = \omega\ b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(1+\omega)(k+3)u}{2v}}. \tag{25}$$

At any time $t = t_0$, we can assume that $p = p_d$ so that

$$\left(a_0 e^{\frac{\omega(k+3)c_1}{2}} (vt - uc_2)^{\frac{\omega(k+3)u}{2v}} + b_0(1+\omega) \right) e^{-\frac{(1+\omega)(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(1+\omega)(k+3)u}{2v}} = 0. \tag{26}$$

The expression for ω will be given by Equation (26).

Now, the expressions for the different cosmological parameters are obtained as given below

Spatial volume:

$$v = e^{\frac{3\alpha+\beta}{2}} = e^{\frac{(k+3)c_1}{2}} (vt - uc_2)^{\frac{(k+3)u}{2v}}. \tag{27}$$

Scalar expansion:

$$\theta = u^i_{;j} = \frac{3\dot{\alpha}}{2} + \frac{\dot{\beta}}{2} = \frac{(k+3)u}{2(vt - uc_2)}. \tag{28}$$

Hubble parameter:

$$H = \frac{\theta}{4} = \frac{(k+3)u}{8(vt - uc_2)}. \tag{29}$$

Deceleration parameter:

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 = \frac{8v}{(k+3)u} - 1. \tag{30}$$

Shear scalar:

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{72}\left(\frac{16vt^2 - 4(3k + 8c_2 + 9)uvt + 3(3k + 4kc_2 + 12c_2 + 9)u^2 + 16uc_2{}^2}{(vt - uc_2)^2} \right). \tag{31}$$

Anisotropic parameter:

$$A_h = \frac{1}{4}\sum_{i=1}^{4}\left(\frac{\Delta H_i}{H}\right)^2 = 3\left(\frac{k-1}{k+3}\right)^2, \tag{32}$$

where $\Delta H_i = H_i - H, (i = 1,2,3,4)$ are the directional Hubble parameters.

Dark energy density parameter:

$$\Omega_d = \frac{\rho_d}{3H^2} = \frac{64}{3}\left(\frac{b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}} (vt - uc_2)^{2-\frac{(1+\omega)(k+3)u}{2v}}}{3(k+3)^2 u^2} \right). \tag{33}$$

Matter density parameter:

$$\Omega_m = \frac{\rho_m}{3H^2} = \frac{64}{3}\left(\frac{a_0 e^{-\frac{(k+3)c_1}{2}}(vt - uc_2)^{2-\frac{(k+3)u}{2v}}}{3(k+3)^2u^2}\right). \tag{34}$$

Overall density parameter:

$$\Omega = \Omega_d + \Omega_m = \frac{64}{3}\left(\frac{\left(a_0 + b_0 e^{-\frac{\omega(k+3)c_1}{2}}(vt - uc_2)^{-\frac{\omega(k+3)u}{2v}}\right)e^{-\frac{(k+3)c_1}{2}}(vt - uc_2)^{2-\frac{(k+3)u}{2v}}}{3(k+3)^2u^2}\right). \tag{35}$$

From [81], the expression for the state finder diagnostic pair $\{r, s\}$ is given by

$$r = 1 + \frac{3\dot{H}}{H^2} + \frac{\ddot{H}}{H^3}. \tag{36}$$

$$s = \frac{r-1}{3\left(q - \frac{1}{2}\right)}. \tag{37}$$

From Equations (29), (36) and (37), we have

$$\{r, s\} = \{1, 0\}. \tag{38}$$

3. Discussion

In this section, for convenience sake and to achieve realistic outcomes, we opt to choose $a_0 = b_0 = c_1 = c_2 = k = 1, u = 2.78$ and $v = \frac{1}{2}$. The discussion on the nature of the parameters with respect to cosmic time t are presented in details with graphs as follows.

From Equations (20) and (21), it is obvious that ρ_d and ρ_m are functions of t. Figure 1 shows that ρ_d is almost consistent throughout whereas ρ_m decreases in the entire course of evolution, which are acceptable scenarios as the ambiguous DE varies slowly or is unchanged with time [3–6], on the other hand, DM diminishes continuously as a result of the galaxies scattering away from one another during expansion [5]. Moreover, when $t \to \infty$, $\rho_m \to 0$. From these, it would be appropriate to conclude that the universe will be progressively dominated by this cryptic DE. Similar increasing dominant nature of DE can also be seen in [36,82,83].

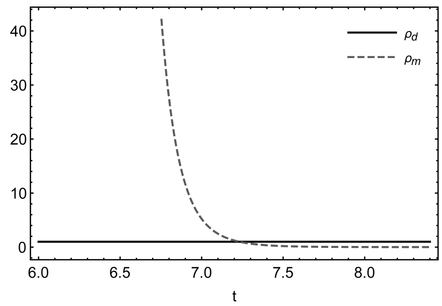

Figure 1. Variation of the energy densities of DE ρ_d and DM ρ_m with t when $a_0 = b_0 = c_1 = c_2 = k = 1, u = 2.78, v = \frac{1}{2}$.

Figure 2 can be regarded as perfect supporting evidence for the present observation of the spatial expansion of the universe. However, at the initial epoch when $t = 0$, $v = 0$. Furthermore, from Figure 3, we can see that θ initially emerges with a large value, decreases with evolution, and finally, tends to become constant after some finite time which is the indication of the Big-Bang scenario [84]. The prediction of a similar scenario with similar cosmological settings can also be seen in [85]. On considering $a_0 = b_0 = c_1 = c_2 = k = 1, u = 2.78, v = \frac{1}{2}$ and assuming the present age of the universe to be $t_0 = 13.8$ Gyr which align with the estimated present age by the most recent Planck 2018 results [24], from Equation (26), the value for EoS parameter is measured to be $\omega = -1$. The Planck 2018 results estimates its value to be $\omega = -1.03 \pm 0.03$ [24]. So, the dark energy candidate we are dealing with is the vacuum energy or the cosmological constant. Moreover, from Figure 1, it can be seen that the dark energy density ρ_d remains almost constant throughout evolution, and from Equation (27), $v \to \infty$ when $t \to \infty$. So, it would be a pertinent fact that the universe has no end; expanding forever, ultimately, leading to the Big Freeze singularity in the far future. In a thermodynamic sense, the model universe will enter a point of minimum temperature and maximum entropy. It will be almost as though all astrophysical process is being smothered, as the fuel for growth and reproduction gets so diffuse that it cannot be used [86]. It will be an ending point characterized by increasing isolation, inexorable decay, and an eons-long fade into darkness [87].

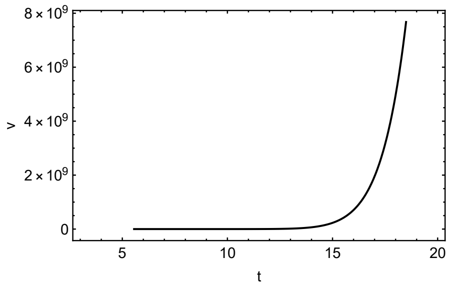

Figure 2. Variation of the spatial volume v with t when $c_1 = c_2 = k = 1, u = 2.78, v = \frac{1}{2}$.

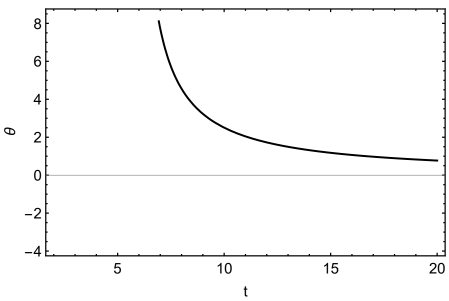

Figure 3. Variation of the expansion scalar θ with t when $c_2 = k = 1, u = 2.78, v = \frac{1}{2}$.

From Figure 4, it is evident that the pressure of DE p_d ranges in the negative plane all through which is in consonance with the mystic property of DE responsible for the accelerated expansion of the universe.

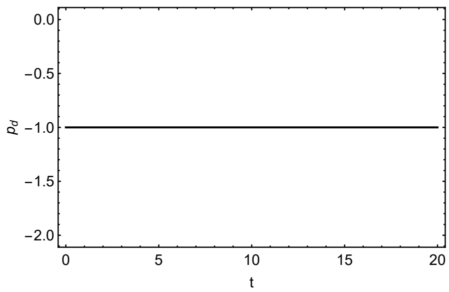

Figure 4. Variation of the DE pressure p_d with t when $b_0 = c_1 = c_2 = k = 1, \omega = -1, u = 2.78, v = \frac{1}{2}$.

From Equation (30), the deceleration parameter q depends on u, v and k. In Table 1, we present different values of q for different values of u, v and k. Recently, Camarena and Marra [88] predict its value as $q = -0.55$, whereas Capozziello et al. [89] estimate the value as $q = -0.644 \pm 0.223$ and $q = -0.6401 \pm 0.187$. With all the values of the constants in Table 1, we obtain the EoS paramter of CC. Since q lies in the range $-1 < q < 0$, the accelerating model universe undergoes exponential expansion [90], in agreement with the present cosmology.

Table 1. Values of deceleration parameter q for different values of u, v and k.

u	v	k	q
2.78	$\frac{1}{2}$	1	−0.64
2.78	$\frac{1}{1.6}$	1	−0.55
2.25	$\frac{1}{2}$	1	−0.55
2.25	$\frac{1}{1.6}$	1.9	−0.54

Figure 5 shows the decreasing nature of Hubble parameter H which is within the limit of the present cosmological scenario [91,92]. Shear scalar σ^2 shows us the rate of deformation of the matter flow within the massive cosmos [93]. The evolution of σ^2 can be seen in Figure 6. It appears to remain constant during the initial epoch, and then it tends to diverge. From these, we can summarize that the model expands with a slow and uniform change of size during the initial evolution, whereas the change becomes faster and faster in late times. This agrees with the present observation of the accelerated expansion of the universe. From Equation (32), the anisotropic parameter $A_h = 0$ for $k = 1$ so that the constructed model is isotropic.

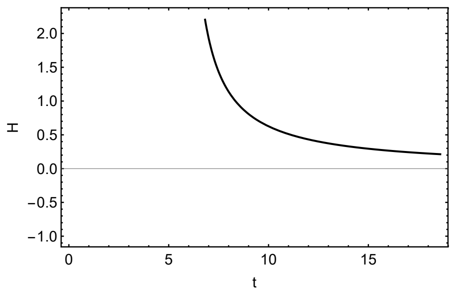

Figure 5. Variation of the Hubble parameter H with t when $c_2 = k = 1, u = 2.78, v = \frac{1}{2}$.

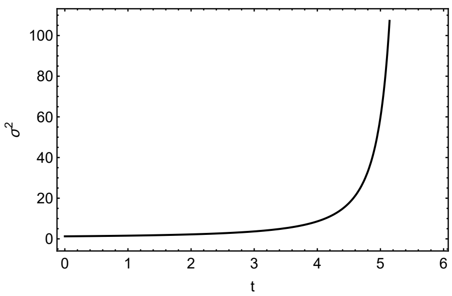

Figure 6. Variation of the shear scalar σ^2 with t when $c_2 = k = 1, u = 2.78$ and $v = \frac{1}{2}$.

Figure 7 shows us the variation of Ω, Ω_d and Ω_m with t. Here, since DE varies slowly or is unchanged with time [3–6], we can see that Ω_d tends to remain constant or increases very slowly. However, Ω_m decreases in the entire course of evolution as a result of the galaxies scattering away from one another leading DM to diminish continuously [5]. Above all, with $k = 1, u = 2.78$ and $v = \frac{1}{2}$, from Equation (35), the overall density parameter is obtained to be $\Omega = 0.97(\approx 1)$. For an exactly flat universe, $\Omega = 1$ [94–96]. Recently, many authors advocate against the belief of an exactly flat universe [94,97–99]. It will be a right conclusion to say that the universe is close to or nearly flat, but not exactly flat [94,99,100]. Above all, the most recent Planck 2018 results [24] obtaining Ω ranging close to unity can be treated as a perfect piece of evidence for a nearly flat universe. Hence, our model obtaining Ω not exactly equal to 1 is justified.

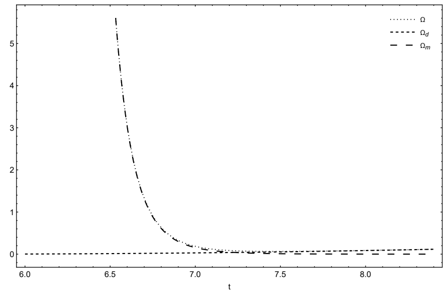

Figure 7. Variation of the overall density parameter Ω, DE density parameter Ω_d and DM density parameter Ω_m with t when $a_0 = b_0 = c_1 = c_2 = k = 1, u = 2.78$ and $v = \frac{1}{2}$.

Lastly, from Equation (38), we can see that the value of the state finder diagnostic pair $\{r, s\} = \{1, 0\}$ which corresponds to the ΛCDM scenario so that the model universe we are considering is a ΛCDM model. Hence, our interacting HDE model can be considered as an alternate cosmological model to the standard ΛCDM model.

4. Stabilization of Extra Dimensions

The study on the stabilization of extra dimensions can be considered as a phenomenological necessity in higher-dimensional models. The discussion on stabilization is mostly confined to particle physics, supersymmetry, supergravity, string theory, and braneworld models. We require a stabilization mechanism to prevent modification of gravity to an experimentally undesirable manner [101]. The stabilization also makes sure the visible 4D universe with a long lifetime [102]. Another benefit of stabilization is that we can ignore any unwanted outcomes of quantum gravity at Planck length distances [103]. One of the most classic solutions for stabilization is the Goldberger–Wise mechanism [104], where stabilization is achieved in the presence of an additional scalar field. In [105], the authors claim that stabilization can be achieved by introducing a potential of the dilaton field. In [106], we can witness a study of an isotropic 3-brane model where stabilization is achieved with the only value of the EoS $\omega(t) = -\frac{2}{3}$. Another observation of stabilization in an isotropic perfect fluid model in 5D with the value of EoS $\omega > -\frac{1}{3}$ can be seen in [107]. In [108], the authors show that the issue of stabilization can be overcome in a theory of gravity involving high-order curvature invariants. The author in [109] obtains stabilization by quantum corrections from massive matter. In [110], we can find the investigation of a class of dilatonic STT where stabilization is achieved by quantum corrections to the effective 4D Ricci scalar. In [111], we can witness an argument calming that stabilization is attained as soon as inflation ends, on the contrary, the authors in [112] assert that inflation ends if stabilization is attained. According to [113], to achieve a realistic theoretical model, we should assume that the visible three dimensions are expanding isotropically, whereas the extra dimensions are contracting (or contracted for a period during the evolution). Similarly, the authors in [114], predict that the extra dimension contracts with the cosmic time. In [115], the hidden extra dimension is related to scalar fields. The work in [116] also represents the size of the extra dimensions in terms of a scalar field. In [117], the authors investigate 4D gauge theories that dynamically generate a 5D, where stabilization is no longer needed. In their works [118], Tosa studies the Kaluza–Klein cosmology for a torus space with a cosmological constant and matter. He predicts that the number of the extra dimensions should be more than 1, and the extra dimensions should be of small size. However, during recent years, many authors have successfully predicted models with just one extra dimension, where stabilization is obtained [119–124]. Additionally, we can also witness large extra dimensions in [125–127], and infinite-volume extra dimensions in the fourth paragraph of this section.

In our work, we have discussed a 5D spherically symmetric cosmological model in general relativity (GR) with the cosmological constant (CC), or in other words, vacuum energy (VE) as the DE candidate. In GR, generally, we cannot find conditions for stabilization, and all dimensions want to be dynamical [116]. In [128], it is mentioned that in an accelerating model with CC, stabilization cannot be obtained. Therefore, in a trial to solve the stabilization problem in GR, we consider two options. The first one is the Casimir energy and the second is the infinite-volume extra dimension, which are discussed below.

Casimir energy is a DE candidate with the ability to drive the late-time accelerated expansion and stabilize the extra dimensions automatically [124,129]. Casimir energy is VE emerging from imposing boundary conditions on the quantum fluctuations of fields and the EoS's of both Casimir energy and CC are of the same form [124]. Further in [124], we can see the interpretation of Casimir energy as CC. Additionally, the author in [130] equates VE with Casimir energy. In [131], Casimir energy is identified with CC. If the CC is to be created from the Casimir energy, then there will be only one extra dimension [132]. Coincidently, in our spherically symmetric cosmological model with the CC as the DE candidate, there is only one extra dimension.

The study on extra dimensions has been widely considered in braneworld models [133–141], one of which is the DGP model [142], which presents an accelerating 5D scenario with an infinite-volume extra dimension. This infinite-volume extra dimension drives the expedited expansion of the universe at late times [143]. The authors in [144,145] assert that with an infinite-volume extra dimension, one does not need stabilization. They further claim that the infinite-volume scenario can explain to us the late time cosmology and the acceleration of the universe driven by DE, which are one of the core components of GR. According to [146], infinite-volume extra dimensions might result in the emergence of DE. Hence, it would be appropriate to conclude that the extra dimension in our study on 5D spherically symmetric cosmological model is of infinite-volume.

One of the most classic solutions for stabilization is the Goldberger–Wise (GW) mechanism [104]. We can witness the application of the GW mechanism in the field of string theory, M-Theory, and Randall and Sundrum (RS) model in the noteworthy works of [147–152]. In these works, the authors consider a 5D static metric with a 4D Poincare symmetry. To obtain stability, they introduce the proper distance and a massive scalar field and show that the effective radion potential has a minimum. Since the Casimir energy (force) provides a natural alternative to the GW mechanism [153], the stabilization mechanism applied in [147–152] might have some sort of relationship with the Casimir energy stabilization approach which we have predicted above. Above all, one may consider it as an advantage above the GW mechanism that the introduction of an ad hoc classical interaction between the branes is not needed in the Casimir energy approach of stabilization [153]. We may note the work in [154] predicting that the Casimir force will not lead to stabilization to the right value unless a tuning of parameters. Fortunately, the work in [153] shows that this conclusion of [154] is not general, and proves that Casimir energy (force) provides a natural alternative to the GW mechanism in the RS model. There might be more advantages or relationships of our predicted stabilization approaches with the GW mechanism, which we would like to find out in our future works.

We have presented two conditions for stabilization of extra dimensions in GR. Probably, our work might be the first to predict such conditions in GR. Nevertheless, these two conditions are toy models which require further in-depth analysis considering different cosmological aspects. We need more investigation on the reliability of considering, within GR, the identification of Casimir energy with cosmological constant, or in other words, vacuum energy. We also need to verify all the possible outcomes of assuming the extra dimension is of infinite volume in a higher-dimensional vacuum energy model within GR.

5. Conclusions

We have analyzed a cosmological model in spherically symmetric space-time in a 5D setting with minimally interacting matter and HDE in SBT. We predict that the expanding isotropic universe will be progressively DE dominated. The pressure of DE is negative all through. We estimate few values of the deceleration parameter and the values are found very close to the recently predicted values. The Hubble parameter H decreases which agrees with the present cosmological scenario. In the initial epoch, the model universe expands with a very slow and uniform change of shape, but after some finite time, the change becomes faster. Then, it again tends to become very slow and uniform after expanding without any deformation for a finite period. The value of the DE EoS parameter is measured to be $\omega = -1$ indicating that the DE we are dealing with is the vacuum energy or the cosmological constant. The value of the overall density parameter is obtained as $\Omega = 0.97(\approx 1)$, which is not exactly equal to 1, since the universe is close to or nearly flat, but not exactly flat. We observe that the model universe starts with the Big-Bang and ends at the Big Freeze singularity. The value of the state finder diagnostic pair obtained corresponds to the ΛCDM model so that our interacting HDE model can be considered as an alternate cosmological model to the standard ΛCDM model. Lastly, we present two conditions to solve the stabilization problem of extra dimension in GR, the first one is the identification of Casimir energy with cosmological constant, or in other words, vacuum energy and the second is assuming the extra dimension is of infinite volume. Nevertheless, these two conditions are toy models which require further in-depth analysis considering different cosmological aspects.

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Higher dimensional phantom dark energy model ending at a de-Sitter phase

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ABSTRACT

In this work, using a spherically symmetric metric, we investigate a minimally interacting holographic dark energy model within the framework of the Saez-Ballester Theory. We predict that the dark energy component dominating the universe is of phantom type, which will lead the model universe to cosmic doomsday (big rip singularity). As the big rip singularity and holographic dark energy are incompatible with each other, we employ a higher dimensional scenario so that the cosmic doomsday is replaced by a de-Sitter phase. The model expands with a slow and uniform change of size during the early evolution, whereas the change becomes faster and faster, agreeing with the present observation of the accelerated expansion. The present values of the Hubble parameter and the dark energy EoS parameter are found to be $H = 67$ and $\lambda = -1.00011$, which agree with the respective values $H_0 = 67.36 \pm 0.54$ kms^{-1}Mpc^{-1}and $\lambda = -1.03 \pm 0.03$ of the most recent Planck 2018 result.

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1. Introduction

Since the profound discovery of dark energy (DE) [1, 2] in 1998, theoretical physicists and cosmologists consider it as one of the most important topics in modern cosmology due to its mysterious nature with a huge negative pressure being responsible for the universe expanding at an expedited rate. This cryptic component is considered to be uniformly permeated and vary slowly or remain unchanged with time [3–6]. With a focus on investigating its nature and application to modern cosmology, cosmologists have made tremendous scientific efforts and are still scrabbling for a perfect answer. From the literature and observations, DE is believed to dominate the massive universe. This qualifies DE as ironically being the dominating component of nature, while also being the least explored. Some studies worth mentioning on this enigmatic dark component of the universe in the last decade are briefly presented in the next paragraph.

In [7], the evolutionary nature of DE is presented. The discussion on the evolution of DE given the latest observational findings is presented in [8]. In [9], the author studied the decaying nature of DE into photons. In [10], the authors obtained DE from a violation of energy conservation. The quantum-mechanical calculation of the DE density is presented in [11]. In [12], the author predicts that galaxies cluster due to the stirring effect of DE. A theoretical investigation of DE as the solution of global warming is presented in [13].

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The atomic-interferometry constraints on DE are discussed in [14]. In [15], the author asserts that the presence of particles with imaginary energy density can lead us to the source of DE. In [16], the author reviews the status of the experimental data on DE. The need for DE with thermodynamic arguments is provided in [17]. Lastly, we can find an interesting study comparing the speed of DE with that of the photon in [18].

To precisely understand the underlying mechanism of the late time accelerated expansion of the universe, cosmologists have adopted two well-appreciated methods. Firstly, different possible forms of DE are developed. Secondly, modifying Einstein's theory of gravitation. Other than these two, many authors have successfully adapted other fascinating ways to explain the amazing expanding phenomenon. In [19], a compelling attempt describing the expanding phenomenon in the absence of DE can be seen. In [20], it is mentioned that acceleration is automatically induced by the Delta Gravity equations, rather than DE. In [21–23], the authors try directly modifying the Friedmann equation empirically to explain the phenomenon. An approach is mentioned in [24], in which the accelerating paradigm is explained by an Ultra Violet Complete Theory. A convincing illustration of the expanding phenomenon by matter only can be seen in [25].

One of the possible forms of DE which have not escaped our attention is holographic dark energy (HDE), introduced by Gerard 't Hooft [26]. It is obtained by the application of the holographic principle [27] to DE. Accordingly, all the physical quantities inside the universe including the energy density of DE can be described by some quantities on the boundary of the universe [28]. Models involving the interaction of HDE and dark matter (DM) or interacting holographic dark energy (IHDE) models are considered to be of paramount importance by many authors. A discussion on an expanding interacting HDE and DM model can be found in [29], where the DE component decays into pressureless DM. In [30], the author discusses an IHDE model asserting that, at present, the universe is dominated by quintessence DE and it will become phantom DE dominated in the near future. In [31], the authors study a minimally IHDE model in a scalar-tensor theory of gravitation experiencing cosmic re-collapse. In [32], the author investigates an IHDE model undergoing accelerated expansion ending at the big rip singularity. In [33], the authors investigate a minimally interacting matter and HDE model with a discussion of singularities and predicting that their model universe expands with the fastest rate and the largest value of the Hubble parameter. The authors in [34] consider an isotropic minimally IHDE model in a Bianchi type-III universe exhibiting early inflation and late-time acceleration. In [35], a minimally IHDE model is discussed in Brans-Dicke theory, where the DE turns out to be of phantom type. In [36], we can witness an IHDE model expanding spatially with a constant overall density parameter. In [37], we can witness an IHDE model free from any initial singularity attaining isotropy at late times. In the last few years, strong arguments have been brought to light asserting that modified gravity can be explained by employing a DE-DM interaction in the Einstein frame [38–42]. Due to the fascinating nature of such interacting models, many fundamental questions have arisen pointing out that there is a lot more physics still undiscovered.

Saez-Ballester Theory (SBT), introduced by Saez and Ballester [43], can be considered to be a very good option for studying DE and the accelerating universe. It is a member of the family of Scalar Tensor Theories (STT) of gravitation. In SBT, the metric potentials are coupled with a scalar field φ. Scalar fields are considered to play key roles in gravitation and cosmology, as they can account for remarkable phenomena like DE, DM, etc. [44]. They can be regarded as a possible contributing factor in the late time acceleration of the universe [45]. STT are a direct generalization and extension of general relativity [46]. STT can be considered as perfect candidates for DE [47]. STT also play a key role in getting rid of the graceful exit problem in the inflationary period [48]. In [49], it is asserted that a scalar field might be responsible for the inflation at the initial epoch. Currently, SBT and general relativity are held to align with observation.

Recently, there has been a growing interest among cosmologists in exploring the DE-DM interaction in the SBT setting. In [50], the authors study an interacting HDE and DM model in SBT, where the expanding model starts with a big bang. The construction of an interacting new HDE model in the framework of SBT can be found in [51]. An investigation of an IHDE model in SBT can be seen in [52], where the authors use a hybrid expansion law and predict a transitioning universe. Reddy [53] investigated an IHDE model in SBT, thereby obtaining three cosmological models. In [54], the observation of an IHDE model in SBT obtaining a transitioning model due to cosmic re-collapse can be seen. Shaikh et al. [55] discussed a model with matter and a modified holographic Ricci DE in SBT. Lastly, in [56], the authors investigate a modified holographic Ricci DE with matter in SBT predicting a quintom-like universe.

The possibility of space-time having more than 4D has fascinated many authors. A higher-dimensional cosmological model was introduced by Kaluza and by Klein [57, 58]. Such models are useful for describing the late time expanding paradigm [59]. The investigation of higher dimensions can be considered as an important task, as the universe might have encountered a higher dimensional phase during its early evolution [60]. According to [61, 62], the additional dimensions might provide an explanation for the flatness and horizon problem. [63], we can witness a study that discusses the evidence for the existence of additional dimensions. Lastly, in [64], the authors assert that the hidden extra dimensions in 5D might correspond to the unknown DE and DM.

We have presented a brief highlight on the past related works [29–37, 50, 51–56] on the IHDE model. Keeping in mind these notable works and the other noteworthy studies mentioned above, we consider a DM-DE interaction in SBT considering a 5D spherically symmetric space-time. In this work, we present a detailed discussion on every cosmological parameter obtained. The definition of the shear scalar and its physical significance are provided. The incompatibility of a big rip singularity with HDE and its elimination in the phantom DE scenario by a de-Sitter phase is discussed. Additionally, we calculate the present values of the Hubble parameter and the dark energy EoS parameter. To obtain realistic results, we make assumptions in accordance with present-day cosmology. The paper is divided into sections. After the introduction, in Sect. 2, we present problem formulations with solutions of the cosmological parameters. In Sect. 3, the solutions are discussed with graphs with the consideration of the recent findings. Lastly, as a summary, concluding remarks are provided in Sect. 4.

2. Formulation of problems with solutions

We start with a consideration of the spherically symmetric metric in 5D as given below

$$ds^2 = dt^2 - e^{\mu}\left(dr^2 + r^2 d\Theta^2 + r^2 \sin^2\Theta d\Phi^2\right) - e^{\delta} dy^2, \tag{1}$$

where μ and δ are cosmic scale factors which are functions of time only.

The Saez-Ballester field equations are given by

$$R_{ij} - \frac{1}{2}g_{ij}R - \omega\varphi^n\left(\varphi_{,i}\varphi_{,j} - \frac{1}{2}g_{ij}\varphi_{,k}\varphi^{,k}\right) = -\left(T_{ij}^m + T_{ij}^{de}\right), \tag{2}$$

where T_{ij}^m and T_{ij}^{de} are the energy momentum tensors for matter and HDE respectively, R and R_{ij} are respectively the Ricci scalar and tensors. The scalar field φ satisfies

$$2\varphi^n\varphi_{;i}^{,i} + n\varphi^{n-1}\varphi_{,k}\varphi^{,k} = 0, \tag{3}$$

where n is an arbitrary constant.

T_{ij}^m and T_{ij}^{de} are given by

$$T_{ij}^m = \rho_m u_i u_j, \tag{4}$$

$$T_{ij}^{de} = (\rho_{de} + p_{de})u_i u_j - g_{ij}p_{de}, \tag{5}$$

where ρ_m and ρ_{de} represent the energy density of matter and HDE respectively, whereas p_{de} is the pressure of the HDE.

By conservation of energy, we have

$$T_{;j}^{m^{ij}} + T_{;j}^{de^{ij}} = 0. \tag{6}$$

Using the co-moving coordinate system, the surviving field equations are obtained as

$$\frac{3}{4}\left(\dot{\mu}^2 + \dot{\mu}\dot{\delta}\right) + \frac{\omega}{2}\varphi^n\dot{\varphi}^2 = \rho, \tag{7}$$

$$\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 + \frac{\ddot{\delta}}{2} + \frac{\dot{\delta}^2}{4} + \frac{\dot{\mu}\dot{\delta}}{2} - \frac{\omega}{2}\varphi^n\dot{\varphi}^2 = -p_{de}, \tag{8}$$

$$\frac{3}{2}\left(\ddot{\mu} + \dot{\mu}^2\right) - \frac{\omega}{2}\varphi^n\dot{\varphi}^2 = -p_{de}. \tag{9}$$

From Eq. (6), we have

$$\ddot{\varphi} + \dot{\varphi}\left(\frac{3\dot{\mu} + \dot{\delta}}{2}\right) + \frac{n}{2}\dot{\varphi}^2\varphi^{-1} = 0, \tag{10}$$

where an overhead dot represents differentiation w.r.t. t.

We assume λ as the EoS parameter of the DE, and hence we have

$$p_{de} = \lambda\rho_{de}. \tag{11}$$

The conservation equation takes the obvious form, as given by

$$\rho_m\left(\frac{3\dot{\mu} + \dot{\delta}}{2}\right) + \dot{\rho}_m + \dot{\rho}_{de} + \rho_{de}(1+\lambda)\left(\frac{3\dot{\mu} + \dot{\delta}}{2}\right) = 0. \tag{12}$$

Due to their minimal interaction, HDE and matter conserve separately, so that by [65–66], Eq. (12) can be written as

$$\rho_m\left(\frac{3\dot{\mu} + \dot{\delta}}{2}\right) + \dot{\rho}_m = 0, \tag{13}$$

$$\rho_{de}(1+\lambda)\left(\frac{3\dot{\mu} + \dot{\delta}}{2}\right) + \dot{\rho}_{de} = 0. \tag{14}$$

Also, we have

$$(\rho + p)\left(\frac{3\dot{\mu} + \dot{\delta}}{2}\right) + \dot{\rho} = 0. \tag{15}$$

From Eqs. (8) and (9), the expression for the cosmic scale factors are obtained as

$$\mu = l_1 - \log(k-t)^{\frac{2}{3}}, \tag{16}$$

$$\delta = m_1 - \log(k-t)^{\frac{2}{3}}, \tag{17}$$

where l_1, m_1 and k are arbitrary constants.

Now, from Eqs. (13), (14), (16) and (17), we have

$$\rho_m = l_0 e^{-\frac{1}{2}(3l_1+m_1)}(k-t)^{\frac{4}{3}}, \tag{18}$$

$$\rho_{de} = m_0 e^{-\frac{1}{2}(1+\lambda)(3l_1+m_1)}(k-t)^{\frac{4}{3}(1+\lambda)}, \tag{19}$$

so that the energy density of our universe is given by

$$\rho = \rho_m + \rho_{de}, \tag{20}$$

where l_0 and m_0 are arbitrary constants.

Again, using Eqs. (16), (17) and (20) in Eq. (15), the pressure of our model universe is obtained as

$$p = \frac{1}{3} l_0 e^{-\frac{1}{2}(3l_1+m_1)}(k-t)^{\frac{4}{3}} + m_0\left(\frac{4\lambda+1}{3}\right) e^{-\frac{1}{2}(1+\lambda)(3l_1+m_1)}(k-t)^{\frac{4}{3}(1+\lambda)}. \tag{21}$$

From Eqs. (11) and (19), the pressure of the DE is given by

$$p_{de} = \lambda m_0 e^{-\frac{1}{2}(3l_1+m_1)(1+\lambda)}(k-t)^{\frac{4}{3}(1+\lambda)}. \tag{22}$$

At any time $t = t_0$, we can assume that $p = p_{de}$ so that, from Eqs. (21) and (22), we have

$$l_0 e^x (k-t_0)^{\frac{4}{3}} + m_0(1+\lambda)e^{(1+\lambda)x}(k-t_0)^{\frac{4}{3}(1+\lambda)} = 0, \tag{23}$$

where $x = -\frac{1}{2}(3l_1 + m_1)$.

Eq. (23) will provide the expression for the EoS parameterλ.

Now, using Eqs. (16) and (17) in Eq. (10), the Saez-Ballester scalar field φ is obtained as follows:

$$\varphi = \left((6+3n)(k-t)^{\frac{7}{3}} - 14c_1\right)^{\frac{2}{2+n}} c_2, \tag{24}$$

where c_1 and c_2 are arbitrary constants.

Finally, the expressions of the different cosmological parameters are obtained as follows.

Spatial volume:

$$v = e^{\frac{3l_1+m_1}{2}}(k-t)^{-\frac{4}{3}} \tag{25}$$

Scalar expansion:

$$\theta = \frac{4}{3}(k-t)^{-1} \tag{26}$$

Hubble's parameter:

$$H = \frac{1}{3}(k-t)^{-1} \tag{27}$$

Shear scalar:

$$\sigma^2 = \frac{2}{9}\left(\frac{1}{k-t} - 1\right)^2 \tag{28}$$

Anisotropic parameter:

$$A_h = 0 \tag{29}$$

Dark energy density parameter:

$$\Omega_{de} = \frac{\rho_{de}}{3H^2} = 3m_0 e^{-\frac{1}{2}(1+\lambda)(3l_1+m_1)}(k-t)^{\frac{2}{3}(5+2\lambda)} \tag{30}$$

Matter density parameter:

$$\Omega_m = \frac{\rho_m}{3H^2} = 3l_0 e^{-\frac{1}{2}(3l_1+m_1)}(k-t)^{\frac{10}{3}} \tag{31}$$

Overall density parameter:

$$\Omega = 3\left(l_0 e^{-\frac{1}{2}(3l_1+m_1)} + m_0 e^{-\frac{1}{2}(1+\lambda)(3l_1+m_1)}(k-t)^{\frac{4\lambda}{3}}\right)(k-t)^{\frac{10}{3}} \tag{32}$$

Jerk parameter:

$$j(t) = 28. \tag{33}$$

3. Discussion

For convenience sake and to obtain realistic results, in this section, we choose fixed values of the arbitrary constants appearing in the solutions, i.e., $l_0 = l_1 = m_0 = m_1 = 1,\ k = 13.80497512437811$.

From Fig. 1, we can observe the decreasing nature of ρ_m, whereas ρ_{de}remains constant throughout. From Fig. 2, we can see that $\Omega$and Ω_{de}tend to become constant after decreasing for a finite period, whereas $\Omega_m$continues to decrease to a larger extent. It may be noted that due to the expansion, galaxies move apart from each other, leading the DM density to diminish gradually [5], whereas DE varies slowly or remains unchanged with time [3, 4, 5, 6]. From these, we have obtained a model, which is DE-dominated, similar to that predicted in [4, 29, 67, 68, 69, 70, 71, 72].

Fig. 3 shows the variation of the time-dependent EoS parameter $\lambda$with cosmic time t. Here, it can be seen that λ starts evolving from the aggressive phantom region and tends to come very close to -1, which aligns with the recent studies [73, 74]. Similar observations of HDE with phantom-like nature can also be seen in the recent works [75, 76]. However, as λappears to evolve due to its time dependence, it attains the value $\lambda = -1$ during evolution [44, 77]. Above all, a phantom model with $\lambda < -1$ should reduce to $\lambda = -1$ in the far future to ensure that cosmological models bypass a future singularity (big rip) thereby, ultimately, leading to the de-Sitter phase [73, 78]. It can also note that in the HDE setting, the big rip singularity is not permitted, because the Planck scale excursion of the UV cutoff in the effective field theory is forbidden, so that the occurrence of the big rip would ruin the theoretical foundation of the HDE scenario [79]. This issue can be solved by employing an extra dimension in the HDE setting, and also the employment of an extra dimension makes HDE models more complete and consistent. The mechanism of replacing a big rip singularity by a de-Sitter phase with the employment of an extra dimension (higher dimension) in the HDE setting can be seen in [79]. Other discussions on replacing a big rip singularity by de-Sitter phase can be seen in [80, 81, 82]. According to the latest Planck 2018 result [83], the present age of the universe is 13.825 ± 0.037 Gyr. With $t_0 = 13.8$Gyr and assuming $l_0 = l_1 = m_0 = m_1 = 1,\ k = 13.80497512437811$, from Eq. (23), the present value of the EoS parameter is obtained to be $\lambda = -1.00011$, which aligns with the value $\lambda = -1.03 \pm 0.03$ of the latest Planck 2018 result [83].

Fig. 4 and Fig. 5 can be considered as the perfect pieces of evidence for the spatial expansion of the universe at an expedited rate. At $t = 0,\ v$ and other related parameters are constant, which indicates that the model doesn't evolve from an initial singularity, whereas, as discussed before, the future big rip singularity is replaced by the de-Sitter phase.

From Fig. 6, it is obvious that the graph of the pressure of DE p_{de} lies in the negative plane during the entire course of evolution, which aligns with the characteristic property of DE, which accounts for the accelerated expansion. From Fig. 7, it is clear that the Hubble parameter $H$of the model universe tends to remain almost constant during the early evolution, so that the model was in an inflationary epoch experiencing rapid exponential expansion [84]. The latest Planck 2018 result [83] estimates the present age of the universe to be 13.825 ± 0.037Gyr. Assuming $t = 13.8$ and $k = 13.80497512437811$, from Eq. (27), the value of Hubble's parameter is measured to be $H = 67$, approximately equal to the value $H_0 = 67.36 \pm 0.54$ kms^{-1}Mpc^{-1}of the latest Planck 2018 result [83].

Fig. 8 shows us the variation of σ^2 with cosmic time t. Initially, σ^2 appears to decrease negligibly, and then it tends to diverge. σ^2

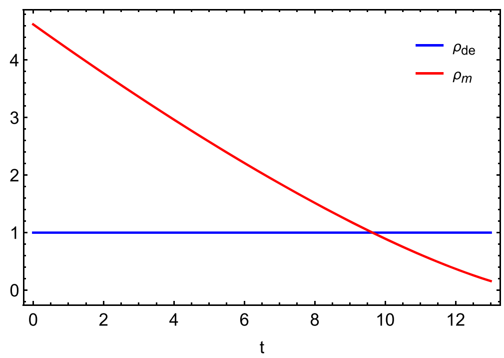

Fig. 1. Energy densities of DE $\rho_{de}$and DM ρ_m with t when $l_0 = l_1 = m_0 = m_1 = 1,\ k = 13.80497512437811$.

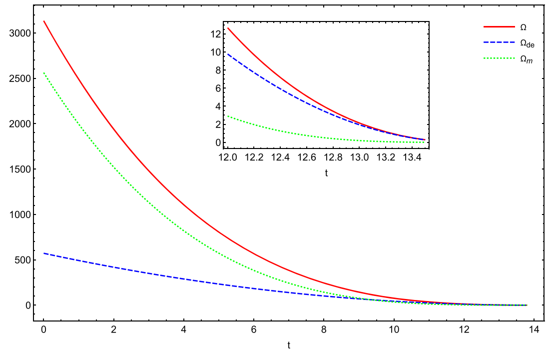

Fig. 2. Overall density parameter Ω, DE density parameter Ω_{de} and DM density parameter $\Omega_m$with t when $l_0 = l_1 = m_0 = m_1 = 1,\ k = 13.80497512437811$.

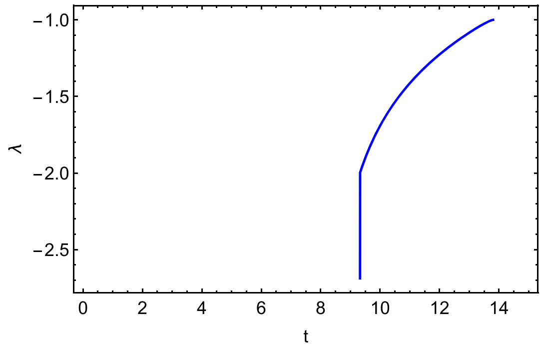

Fig. 3. EoS parameter λ with t when $l_0 = l_1 = m_0 = m_1 = 1,\ k = 13.80497512437811$.

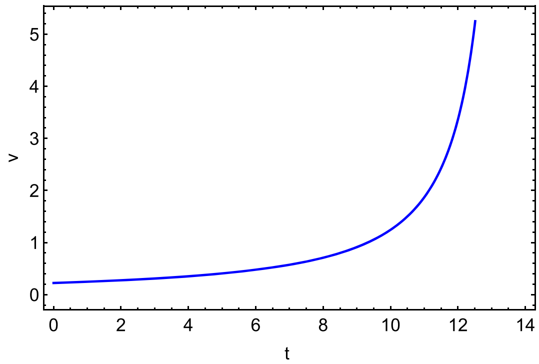

Fig. 4. Spatial volume v with t when $l_1 = m_1 = 1,\ k = 13.80497512437811$.

shows us the rate of deformation of the matter flow within the massive cosmos [85]. From Eq. (29), the anisotropic parameter $A_h = 0$. So, we can sum up that the universe is isotropic and expands with a slow and uniform change of size in the early evolution, whereas the change tends to become faster at late times. This is in agreement with the present observation of the accelerated expansion of the universe.

Fig. 9 shows the variation the Saez-Ballester scalar field $\varphi$with cosmic time t when $c_1 = c_2 = 1$ (both $c_1,\ c_2 > 0$), whereas Fig. 10 shows the variation when $c_1 = -1,\ c_2 = 1$ ($c_1 < 0, c_2 > 0$). In both cases, a real value of φ can't be obtained for $n = -2$. In Fig. 9, we can see the decreasing nature of φ. However, when $n = -1$, it decreases to a minimum value and then increases to attain a constant positive value. It decreases to become negative when $n = 0$. In Fig. 10, φ decreases and is positive throughout. When $n = -1$, it tends to attain a constant positive value after decreasing for a finite time. When $n = -3$, it attains its maximum and minimum values during

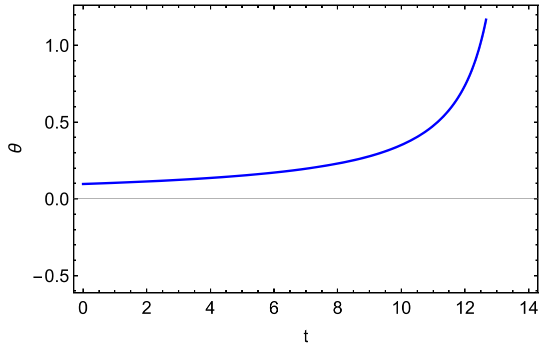

Fig. 5. Scalar expansion θ with t when $k = 13.80497512437811$.

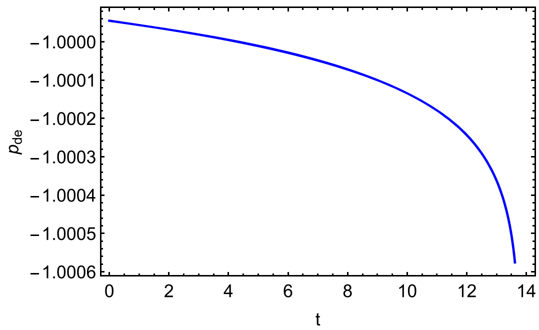

Fig. 6. DE pressure p_{de} with t when $l_1 = m_0 = m_1 = 1,\ k = 13.80497512437811$.

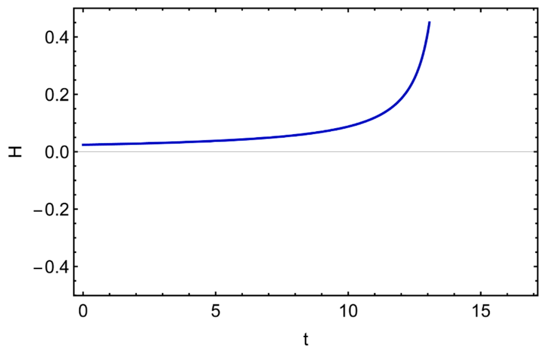

Fig. 7. Hubble's parameter $H$with t when $k = 13.80497512437811$.

the evolution. Hence, in both cases, when $n = -1$, φ tends to attain almost the same large positive constant, which might be the reason for the phantom-like nature of the DE at present. This observation is somewhat similar to that of [86] where, after both increasing and decreasing, the scalar field tends to attain a positive constant value.

Lastly, from Eq. (33), the value of the jerk parameter is obtained to be $j(t) = 28$. It can be used as a tool to describe the closeness of models to the standard ΛCDM model. Its value for the standard ΛCDMmodel is $j(t) = 1$.

4. Conclusions

In this work, we have investigated an interacting model of HDE and matter in a spherically symmetric space-time in a 5D setting within the framework of SBT. We have obtained an accelerating model where the HDE with phantom-like nature dominates the universe. The model doesn't evolve from an initial singularity. To preserve the theoretical foundation of the HDE scenario, an extra

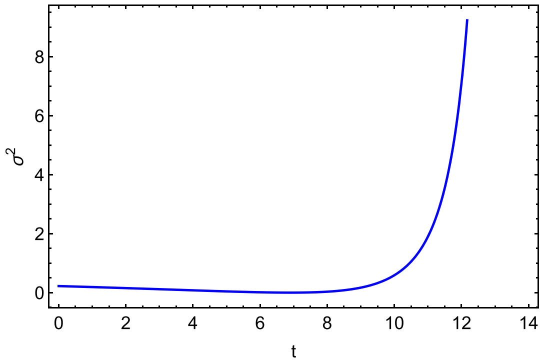

Fig. 8. Variation of the shear scalar σ^2 with t when $k = 13.80497512437811$.

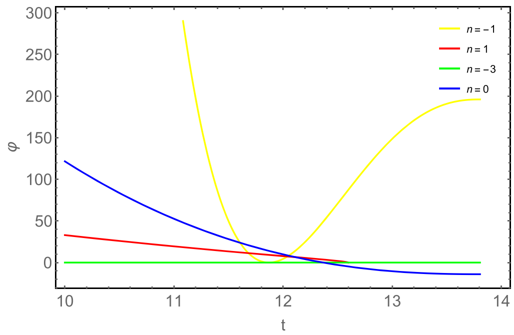

Fig. 9. Variation of the Saez-Ballester scalar field φ with t when $c_1 = c_2 = 1$.

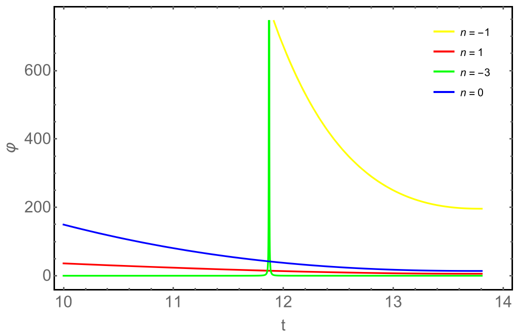

Fig. 10. Variation of the Saez-Ballester scalar field φ with t when $c_1 = -1, c_2 = 1$.

dimension is employed. In the far future, the DE departs from phantom-like nature to a cosmological constant, thereby bypassing a future singularity and ultimately leading to the de-Sitter phase. The universe is predicted to be isotropic. At $t = 13.8$Gyr, the approximate present age of the universe, the values of the Hubble parameter and the DE EoS parameter are found to be $H = 67$ and $\lambda = -1.00011$, which agree with the respective values $H_0 = 67.36 \pm 0.54\,\mathrm{kms}^{-1}\mathrm{Mpc}^{-1}$and $\lambda = -1.03 \pm 0.03$ of the latest Planck 2018 result [83]. It is predicted that the model expands with a slow and uniform change of size in the early evolution, whereas the change tends to become faster at late times. We observe that when $n = -1$, the Saez-Ballester scalar field φ tends to attain a positive constant value in the course of evolution.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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A higher dimensional cosmological model for the search of dark energy source

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With due consideration of reasonable cosmological assumptions within the limit of the present cosmological scenario, we have analyzed a spherically symmetric metric in 5D setting within the framework of Lyra manifold. The model universe is predicted to be a DE model, dominated by vacuum energy. The model represents an oscillating model, each cycle evolving with a big bang and ending at a big crunch, undergoing a series of bounces. The universe is isotropic and undergoes super-exponential expansion. The value of Hubble's parameter is measured to be $H = 67.0691$ which is very close to $H_0 = 67.36 \pm 0.54\,\mathrm{km\,s^{-1}Mpc^{-1}}$, the value estimated by the latest Planck 2018 result. A detailed discussion on the cosmological parameters obtained is also presented with graphs.

Keywords: Spherically symmetric; five-dimension; Lyra manifold; dark energy.

Mathematics Subject Classification 2020: 83CXX, 83F05, 83C56

1. Introduction

The enigmatic dark energy (DE) has gained the reputation of being one of the most discussed topics of paramount importance in cosmology since its profound discovery in 1998 [1, 2]. Its property with a huge negative pressure with repulsive gravitation causing the universe to expand at an expedited rate still remains a mystery. It is believed to uniformly permeate throughout the space and vary slowly or almost

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consistent with time [3–6]. Cosmologists around the world have invested tremendous scientific efforts with a strong focus to hunt its origin and are still scrabbling for a perfect answer. Different authors have put forward their own versions of the answer with convincing evidences in support. Some worth mentioning studies on the search of the root of DE which have not escaped our attention in the past few years are briefly discussed below.

In [7], the authors assert that emergent D-instanton might lead us to the root of DE. Scalar field theory as dark energy of the universe is presented in [8]. In [9], the authors investigate the evolution of the DE using a non-parametric Bayesian approach in the light of the latest observation. In [10], the author claims that vacuum condensate can indicate us the origin of DE. A study on neutrino mixing as the origin of DE is presented in [11]. In [12], the explanation for DE with pure quantum mechanical method is presented. In [13], it is mentioned that DE emerges due to condensation of fermions formed during the early evolution. The explanation of a physical mechanism as a source of DE can be seen in [14]. A cosmological model involving an antineutrino star is proposed in [15] in an attempt to find the origin of DE. In [16], DE is obtained from the violation of energy conservation. A unified dark fluid is obtained as a source of DE in [17]. Finally, in [18], the author claims that the presence of particle with imaginary energy density can lead us to the source of DE.

From literatures and observations [4, 19–25], it is obvious that the massive universe is dominated by the mystic DE with negative pressure and positive energy density. This qualifies DE a completely irony of nature as the dominating component is also the least explored. So, there a lot more hidden physics behind this dark entity yet to be discovered. Contradicting the condition of positive energy density, it is surprising that many authors have come up with the notion of negative energy density (NED) with convincing and fascinating arguments in support. In [4], the authors predicted a condition in which NED is possible only if the DE is in the form of vacuum energy. Besides defying the energy conditions of GR, NED also disobeys the second law of thermodynamics [26]. However, the condition should be solely obeyed on a large scale or on a mean calculation, thereby neglecting the probable violation on a small scale or for a short duration, in relativity [27–35]. Hence, in the initial epoch, if there were circumstances of defiance for a short duration measured against the present age which is estimated to be 13.825 ± 0.037 Gyr by the latest Planck 2018 result [36], it will remain as an important part in the course of evolution. In [37], under certain conditions, a repelling negative gravitational pressure can be seen with NED. It further mentions of a repelling negative phantom energy with NED. Energy density assuming negative value with equation of state (EoS) parameter $\omega < -1$ can be found in [38]. In [39], the author asserts that the universe evolves by inflation when the coupled fluid has NED in the initial epoch. The authors in [40] discuss negative vacuum energy density in Rainbow Gravity. According to [41], the introduction of quantized matter field with NED to energy momentum tensor might by pass cosmological singularity. In [42], the author investigates models which evolved with NED in the infinite past.

In [43], NED is discussed where the authors assert that their models evolve with a bounce. The authors continued that there might be bounces in the future too. Finally, an accelerating universe with NED is studied in [44].

The sessions of the Prussian Academy of Science during the four Thursdays of the month of November 1915 can be marked as the most memorable moments in the life of the great Einstein. On 4th, 11th, 18th and 25th of the month, he presented four of his notable communications [45–48] at the sessions, which led to the foundation of GR. Since then, a number of authors have been exploring gravitation in different settings. In Weyl's work of 1918 [49], we can witness the first trial to extend GR with the aim of bringing together gravitation and electromagnetism geometrically. Similar to that of Weyl, Lyra's modification [50] by proposing a gauge function into the structureless manifold provides one of the well-appreciated alternate or modified theories of gravitation. The static model with finite density in Lyra's modified Riemannian geometry is similar to the static Einstein model [51]. The scalar-tensor treatment based on Lyra's geometry yields the same effects, as GR, under certain limits [52]. Lyra's work is further extended by other well-known authors [53–57]. Cosmologists choose to opt alternate or modified theories of gravitation in order to precisely understand the underlying mechanism of the late time expedited expansion of the universe. Many other authors too have succeeded in developing fascinating and worth appreciating modified theories which have served the purpose of explaining the expanding paradigm in a quite convincing way [58–66].

During the past few years, there has been an increasing interest among cosmologists to study the ambiguous DE paired with Lyra Manifold (LM). Recently, in [67], the authors investigate the existence of Lyra's cosmology with interaction of normal matter and DE. In [68], we can witness a DE model in an LM which proves that the expansion paradigm can be illustrated in the absence of a negative pressure energy component. It is further mentioned that DE is naturally of geometrical origin. The investigation of a two-component DE model in LM can be seen in [69]. In [70], we can find a cosmological model of anisotropic DE paired with LM which is in consonant with the present observation. The study of a magnetized DE model in the Lyra setting can be seen in [71]. A discussion on the effect of DE on model with linear varying deceleration parameter in LM can be found in [72]. In [73], the authors present the isotropization of DE distribution in LM. In [74], Kaluza–Klein DE model is studied in LM thereby obtaining an exponentially expanding universe. In [75], we can see a DE model in LM where constant deceleration parameter is assumed. Brans–Dicke scalar field as a DE candidate in LM is illustrated in [76]. In [77], a DE model with quadratic EoS is presented in the framework of LM. Finally, in [78], the authors search for the existence of Lyra's cosmology with minimal interaction between DE and normal matter and obtain that the time-varying displacement $\beta(t)$ co-relates with the nature of cosmological constant $\Lambda(t)$.

The chance of space-time having more than 4D has captivated many authors. This has ignited a spark of interest among cosmologists and theorological physicists so that, in the past few decades, there has been a trend among authors opting to choose higher dimensional space-time to study cosmology. Higher dimensional model was introduced in [79, 80] in an attempt to unify gravity with electromagnetism. Higher dimensional model can be regarded as a tool to illustrate the late time expedited expanding paradigm [81]. Investigation of higher dimensional space-time can be regarded as a task of paramount importance as the universe might have come across a higher dimensional era during the initial epoch [82]. In [83, 84], it is mentioned that extra dimensions generate huge amount of entropy which gives possible solution to flatness and horizon problem. In [85], it is asserted that the detection of a time-varying fundamental constant can possibly show us the proof for extra dimensions. Since we are living in a 4D space-time, the hidden extra dimension in 5D is highly likely to be associated with the invisible DM and DE [86]. Few of the worth mentioning works on higher dimensional space-time during the last few years can be seen in [20, 87–96].

Keeping in mind the above notable works by different authors, we have analyzed a spherically symmetric metric in 5D setting within the framework of LM with the aim of predicting the sible source of DE. Here, we observe the field equations with due consideration of reasonable cosmological assumptions within the limit of the present cosmological scenario. The paper has been structured into sections. In Sec. 2, in addition to obtaining the solutions of the field equations, the cosmological parameters are also solved. In Sec. 3, the physical and kinematical aspects of our model are discussed with graphs. Considering everything, a closing remark is presented in Sec. 4.

2. Formulation of Problem with Solutions

The five-dimensional spherically symmetric metric is given by

$$ds^2 = dt^2 - e^{\mu}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) - e^{\delta} dv^2, \tag{1}$$

where $\mu = \mu(t)$ and $\delta = \delta(t)$ are cosmic scale factors.

The modified Einstein's field equations in Lyra geometry appear in the form

$$R_{ij} - \frac{1}{2} g_{ij} R + \frac{3}{2}\varphi_i \varphi_j - \frac{3}{4} g_{ij} \varphi_k \varphi^k = -T_{ij}, \tag{2}$$

where φ_i is the displacement vector and other symbols have their usual meaning as in Riemannian geometry. The displacement vector φ_i takes the time-dependent form

$$\varphi_i = (\beta(t), 0, 0, 0, 0). \tag{3}$$

The assumption that φ_i is time independent i.e. constant is vague as there is no specific mathematical or physical explanation showing that a constant displacement vector contributes to the late time acceleration of the universe [52]. Above all, assuming displacement vector field as a constant is just for convenience sake without any scientific reason [96].

The energy momentum tensor $T_{i\,j}$, considered as a perfect fluid, in the co-moving coordinates, is given by

$$T_{i\,j} = (\rho + p)u_i u_j - p g_{i\,j}, \tag{4}$$

where ρ and p respectively represent the energy density and isotropic pressure of the matter source. The five velocity vector u^i satisfies

$$u^i u_i = 1, \quad u^i u_j = 0. \tag{5}$$

Now, the surviving field equations are obtained as follows:

$$\frac{3}{4}(\dot{\mu}^2 + \dot{\mu}\dot{\delta} - \beta^2) = \rho, \tag{6}$$

$$\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 + \frac{\ddot{\delta}}{2} + \frac{\dot{\delta}^2}{4} + \frac{\dot{\mu}\dot{\delta}}{2} + \frac{3}{4}\beta^2 = -p, \tag{7}$$

$$\frac{3}{2}(\ddot{\mu} + \dot{\mu}^2) + \frac{3}{4}\beta^2 = -p, \tag{8}$$

where an overhead dot represents differentiation with respect to t.

From continuity equation, we have

$$\dot{\rho} + \frac{3}{2}\dot{\beta}\beta + 3H\left(\rho + p + \frac{3}{2}\beta^2\right) = 0. \tag{9}$$

From [94], assuming that β and ρ are independent without any interaction, Eq. (9) can be separately written as

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{10}$$

$$\dot{\beta}\beta + 3H\beta^2 = 0, \tag{11}$$

where H is Hubble's parameter.

From Eqs. (7) and (8), the expression for cosmic scale factors are obtained as

$$\mu = l - \log(k - t)^{\frac{2}{3}}, \tag{12}$$

$$\delta = m - \log(k - t)^{\frac{2}{3}}, \tag{13}$$

where l, m, k are arbitrary constants.

Now, we obtain the expression for the cosmological parameters as follows:

Spatial volume:

$$v = e^{\frac{3\mu+\delta}{2}} = e^{\frac{3l+m}{2}}(k - t)^{-\frac{4}{3}}. \tag{14}$$

Scale factor:

$$a(t) = v^{\frac{1}{4}} = e^{\frac{3l+m}{8}}(k - t)^{-\frac{1}{3}}. \tag{15}$$

Scalar expansion:

$$\Theta = u^i_{;j} = \frac{3\dot{\mu}}{2} + \frac{\dot{\delta}}{2} = \frac{4}{3}(k-t)^{-1}. \tag{16}$$

Hubble's parameter:

$$H = \frac{\Theta}{4} = \frac{1}{3}(k-t)^{-1}. \tag{17}$$

Deceleration parameter:

$$q = \frac{d}{dt}\left(\frac{1}{H}\right) - 1 = -4. \tag{18}$$

With $\Delta H_i = H_i - H, (i = 1, 2, 3, 4)$ representing the directional Hubble's parameters, anisotropic parameter A_h is defined as

$$A_h = \frac{1}{4}\sum_{i=1}^{4}\left(\frac{\Delta H_i}{H}\right)^2 = 0. \tag{19}$$

Shear Scalar:

$$\sigma^2 = \frac{1}{2}\sigma_{i\,j}\sigma^{i\,j} = \frac{1}{2}\sum_{i=1}^{4}(H_i^2 - 4H) = \frac{2}{9}\left(1 - \frac{1}{k-t}\right)^2. \tag{20}$$

From Eqs. (11) and (17), the expression for displacement vector is obtained as

$$\beta = d(t-k), \tag{21}$$

where d is an arbitrary constant.

From Eqs. (6), (12), (13) and (21), the expression for energy density is obtained as

$$\rho = \frac{3}{4}\left(\frac{8 - (3d(k-t)^2)^2}{9(k-t)^2}\right). \tag{22}$$

From Eqs. (8), (12) and (13), the expression of pressure is obtained as

$$p = -\frac{8(1 + (k-t)^2) + (3d(k-t)^3)^2}{12(k-t)^4}. \tag{23}$$

3. Discussion

For convenience sake and to obtain realistic results, specific values of the constants are chosen i.e. $l = m = 1, d = -1, k = 3.45497$ and the variations of some of the physical parameters with cosmic time t are provided as figures in this section. A scale of 1 Unit = 4 Gyr is taken along the time axis of each graph so that the point $t = 3.45$ corresponds to 13.8 Gyr which align with 13.825 ± 0.037 Gyr, the present age of the universe estimated by the latest Planck 2018 result [36].

Figures 1 and 2 can be regarded as the perfect evidences of the present spacial expansion of the universe at an expedited rate. From Figs. 3 and 4, we can witness a transition of the energy density ρ of the model universe from being negative

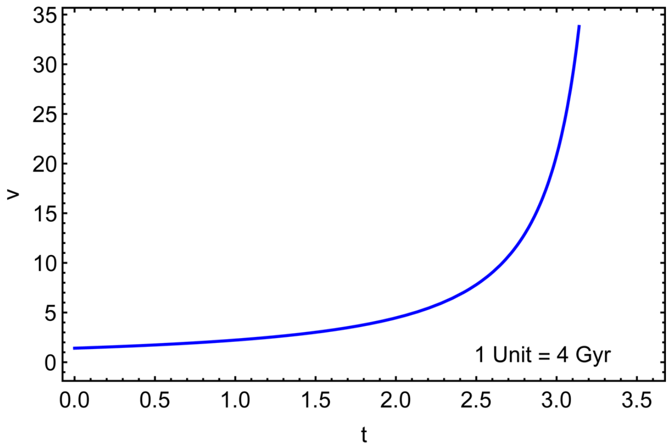

Fig. 1. Variation of spatial volume v with time t when $l = m = 1, k = 3.45497$ showing its increasing nature throughout the evolution.

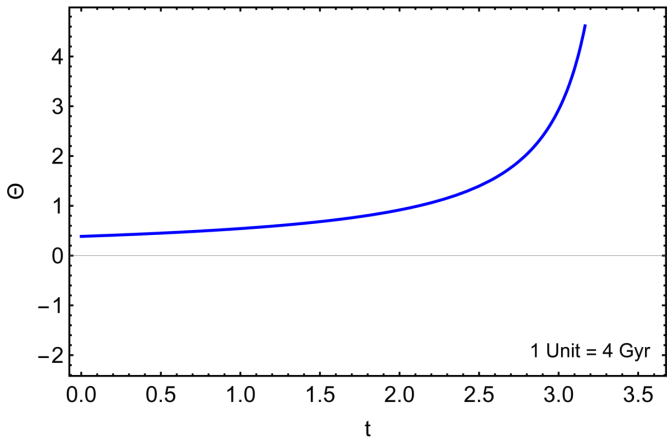

Fig. 2. Variation of scalar expansion Θ with time t when $l = m = 1, k = 3.45497$ showing its increasing nature throughout the evolution.

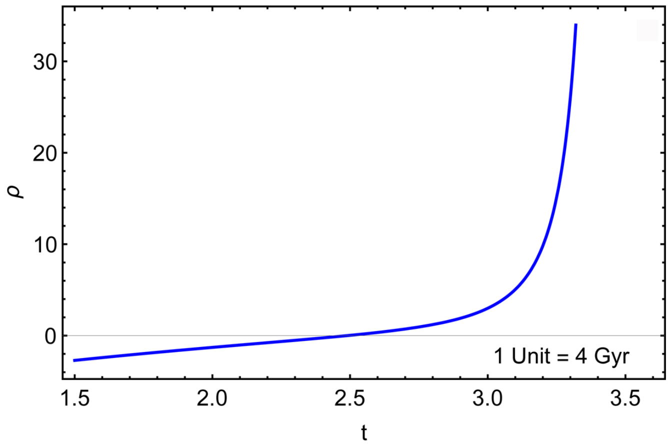

Fig. 3. Variation of energy density ρ with time t when $l = m = 1, k = 3.45497$ showing its transition from being negative to positive during evolution.

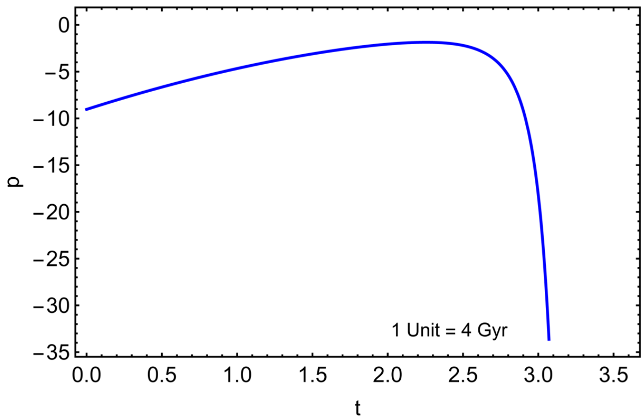

Fig. 4. Variation of pressure p with time t when $l = m = 1, k = 3.45497$ showing its negative nature all through.

to positive during the course of evolution whereas the pressure p of the model is negative all through. In short, the universe expands at an expedited rate with both ρ and p negative. This negative p can be regarded as the indication of the presence of DE. In this scenario, we can predict that DE in the form of vacuum energy is dominating the model, as mentioned in [4], NED is possible only if the DE is in the form of vacuum energy. When $t \to \infty$, both v and $\theta \to 0$ showing that in the far future, the expanding phenomenon will cease, the universe will be dominated by gravity, resulting to collapse and ultimately ending at the big crunch singularity. This may be supported by the fact that DE density may decrease faster than matter leading DE to vanish at $t \to \infty$ [6]. Additionally, when $t \to \infty$, ρ again starts to become negative. In this condition, due to the presence of NED, from [43], we can assume that the model universe represents an oscillating model, each cycle evolving with a big bang and ending at a big crunch, undergoing a series of bounces. Additionally, from Fig. 5, it is clear that the displacement vector β is a decreasing function of time. Here, we can assert that β acts as the time-dependent cosmological constant [1, 2, 53, 98]. Hence, it is fascinating to observe that LM itself can be regarded as a DE model.

Accelerated expansion can be attained when $-1 < q < 0$ whereas $q < -1$ causes super-exponential expansion [99]. Figure 6 shows that the deceleration parameter q is a negative constant -4 all through indicating that the model universe undergoes super-exponential expansion in the entire course of evolution. It may be noted that in higher dimensional theory with cosmological constant, super-exponential inflation (expansion) can be attained if H increases with t [100–102]. In our case, H is increasing as shown in Fig. 7. Shear scalar σ^2 provides us the rate of deformation of the matter flow within the massive cosmos [103]. From Fig. 8, we can see that σ^2 evolves almost constantly, then diverges after some finite time. From Eq. (19), the anisotropic parameter $A_h = 0$. From these, we can sum up that initially, the isotropic universe expands with a slow and uniform change of shape, but after some finite time, the change becomes faster.

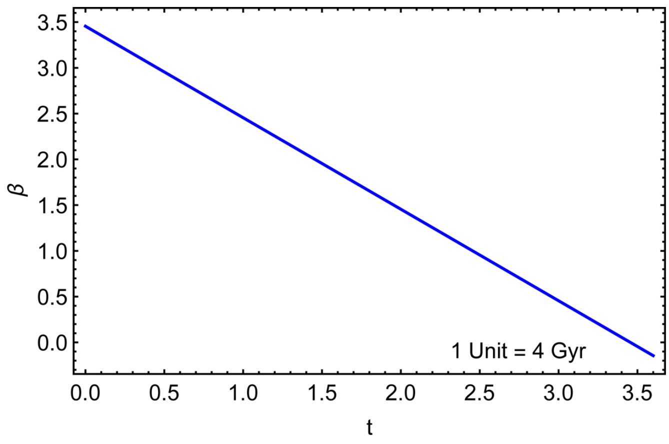

Fig. 5. Variation of displacement vector β with time t when $d = -1, k = 3.45497$ showing its decreasing nature.

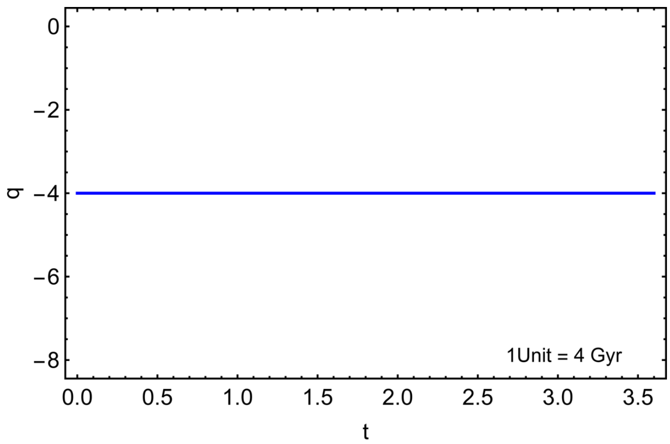

Fig. 6. Variation of deceleration parameter q with time t when $l = m = 1, k = 3.45497$ showing that it is a negative constant -4 all through.

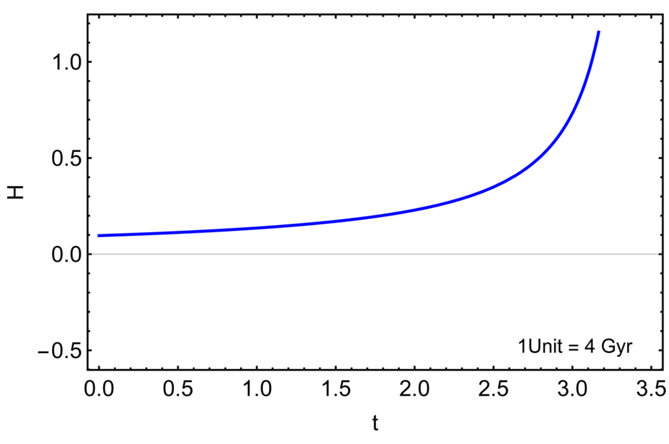

Fig. 7. Variation of Hubble's parameter H with time t when $k = 3.45497$.

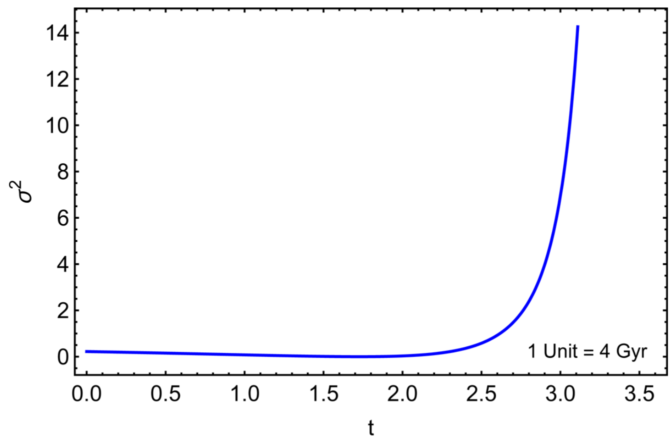

Fig. 8. Variation of shear scalar σ^2 with time t when $l = m = 1, k = 3.45497$.

Finally, with a scale of 1 Unit = 4 Gyr, the point $t = 3.45$ corresponds to 13.8 Gyr which align with 13.825 ± 0.037 Gyr, the present age of the universe estimated by the latest Planck 2018 result [36]. At the point $t = 3.45$ and assuming $k = 3.45497$, from Eq. (17), the numeric value of Hubble's parameter is measured to be $H = 67.0691$ which is very close to $H_0 = 67.36 \pm 0.54\,\mathrm{kms^{-1}Mpc^{-1}}$, the value estimated by the latest Planck 2018 result [36].

4. Conclusion

With due consideration of reasonable cosmological assumptions within the limit of the present cosmological scenario, we have analyzed a spherically symmetric metric in 5D setting within the framework of LM. The model universe is predicted to be a DE model, dominated by vacuum energy. The displacement vector also acts as the time-dependent DE. The model represents an oscillating model, each cycle evolving with a big bang and ending at a big crunch, undergoing a series of bounces. Our universe undergoes super-exponential expansion in the entire course of evolution. Initially, the isotropic universe expands with a slow and uniform change of shape, but after some finite time, the change becomes faster. Then, the change slows down and tends to become uniform after expanding without any deformation of the matter flow for a finite time period. Finally, Hubble's parameter is measured to be $H = 67.0691$ which is very close to $H_0 = 67.36 \pm 0.54\,\mathrm{km\,s^{-1}Mpc^{-1}}$, the value estimated by the latest Planck 2018 result [36]. We have constructed a model in LM appearing as a DE model; nonetheless, the work we have put forward is just a toy model. The model needs further deep study considering all the observational findings, which will be our upcoming work.

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$f(R, T)$Gravity model behaving as a dark energy source

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ABSTRACT

Within the limits of the present cosmological observations in $f(R, T)$gravity theory, we have analyzed a spherically symmetric space-time in 5D setting. The field equations have been carefully studied considering reasonable cosmological assumptions to obtain exact solutions. We have obtained an isotropic model universe undergoing super-exponential expansion. It is predicted that the model universe behaves like a dark energy (vacuum energy) model. In the present scenario, the model evolves with a slow and uniform change of shape. It is observed that the universe is close to or nearly flat. The model is free from an initial singularity and is predicted to approach the de-Sitter phase dominated by vacuum energy or cosmological constant in the finite-time future. A comprehensive discussion on the cosmological parameters obtained in view of the recent studies is presented in detail with graphs.

1. Introduction

The ambiguous dark energy (DE) has been regarded as one of the most tantalizing topics in cosmology since its profound discovery in 1998 (Riess, 1998; Perlmutter, 1999). It is considered to be the reason behind the late time expanding universe at an expedited rate due to its huge negative pressure with repulsive gravitation. It is uniformly permeated throughout the space and vary slowly or almost consistent with time (Chan, 2015b; Peebles and Ratra, 2003; Carroll, 2001a; 2001b). Cosmologists all over the map have conducted a series of studies with the aim of hunting its origin and are still scrabbling for a perfect answer. Some worth mentioning such studies that have not escaped our notice in the recent years are briefly discussed below.

In Singh and Kar (2019), the authors assert that emergent D-instanton might indicate us a hint to the root of DE. A cosmological model associated with an antineutrino star is constructed by Neiser (2020) in order to search the origin of DE. In Dikshit (2019), the author presents an explanation for DE with pure quantum mechanical method. In Huterer and Shafer (2017), the investigation of the twenty years old history of DE and the current status can be seen. The authors in Wang et al. (2018) study the evolution of the DE using a non-parametric Bayesian approach in the light of the latest observation. In Capolupo (2018), the author claims that vacuum condensate can provide us the origin of DE. According to Josset et al. (2017), DE is originated from the violation of energy conservation. A unified dark fluid is obtained as a source of DE by Tripathy et al. (2015). The presence of particle with imaginary energy density can lead us to the source of DE (Chan, 2015a). The explanation of a physical mechanism as a source of DE is presented by Gontijo (2012). Lastly, in Alexander et al. (2010), DE evolves as a result of the condensation of fermions formed during the early evolution.

It is an obvious fact that the universe is dominated by the cryptic DE with negative pressure and positive energy density (Carroll, 2001a; Law, 2020; Singh and Singh, 2019a; Agrawal et al., 2018; Ray et al., 2013; Straumann, 2007; de Araujo, 2005; Wu and Yu, 2005). This qualifies DE a completely irony of nature as the dominating component is also the least explored. As against the positive energy density condition, it is fascinating to see many authors introducing the concept of the possibility of negative energy density (NED) with convincig arguments in support. In Ijjas and Steinhardt (2019), the authors discuss NED where models evolve with a bounce. The authors continued that there might be bounces in the future too. The discussion of negative vacuum energy density in Rainbow Gravity can be seen in Wong et al. (2019). In Nemiroffa et al. (2015), we can witness, under certain conditions, a repelling negative gravitational pressure with NED. Further, we can find a repelling negative phantom energy with NED. In Fay (2014), the author claims that the universe evolves by inflation when the coupled fluid has NED in the initial epoch. An accelerating universe with NED is studied by Sawicki and Vikman (2013). In Macorra and German (2004), we can find an explaination of energy density with negative value with equation of state parameter (EoS) $\omega < -1$. The author in Carroll (2001a)predicts that NED is possible only if the DE is in the form of vacuum energy. In Huang (1990), the investigation of models which evolved with NED in the infinite past can

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be found. According to Parker and Fulling (1973), the introduction of quantized matter field with NED to energy momentum tensor might by pass cosmological singularity. Besides defying the energy conditions of GR, NED also disobeys the second law of thermodynamics (Hawking and Ellis, 1973). However, the condition should be solely obeyed on a large scale or on a mean calculation, thereby neglecting the probable violation on a small scale or for a short duration, in relativity (Fewster, 2012; Graham and Olum, 2003; Visser and Barcelo, 2000; Pfenning and Ford, 1998; Helfer, 1998b; 1998a; Ford and Roman, 1996; Roman, 1986; Epstein et al., 1965). Hence, in the initial epoch, if there were circumstance of defiance for a short duration measured against the present age of 13.830 ± 0.037Gyr estimated by the latest Planck 2018 result (Planck collabration, 2019), it will remain as an important part in the course of evolution.

In the present cosmology, authors prefer to opt alternate or modified theories of gravity in order to precisely understand the underlying mechanism of the late time expedited expansion of the universe. One such well appreciated modified theory is the $f(R, T)$gravity introduced by Harko et al. (2011) in which the gravitational Lagrangian is represented by an arbitrary function of the Ricci scalar $R$and the trace $T$of the energy-momentum tensor. In the past few years, this theory has captivated many cosmologists and theoretical physicists as it presents a natural gravitational substitute to DE (Chirde and Shekh, 2019). Recently, Myrzakulov (2020) studies the theory and predicts the conditions to obtain expanding universe in the absence of any dark component. In Sahoo et al. (2020), the authors investigate a mixture of barotropic fluid and DE in $f(R, T)$gravity where the model evolves from the Einstein static era and approaches ΛCDM. In Singh and Singh (2019b), the study of cosmological dynamics of DE within the theory can be seen. Pawar et al. (2019)study a modified holographic Ricci DE model in the theory obtaining a singularity free model. The authors in Zia et al. (2018) investigate $f(R, T)$gravity discussing future singularities in DE dominated universe. In Srivastava and Singh (2018), we can find a discussion of new holographic DE model in $f(R, T)$gravity thereby obtaining ΛCDM in the late times. In Fayaz et al. (2016), the examination of ghost DE model within the theory can be seen, predicting model behaving as phantom or quintessence like nature. The investigation of cosmological models within the theory without DE is observed in Sun and Huang (2016). The authors in Mishra et al. (2016b)and Singh and Kumar (2016) study the relation of the theory with DE. Houndjo and Piattella (2012) present a reconstruction of the theory from holographic DE. The study of cosmological model in $f(R, T)$gravity obtaining DE induced cosmic acceleration can be seen in Mishra et al. (2016a). Zubair et al. (2016) discuss Bianchi space-time within the theory with time-dependent deceleration parameter. Ahmed et al. (2016) investigate a model in which the cosmological constant is considered as a function of T. The authors in Rao and Rao (2015)discuss a higher dimensional anisotropic DE model within the theory obtaining the EoS parameter $\omega = -1$. Jamil et al. (2012)construct models within the theory asserting that dust fluid leads to ΛCDM. Houndjo (2012)predicts a model in $f(R, T)$gravity that transit from matter dominated to accelerating phase. From these worth appreciating studies, it won't be a wrong guess to sum up that there must be some sort of hidden correspondences between the pair of DE and $f(R, T)$gravity theory. Consequently, in this work, we will try to find out if $f(R, T)$gravity theory itself behaves as a DE source.

The possibility of space-time possessing with more than 4D has fascinated many authors. In the recent years, there has been a trend of preferring higher dimensional space-time to study cosmology. Higher dimensional model was introduced by Kaluza (1921) and Klein (1926) in an effort to unify gravity with electromagnetism. Higher dimensional model can be regarded as a tool to illustrate the late time expedited expanding paradigm (Banik and Bhuyan, 2017). Investigation of higher dimensional space-time can be regarded as a task of paramount importance as the universe might have come across a higher dimensional era during the initial epoch (Singh et al., 2004). Marciano (1984) asserts that the detection of a time varying fundamental constant can possibly show us the proof for the extra dimension. According to Alvax and Gavela (1983) and Guth (1981), the extra dimension generates a huge amount of entropy which gives possible solutions to the flatness and horizon problem. Since we are living in a 4D space-time, the hidden extra dimension in 5D is highly likely to be associated with the invisible DM and DE (Chakraborty and Debnath, 2010).

Keeping in mind the above notable works by different authors, we have analysed a spherically symmetric metric in 5D setting within the framework of $f(R, T)$gravity with a focus to predict a possible source of DE. Here, we observe the field equations with due consideration of reasonable cosmological assumptions within the limit of the present cosmological scenario. The paper has been structured into sections. In Sect. 2, the field equations of the theory are discussed. In Sect. 23, in addition to obtaining the solutions of the field equations, the cosmological parameters are also solved. In Sect. 34, the physical and kinematical aspects of our model are discussed with graphs. Considering everything, a closing remark is presented in Sect. 45.

2. The field equations of $f(R, T)$gravity theory

The action of $f(R, T)$gravity theory is given by

$$S = \int \left(\frac{1}{16\pi} f(R, T) + \mathcal{L}_m\right)\sqrt{-g}\, d^4x \tag{1}$$

where $g \equiv det(g_{ij})$,$f$is an arbitrary function of the Ricci scalar $R = R(g)$and the trace $T = g^{ij}T_{ij}$of the energy-momentum tensor of matter T_{ij}defined by Koivisto (2006) as

$$T_{ij} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\,\mathcal{L}_m)}{\delta g^{ij}} \tag{2}$$

Here, the matter Lagrangian density $\mathcal{L}_m$is assumed to rely solely on g_{ij}so that we obtain

$$T_{ij} = g_{ij}\mathcal{L}_m - 2\frac{\partial \mathcal{L}_m}{\partial g^{ij}} \tag{3}$$

The action $S$is varied w.r.t. the metric tensor $g^{ij}$and hence, the field equations of $f(R, T)$gravity are given by

$$f_R(R, T)R_{ij} - \frac{1}{2} f(R, T)g_{ij} + (g_{ij}\Box - \nabla_i\nabla_j)f_R(R, T) = 8\pi T_{ij} - f_T(R, T)T_{ij} - f_T(R, T)\Theta_{ij} \tag{4}$$

where

$$\Theta_{ij} = -2\, T_{ij} + g_{ij}\mathcal{L}_m - 2g^{lk}\frac{\partial^2 \mathcal{L}_m}{\partial g^{ij}\partial g^{lk}} \tag{5}$$

Here, the subscripts appearing in $f$represent the partial derivative w.r.t. Ror $T$and $\Box \equiv \nabla^i\nabla_i$, ∇_ibeing the covariant derivative.

With $\rho$and $p$respectively representing the energy density and pressure such that the five velocity u^isatisfies $u^i u_i = 1$and $u^i\nabla_j u_i = 0$,we opt to use the perfect fluid energy-momentum tensor of the form

$$T_{ij} = (p + \rho)u_i u_j - p g_{ij} \tag{6}$$

We assume that $\mathcal{L}_m = -p$so that equation (5) is reduced to

$$\Theta_{ij} = -2T_{ij} - p g_{ij} \tag{7}$$

In general, the field equations of $f(R, T)$gravity also rely on the physical aspect of the matter field and consequently, there exists three classes of field equations as follows

$$f(R, T) = \begin{cases} R + 2f(T) \\ f_1(R) + f_2(T) \\ f_1(R) + f_2(R)f_3(T) \end{cases} \tag{8}$$

Our study will be dealing with the class $f(R, T) = R + 2f(T)$,where

$f(T)$represents an arbitrary function so that the field equations of the modified theory is be reduced to

$$R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi T_{ij} + 2f'(T)T_{ij} + \{2pf'(T) + f(T)\}g_{ij} \quad (9)$$

where the prime indicates differentiation w.r.t. $T$and we assume that $f(T) = \lambda T$,where $\lambda$is an arbitrary constant.

3. Formulation of the problem and solutions

The five-dimensional spherically symmetric metric is given by

$$ds^2 = dt^2 - e^{\mu}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) - e^{\delta}dv^2 \quad (10)$$

where $\mu = \mu(t)$and $\delta = \delta(t)$are cosmic scale factors.

Now, using co-moving co-ordinates, the surviving field equations are obtained as follows

$$-\frac{3}{4}(\dot{\mu}^2 + \dot{\mu}\dot{\delta}) = (8\pi + 3\lambda)\rho - 2p\lambda \quad (11)$$

$$\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 + \frac{\ddot{\delta}}{2} + \frac{\dot{\delta}^2}{4} + \frac{\dot{\mu}\dot{\delta}}{2} = (8\pi + 4\lambda)p - \lambda\rho \quad (12)$$

$$\frac{3}{2}(\ddot{\mu} + \dot{\mu}^2) = (8\pi + 4\lambda)p - \lambda\rho \quad (13)$$

where an overhead dot indicates differentiation w.r.t. t.

From Eqs. (12) and (13), the expressions for the cosmic scale factors are obtained as

$$\mu = a - 3\log(2(k - 3t)) \quad (14)$$

$$\delta = b - 3\log(2(k - 3t)) \quad (15)$$

where a, b, $k$are arbitrary constants.

Now, the expressions for spatial volume v,scalar expansion θ,Hubble's parameter H,deceleration parameter q,shear scalar $\sigma^2$and anisotropic parameter $A_h$are obtained as follows.

$$v = e^{\frac{3a+b}{2}}(2(k - 3t))^{-6} \quad (16)$$

$$\theta = 18(k - 3t)^{-1} \quad (17)$$

$$H = \frac{9}{2}(k - 3t)^{-1} \quad (18)$$

$$q = -1.7 \quad (19)$$

$$\sigma^2 = \left(\frac{27 - 2(k - 3t)}{18(k - 3t)}\right)^2 \quad (20)$$

$$A_h = 0 \quad (21)$$

From Eqs. (11) and (13), the expressions for the pressure $p$and the energy density $\rho$of the model universe are respectively obtained as

$$p = \frac{243\lambda - 324(8\pi + 3\lambda)}{4(k - 3t)^2(-5\lambda^2 - 32\pi^2 - 28\pi\lambda)} \quad (22)$$

$$\rho = \frac{(8\pi + 4\lambda)}{\lambda}\left(\frac{243\lambda - 324(8\pi + 3\lambda)}{4(k - 3t)^2(-5\lambda^2 - 32\pi^2 - 28\pi\lambda)}\right) - \frac{162}{\lambda(k - 3t)^2} \quad (23)$$

The expression for the scalar curvature $R$is obtained as

$$R = \frac{513}{(k - 3t)^2} \quad (24)$$

4. Discussions

For convenience sake and to obtain realistic results, specific values of the arbitrary constants involved are chosen i.e., $a = b = 1$, $k = 15$and $\lambda = -5.06911$and -12.5856. The graphs of the cosmological parameters w.r.t. cosmic time $t$are presented with the detailed discussion in view of the latest observations.

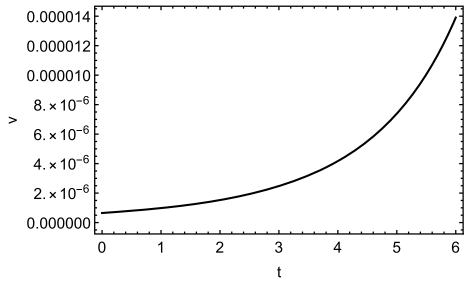


Fig. 1. Variation of the spatial volume $v$with t when $a = b = 1$, $k = 15$.

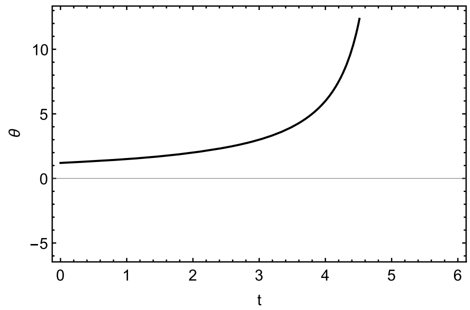
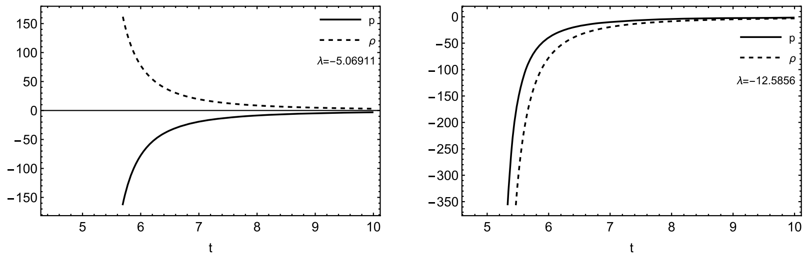


Fig. 2. Variation of the expansion scalar $\theta$with t when $a = b = 1$, $k = 15$.

Fig. 1 and Fig. 2 can be regarded as the perfect evidences for the present spatial expansion at an expedited rate. When $t \to 0$,$v$and other related parameters are constants ($\neq 0$), implying that the model universe doesn't evolve from an initial singularity.

Fig. 3 (a) shows the variation of the pressure $p$and energy density $\rho$when $a = b = 1$, $k = 15$, $\lambda = -5.06911$. From the graph, it is obvious that the model is experiencing accelerated expansion with negative $p$and positive ρ. Here, the model evolves with a large $\rho$and it converges to become constant at late times. This phenomenon is a clear indication of the presence of DE as the present cosmology believes that the late time accelerating universe is due to the dominant and slowly varying or constant DE with negative pressure and positive energy density (Carroll, 2001a; Law, 2020; Singh and Singh, 2019a; Agrawal et al., 2018; Ray et al., 2013; Straumann, 2007; de Araujo, 2005; Wu and Yu, 2005). In order to predict the nature, the graph of $\omega = \frac{p}{\rho}$which is the DE EoS parameter is plotted in Fig. 4 which shows that $\omega = -1$. Hence, we can sum up that the $f(R, T)$gravity model we have constructed turns out to be a DE model, DE in the form of vacuum energy or the cosmological constant. Fig. 3(b) shows the variation of the pressure $p$and energy density $\rho$when $a = b = 1$, $k = 15$, $\lambda = -12.5856$. In this case, the model undergoes expansion at an expedited rate with $p$and ρboth negative. This negative $p$can be regarded as the indication of the presence of DE. In this scenario too, we can predict that DE in the form of vacuum energy is dominating the model, as predicted by Carroll (2001a), NED is possible only if the DE is in the form of vacuum energy. Hence, in both the cases, it is fascinating to see that $f(R, T)$gravity model behaves as a DE (vacuum energy) model. We have not considered the case when $\lambda > 0$as it yields positive pressure which is not reliable in the present scenario.

Accelerated expansion can be attained when $-1 < q < 0$whereas $q < -1$causes super-expansion (Singh and Bishi, 2017). Fig. 5 shows that the deceleration parameter $q$is a negative constant -1.7 all through indicating that the model universe undergoes super-exponential expansion in the entire course of evolution. Fig. 6 shows that the Hubble's



(a) p and ρ with t when $\lambda = -5.06911$. (b) p and ρ with t when $\lambda = -12.5856$.

Fig. 3. Variation of pressure $p$and energy density $\rho$when $a = b = 1$, $k = 15$, $\lambda = -5.06911$and $-$ 12.5856.

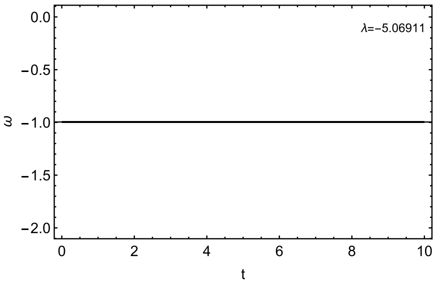


Fig. 4. Variation of the DE EoS $\omega$with $t$when $a = b = 1$, $k = 15$, $\lambda = -5.06911$.

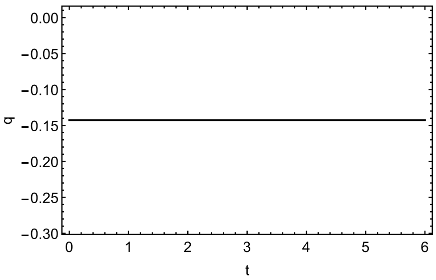


Fig. 5. Variation of the deceleration parameter $q$with t.

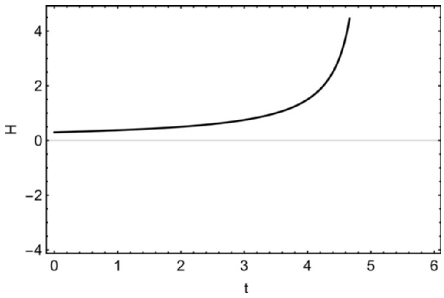


Fig. 6. Variation of the Hubble's parameter $H$with t when $k = 15$.

parameter Happears to remain almost constant in the early evolution so that our universe was in the inflationary epoch experiencing rapid exponential expansion (Crevecoeur, 2017).

Shear scalar σ^2provides us the rate of deformation of the matter flow within the massive cosmos (Ellis and van Elst, 1999). From figure 7, we can see that σ^2evolves constantly, then diverges after some finite time

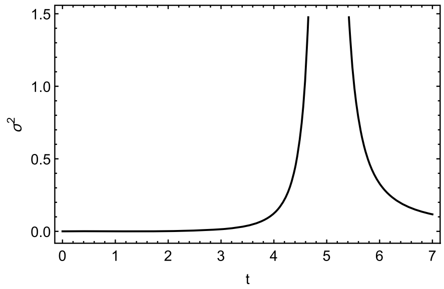


Fig. 7. Variation of the scalar scalar $\sigma^2$with t when $k = 15$.

and again converges to become constant after vanishing for a finite period. From equation (21), the anisotropic parameter $A_h = 0$. From these, we can sum up that initially, the isotropic universe expands with a slow and uniform change of shape, but after some finite time, the change becomes faster. Then, the change slows down and tends to become uniform after expanding without any deformation of the matter flow for a finite time period.

Fig. 8 shows the decreasing nature of the scalar curvature $R$with cosmic time t. Similar observation can also be seen in the recent studies (Pavlovic and Sossich, 2017; Pashitskii and Pentegov, 2016). It tends to become constant in the future. At $t = 13.8$Gyr which align with 13.830 ± 0.037Gyr, the approximate present age of the universe estimated by the latest Planck 2018 result (Planck collabration, 2019), the value of the scalar curvature is obtained as $R = 0.72$. $R = 0$corresponds to an exactly flat expanding universe (Gueorguiev and Maeder, 2020; Kleban and Senatore, 2016; Bevelacqua, 2006). However, in the recent years, arguments against the notion of exactly flat universe have been put forward by many authors (Valentino et al., 2020),Javed et al., (Khodadi et al., 2015; Nashed and Hanafy, 2014). In the present scenario, the universe is assumed to be close to or nearly flat, but not

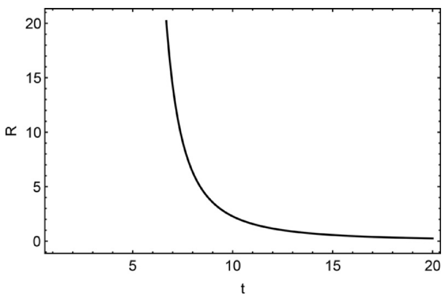


Fig. 8. Variation of the scalar curvature $R$with t when $k = 15$.

exactly flat (Nashed and Hanafy, 2014; Adler and Overduin, 2005; Levin and Freese, 1994). Additionally, the latest Planck 2018 results (Planck collabration, 2019) estimating the value of overall density parameter Ωranging close to unity can also regarded as an evidence for nearly flat universe, as for an exactly flat universe, $\Omega = 1$(Khodadi et al., 2015; Levin and Freese, 1994; Holman, 2018). Hence, our model obtaining a small $R = 0.73$is justified. (Tiwari, 2016) and (Kim et al., 2003), in their studies, assert that $R$is constant for de-Sitter phase. So, the reason for R becoming constant in the future can be regarded as an indication for the model approaching the de-Sitter phase dominated by vacuum energy or cosmological constant in the finite time future avoiding singularity. According to Falls et al. (2018), accelerated expansion will lead $R$to approach a nearly constant value so that the universe behaves in the same manner as a de-Sitter universe in the future. Many other authors have also asserted that the expanding universe will end at the de-Sitter phase dominated by vacuum energy, avoiding singularity. Basilakos et al. (2018); Carneiro (2006); Dymnikova (2019); Dyson et al. (2002); Nojiri and Odintsov (2004); Krauss and Starkman (2000); Markkanen (2018); Sakharov (1966); Starobinsky (2000); Zilioti et al. (2018).

5. Conclusions

Within the framework of $f(R, T)$gravity, we have analysed a spherically symmetric space-time in 5D setting. We have obtained an isotropic model universe undergoing super-exponential expansion. The variation of the pressure $p$and energy density $\rho$with cosmic time $t$are analysed when $\lambda = -5.06911$and -12.5856. In both the cases, it is fascinating to see that our $f(R, T)$gravity model behaves as a DE (vacuum energy) model. The model is free from an initial singularity. The model expands with a slow and uniform change of shape, but after some finite time, the change becomes faster. Then, the change slows down and tends to become uniform after expanding without any deformation of the matter flow for a finite time period. The scalar curvature $R$is decreasing with time which is consistent with the recent studies. The model is predicted to approach the de-Sitter phase dominated by vacuum energy or cosmological constant in the finite time future avoiding singularity. We have constructed a model where $f(R, T)$gravity theory itself behaves as a DE (vacuum energy) model; nonetheless, the work we have put forward is just a toy model. The model needs further deep study considering all the observational findings, which will be our upcoming work.

CRediT authorship contribution statement

Pheiroijam Suranjoy Singh: Writing - original draft, Software, Investigation, Visualization, Investigation, Data curation, Formal analysis, Conceptualization. **Kangujam Priyokumar Singh:** Supervision, Validation, Resources, Methodology, Writing - review & editing.

Declaration of Competing Interest

None.

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Dark energy on higher dimensional spherically symmetric Brans–Dicke universe

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ABSTRACT

In this work, we have presented a cosmological model in five dimensional spherically symmetric space-time with energy momentum tensors of minimally interacting fields of dark matter and holographic dark energy in Brans–Dicke theory. Under some realistic assumptions in consistent with the present cosmological observations, we have analyzed the field equations to obtain their exact solutions. With particular choices of the constants involved, the values of the overall density parameter and the Hubble's parameter are obtained to be very close to the latest observational values. We obtain a model universe experiencing super exponential expansion which will be increasingly dark energy dominated in the far future. A comprehensive presentation of the physical as well as kinematical aspects of the parameters, including future singularity, in comparison with the present observational findings is also provided.

1. Introduction

Topics on the accelerated expansion of the universe have attracted wide attention from many theoretical physicists and cosmologists around the world energizing them for further investigations and many clear and convincing evidence have been produced in support. This accelerated expansion is explained by the so-called dark energy [1,2], a completely mysterious form of energy with an exotic property of negative pressure which generates a negative gravity that causes the acceleration by emitting a strong repulsive force resulting in an anti-gravity effect. This uniformly distributed mystical component dominating the universe is slowly varying with time and space [3–5]. Since its discovery, it has become one of the most discussed topics among the cosmological society and great scientific efforts have been invested in order to explore its bizarre nature, properties, future characteristics and applications to modern cosmology. In [6], the authors obtained that the universe might be dark energy dominant or free from dark energy in future time. Steinhardt [7] studied the quintessential introduction to dark energy. In [8], the authors investigated wet dark fluid, a dark energy candidate. In [9], the authors studied an axially symmetric cosmological model in the presence of anisotropic dark energy in which the solutions obtained could give us an appropriate description of the evolution of the universe. Dark Energy Survey Collaboration [10] describes the future prospect and discovery potential of the Dark Energy Survey (DES) beyond cosmological studies. In [11], the authors examined if dark energy could neutralize the global warming. In [12], the authors put forward interesting explanations to show that Lyra's manifold could be the hidden source of dark energy. Nair and Jhingana [13] examined whether dark energy is evolving or not. Abbott [14] provides us the first public data release of the DES. Risaliti and Lusso [15] observed that the dark energy density is increasing with time. According to [16], there are cosmologists who doubt if dark energy is behind the increasing expansion of the universe and the author analysed the arguments. Lastly, in [17], the authors have hunted down the origin

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of dark energy as far back as Newton and Hooke and presented a comprehensive summary of 90 years old history of the cosmological constant.

To understand this dark component as precisely as possible to obtain hints as to exactly predict its nature and properties, cosmologists have opted for analysing the equation of state (EoS) parameter ω which is the ratio of the pressure to the density of the dark energy. In recent years, different authors have calculated different viable limits on the value of ω with strong evidence in support. In [18], the authors have calculated two different ranges $-1.61 < \omega < -0.78$ and $-1.67 < \omega < -0.62$ on two different situations. In [19], a bound of $-1.38 < \omega < -0.82$ is measured whereas according to the latest Planck 2018 results [20], the value of ω is measured to be $\omega = -1.03 \pm 0.03$.

We can describe the accelerated expansion of the universe by two approaches: (i) Dark energy approach in which different viable candidates of dark energy are developed (ii) Modified theories of gravitation approach in which Einsteinâs theory of gravitation is modified to many optimized forms. Besides these approaches, many authors have put forward other possible ways to explain the late time acceleration of the universe. It is shown that the acceleration of the universe is the result of the back reaction of cosmological perturbations, rather than the effect of a negative pressure dark energy fluid or a modification of general relativity [21]. In Gorji [22], it can be seen that the late time cosmic acceleration is addressed by the infrared corrections. An interesting explanation of cosmic acceleration using only dark matter and ordinary matter can be seen in [23]. An approach is also suggested in [24] where the late time cosmic acceleration is obtained from an Ultraviolet Complete Theory.

The natural candidate for dark energy is the cosmological constant or the vacuum energy with $\omega = -1$. But, the vacuum energy fails to illustrate many riddles of physics, one of which worth mentioning is the coincidence problem [25] in which the similar densities, at the present epoch, of the differently evolved dark energy and dark matter remains a mystery. Therefore, many other viable candidates of dark energy have been introduced [26]. Cosmologist started to construct models which involve the interaction of these two dark components to explain the small value of Λ [27,28]. Afterwards, these constructed models were found applicable to mollify the coincidence problem [29–32]. During the last decade, evidence have been put forward which confirm that modified gravity can be presented in terms of interaction of these two dark components in the Einstein frame [33–37]. This can enable us to broaden the gravitational theory beyond the breadth of general relativity if we can figure out the specific interaction term. Recently, great scientific efforts have been utilized to study the dark energy-dark matter interaction, for both theoretical and observational point of view, in the holographic dark energy setting [38–47]. Holographic dark energy, a consequence of the application of the holographic principle [48] to the repulsive dark entity, was introduced by Gerardt Hooft [49]. Interacting models involving this dark holographic entity and matter in spherically symmetric space-time were studied in [47,50]. The mysterious nature of these two dark components have arisen many fundamental questions indicating that there are many new physics yet to be uncovered.

In the past few decades, many modified theories of gravitation challenging Einsteins theory have been put forward and these theories succeeded to fit the present cosmological trends in a quite satisfactory way, a handful of which that have not escaped our notice are Weyl's theory [51], Lyra geometry [52], Brans–Dicke theory [53], $F(R)$ gravity [54], $F(R, t)$ gravity [55], Mimetic $F(R)$ gravity [56] etc. Brans–Dicke theory of gravitation has become one of the favourite choices among many cosmological audiences and enormous efforts have been employed to study its modern cosmological aspects [57–65]. In this theory, a metric tensor g_{ij} is introduced along with a scalar filed φ which represents the space-time varying gravitational constant. In Einstein theory, gravity is explained by the lone entity – the space-time metric tensor or, in simple word, geometry. Whereas, in this modified theory, all matters are the reason for the gravitational behaviour of φ, so that, in this logic, it can be treated as a modification from purely geometric to geometric-scalar nature and thus, becoming a part of the family of scalar-tensor theory.

Brans–Dicke theory can be of good choice to study dark energy and the expansion of the universe. It can be considered as the most natural choice of the scalar-tensor generalization of general relativity due to its easiness and is less stringent than general relativity. Above all, the scalar field and the theory itself are of classical origin and can be considered as viable candidates to contribute in the late time evolution of the universe [66]. Some authors [67,68] have shown that Brans–Dicke theory or its modified versions are also the possible agents generating the present cosmic acceleration. It has also been shown that the theory can potentially generate sufficient acceleration in the matter dominated era [69]. In most of the studies in the Brans–Dicke theory setting, it can be seen that the accelerated expansion of the universe needs a very small value of ω, in the order of unity [70] and to be negative. It is shown that if the Brans–Dicke scalar field interacts with the dark matter, a generalized Brans–Dicke theory may cause the acceleration of the universe even with a high value of ω [71]. Interestingly, in [72], it is shown that the theory is essentially equivalent to a dark energy model. At present, both Brans–Dicke theory and general relativity are generally held to be in agreement with observation.

In [38], the authors studied a spatially homogeneous and anisotropic Bianchi type-V universe filled with minimally interacting fields of holographic dark energy and matter obtaining a universe which decelerate initially and accelerate in infinite time. In [39,40], the authors examined interacting models in Bianchi type-I and Bianchi type-V universe respectively showing that for suitable choice of interaction between matter and dark energy, there is no coincidence problem. In [41], we can find an interacting models between the two dark components in Brans–Dicke theory setting and the authors obtained a model that exhibits early inflation and late time acceleration. [42] presents an five dimensional interaction model in Brans–Dicke theory obtaining an anisotropic universe. In the paper, the authors further mentioned that their universe will become isotropic in finite time due to cosmic re-collapse. In [43], we can find an interacting model in a five dimensional spherically universe where the model experiences a transition from decelerated to accelerated phase due to cosmic re-collapse. In [46], the authors studied dark energy and matter using a relation between metric potentials and an equation of state representing disordered orientation obtaining the flat ΛCDM model as a particular case.

Inspired by the above studies, in this research, the minimal interaction model of the two dark entities has been presented with a five dimensional spherically symmetric space-time in Brans–Dicke theory of gravitation. Here, we consider some reasonable assumptions in agreement with the present cosmological observations. With particular choices of the constants involved, the values of the overall density parameter and the Hubble's parameter are obtained to be very close to the latest observational values. We obtain a model universe experiencing super exponential expansion which will be increasing dark energy dominated in the far future. We also obtain that the model universe will face big crunch singularity in the far future. The paper has been structured into sections. In Section 2, the formulation of the problem is presented along with the solutions of the field equations. Related cosmological parameters are also solved in this section. In Section 3, the graphs of the parameters are plotted and the physical and kinematical aspects of our model in comparison with the present observational findings are discussed. Considering everything, a concluding note is provided in Section 4.

2. Formulation of the problem and solutions

For our universe, we consider the spherically symmetric metric in the form

$$ds^2 = dt^2 - e^{\alpha}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) - e^{\beta} dy^2 \tag{1}$$

where α and β are cosmic scale factor which are functions of time only.

The Brans–Dicke field equations are written as follows.

$$R_{ij} - \frac{1}{2}g_{ij}R + \omega\varphi^{-2}\left(\varphi_{,i}\varphi_{,j} - \frac{1}{2}g_{ij}\varphi_{,k}\varphi^{,k}\right) + \varphi^{-1}(\varphi_{i;j} - g_{ij}\varphi^{k}_{;k}) = -8\pi\varphi^{-1}(T_{ij} + S_{ij}) \tag{2}$$

where φ is the Brans–Dicke scalar field and T_{ij} and S_{ij} are respectively the energy momentum tensors for matter and holographic dark energy, whereas R is the Ricci scalar and R_{ij} is the Ricci tensor

The field equations given by Eq. (2) were also studied in five dimension in spherically symmetric setting by some renowned authors in [46,50].

In our study, we define T_{ij} and S_{ij} as follows.

$$T_{ij} = \rho_m u_i u_j \tag{3}$$

$$S_{ij} = (\rho_d + p_d)u_i u_j - g_{ij}p_d \tag{4}$$

where ρ_m is the energy density of matter whereas ρ_d and p_d are respectively the energy density and the pressure of the holographic dark energy.

The wave equation satisfied by the scalar field is written as

$$\varphi^{k}_{;k} = 8\pi(3 + 2\omega)^{-1}(S + T) \tag{5}$$

The energy conservation equation in its obvious form is given by

$$S^{ij}_{;j} + T^{ij}_{;j} = 0 \tag{6}$$

We consider the co-moving co-ordinate system so that the flow vector satisfies the relation

$$g_{ij}u^i u^j = 1 \tag{7}$$

Now, we obtain the field equations as follows

$$\frac{3}{4}(\dot{\alpha}^2 + \dot{\alpha}\dot{\beta}) - \frac{\omega}{2}\frac{\dot{\varphi}^2}{\varphi^2} + \frac{\dot{\varphi}}{\varphi}\left(\frac{3\dot{\alpha} + \dot{\beta}}{2}\right) = 8\pi\varphi^{-1}(\rho_m + \rho_d) \tag{8}$$

$$\ddot{\alpha} + \frac{3}{4}\dot{\alpha}^2 + \frac{\ddot{\beta}}{2} + \frac{\dot{\beta}^2}{4} + \frac{\dot{\alpha}\dot{\beta}}{2} + \frac{\omega}{2}\frac{\dot{\varphi}^2}{\varphi^2} + \frac{\ddot{\varphi}}{\varphi} + \frac{\dot{\varphi}}{\varphi}\left(\dot{\alpha} + \frac{\dot{\beta}}{2}\right) = -8\pi\varphi^{-1}p_d \tag{9}$$

$$\frac{3}{2}(\ddot{\alpha} + \dot{\alpha}^2) + \frac{\omega}{2}\frac{\dot{\varphi}^2}{\varphi^2} + \frac{\ddot{\varphi}}{\varphi} + \frac{3}{2}\frac{\dot{\varphi}}{\varphi}\dot{\alpha} = -8\pi\varphi^{-1}p_d \tag{10}$$

And Eq. (6) gives

$$\ddot{\varphi} + \dot{\varphi}\left(\frac{3\dot{\alpha} + \dot{\beta}}{2}\right) = 8\pi(3 + 2\omega)^{-1}(\rho_m + \rho_d - 4p_d) \tag{11}$$

where an overhead dot represents differentiation with respect to time t.

Taking ω as the equation of state (EoS) parameter of holographic dark energy, we have

$$p_d = \omega\rho_d \tag{12}$$

Then, the conservation equation takes the form

$$\dot{\rho}_d + (1 + \omega)\left(\frac{3\dot{\alpha} + \dot{\beta}}{2}\right)\rho_d + \dot{\rho}_m + \rho_m\left(\frac{3\dot{\alpha} + \dot{\beta}}{2}\right) = 0 \tag{13}$$

Since the holographic dark energy and matter are interacting minimally, both the components will conserve separately. Thus, from [73,74], we can write

$$\dot{\rho}_m + \rho_m\left(\frac{3\dot{\alpha} + \dot{\beta}}{2}\right) = 0 \tag{14}$$

$$\dot{\rho}_d + (1 + \omega)\rho_d\left(\frac{3\dot{\alpha} + \dot{\beta}}{2}\right) = 0 \tag{15}$$

Also, we have

$$\dot{\rho} + (\rho + p)\left(\frac{3\dot{\alpha} + \dot{\beta}}{2}\right) = 0 \tag{16}$$

Now, from Eqs. (9) and (10), we have

$$\frac{1}{2}\ddot{\alpha} + \frac{3}{4}\dot{\alpha}^2 - \frac{\ddot{\beta}}{2} - \frac{\dot{\beta}^2}{4} - \frac{\dot{\alpha}\dot{\beta}}{2} + \frac{1}{2}\frac{\dot{\varphi}}{\varphi}(\dot{\alpha} - \dot{\beta}) = 0 \tag{17}$$

From Eq. (14), we have

$$\rho_m = a_0 e^{-\left(\frac{3\alpha+\beta}{2}\right)} \tag{18}$$

Similarly, we have

$$\rho_d = b_0 e^{-(1+\omega)\left(\frac{3\alpha+\beta}{2}\right)} \tag{19}$$

where a_0 and b_0 are an arbitrary constants.

From Eqs. (9) and (10), we get

$$\alpha = a_1 - \log(c_1 - t)^{\frac{2}{3}} \tag{20}$$

$$\beta = b_1 - \log(c_1 - t)^{\frac{2}{3}} \tag{21}$$

where a_1, b_1 and c_1 are arbitrary constants.

Thus, from Eqs. (18)–(21), we obtain

$$\rho_m = a_0 e^{-\frac{1}{2}(3a_1+b_1)}(c_1 - t)^{\frac{4}{3}} \tag{22}$$

$$\rho_d = b_0 e^{-\frac{1}{2}(3a_1+b_1)(1+\omega)}(c_1 - t)^{\frac{4}{3}(1+\omega)} \tag{23}$$

Now, using Eqs. (12), (20)–(23) in Eq. (11), we obtain the expression of the scalar field φ as

$$\varphi = M_0(c_1 - t)^{\frac{10}{3}} + N_0(c_1 - t)^{\frac{10}{3}+\frac{4\omega}{3}} \tag{24}$$

where

$$M_0 = \frac{36}{65}\pi(3 + 2\omega)^{-1}a_0 e^{-\frac{1}{2}(3a_1+b_1)} \tag{25}$$

and

$$N_0 = 8\pi b_0(1 - 4\omega)(3 + 2\omega)^{-1}\left(\frac{10}{3} + \frac{4\omega}{3}\right)^{-1}\left(\frac{11}{3} + \frac{4\omega}{3}\right)^{-1} e^{-\frac{1}{2}(1+\omega)(3a_1+b_1)} \tag{26}$$

From Eqs. (22) and (23), we obtain the expression for energy density ρ as

$$\rho = \rho_m + \rho_d = a_0 e^{-\frac{1}{2}(3a_1+b_1)}(c_1 - t)^{\frac{4}{3}} + b_0 e^{-\frac{1}{2}(1+\omega)(3a_1+b_1)}(c_1 - t)^{\frac{4}{3}(1+\omega)} \tag{27}$$

Using Eqs. (20), (21) and (27) in Eq. (16), we obtain the expression for the pressure as

$$p = \frac{1}{3}a_0 e^{-\frac{1}{2}(3a_1+b_1)}(c_1 - t)^{\frac{4}{3}} + \left(\frac{1}{3} + \frac{4\omega}{3}\right)b_0 e^{-\frac{1}{2}(1+\omega)(3a_1+b_1)}(c_1 - t)^{\frac{4}{3}(1+\omega)} \tag{28}$$

From Eqs. (12) and (23), the pressure of dark energy is given by

$$p_d = \omega b_0 e^{-\frac{1}{2}(3a_1+b_1)(1+\omega)}(c_1 - t)^{\frac{4}{3}(1+\omega)} \tag{29}$$

Now, at any time $t = t_0$, we can take

$$p = p_d \tag{30}$$

Therefore, from Eqs. (28), (29) and (30), we get

$$\left(a_0 e^{k} + b_0(1+\omega)e^{k(1+\omega)}(c_1 - t_0)^{\frac{4\omega}{3}}\right)(c_1 - t_0)^{\frac{1}{3}} = 0, \text{ where } k = -\frac{1}{2}(3a_1 + b_1) \tag{31}$$

Eq. (31) will give us the expression for the EoS parameter ω.

Now, we obtain the values of the different cosmological parameters as follows

Spatial volume:

$$v = e^{\frac{3a_1+b_1}{2}}(c_1 - t)^{-\frac{4}{3}} \tag{32}$$

Scalar expansion:

$$\theta = \frac{4}{3}(c_1 - t)^{-1} \tag{33}$$

Hubble's parameter:

$$H = \frac{1}{3}(c_1 - t)^{-1} \tag{34}$$

Deceleration parameter:

$$q = -4 \tag{35}$$

Shear scalar:

$$\sigma^2 = \frac{2}{9}\left(1 - \frac{1}{c_1 - t}\right)^2 \tag{36}$$

Anisotropic parameter:

$$A_h = 0 \tag{37}$$

Dark energy density parameter:

$$\Omega_d = \frac{\rho_d}{3H^2} = 3b_0 e^{-\frac{1}{2}(3a_1+b_1)(1+\omega)}(c_1 - t)^{\frac{2}{3}(5+2\omega)} \tag{38}$$

Matter density parameter:

$$\Omega_m = \frac{\rho_m}{3H^2} = 3a_0 e^{-\frac{1}{2}(3a_1+b_1)}(c_1 - t)^{\frac{10}{3}} \tag{39}$$

Overall density parameter:

$$\Omega = \Omega_d + \Omega_m = 3\left(a_0 e^{-\frac{1}{2}(3a_1+b_1)} + b_0(c_1 - t)^{\frac{4\omega}{3}} e^{-\frac{1}{2}(3a_1+b_1)(1+\omega)}\right)(c_1 - t)^{\frac{10}{3}} \tag{40}$$

Jerk parameter:

$$j(t) = q + 2q^2 - \frac{\dot{q}}{H} = 28 \tag{41}$$

3. Physical and kinematical properties

For different values of the constants involved, we will obtain different graphs. So, we opt to take particular values of the constants i.e., $a_0 = b_0 = a_1 = b_1 = 1$, $c_1 = 14.301443981790266$ and plot the graphs of some of the parameters showing their variations with time as shown in the figures of this section.

From Eqs. (22) and (23), it is obvious that energy densities of matter ρ_m and dark energy ρ_d are functions of cosmic time. To examine their nature, we plot their graphs showing their variations with cosmic time t as shown in Fig. 1. Here, it can be seen that ρ_m decreases throughout the evolution, as with the expansion of the universe, the galaxies get farther away from each other so that the matter density continues to diminish [4]. But, ρ_d tends to increase very slowly or is nearly unchanged. This may be a result of this anti-gravity dark component varying slowly with time and space [3–5]. So, our model universe will be increasingly dominated by dark energy in the far future.

From Fig. 2, it can be clearly seen that the pressure of dark energy varies in the negative region throughout the evolution which is in agreement with the exotic property of dark energy that causes the universe to expand.

Figs. 3 and 4 respectively show that spatial volume v and the scalar expansion θ increases with cosmic time showing the accelerating spatial expansion of the universe. From Eq. (32), the spatial volume v of the universe is constant ($v \neq 0$) at time $t = 0$. Also, other related parameters are also constant at time $t = 0$. These show that our universe is free from initial singularity at time $t = 0$. But, when $t \to \infty$, both v and $\theta \to 0$ which indicates that after an infinite period of time, there will be a phase transition in which the expansion of the universe will cease. This may be supported by the fact that dark energy which causes the expansion of the universe

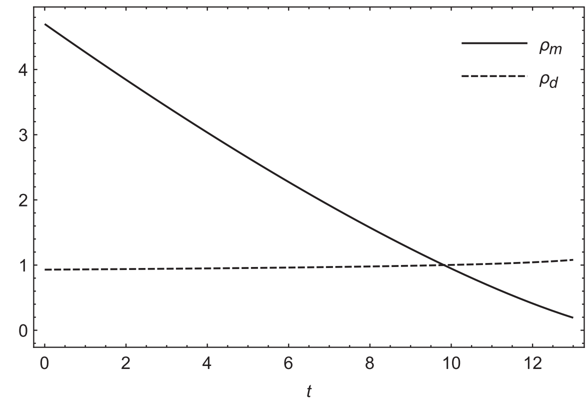

Fig. 1. Variation of energy densities of dark energy ρ_d and matter ρ_m with time t when $a_0 = b_0 = a_1 = b_1 = 1$, $c_1 = 14.301443981790266$ showing that ρ_m decreases throughout the evolution whereas ρ_d tends to increase very slowly or is nearly unchanged.

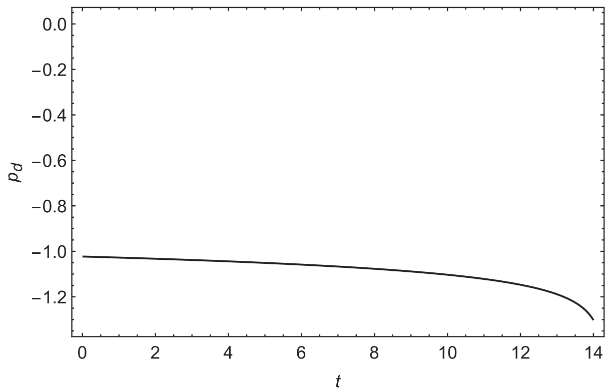

Fig. 2. Variation of pressure of dark energy p_d with time t when $a_0 = b_0 = a_1 = b_1 = 1$, $c_1 = 14.301443981790266$ showing that p_d varies in the negative region throughout evolution.

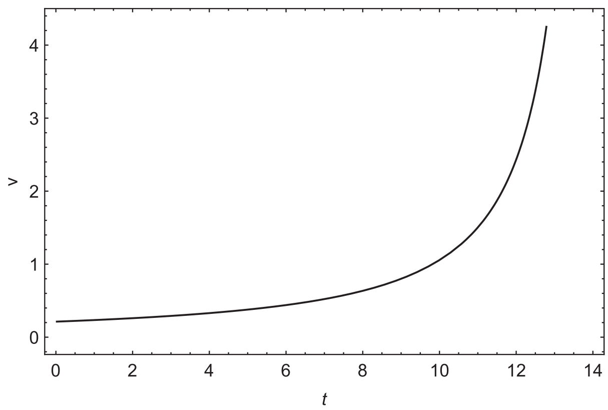

Fig. 3. Variation of volume v with time t when $a_0 = b_0 = a_1 = b_1 = 1$, $c_1 = 14.301443981790266$.

varies slowly with time and space [3–5]. Also, the energy density of dark energy may decreases faster than that of matter leading to the disappearance of dark energy at $t \to \infty$ [5]. Then, our model universe will expand up to a finite degree; the expansion will tend to decrease. So, in the far future, this would lead our universe to be dominated by gravity causing it to shrink; finally collapsing resulting to the big crunch singularity.

The universe experiences accelerated expansion when deceleration parameter q assumes value in the range $(-1, 0)$. But, if $q < -1$,

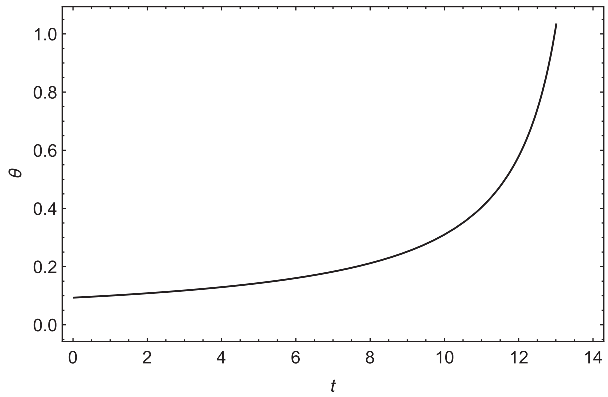

Fig. 4. Variation of scalar expansion θ with time t when $a_0 = b_0 = a_1 = b_1 = 1$, $c_1 = 14.301443981790266$.

then super exponential expansion [75] occurs. From Fig. 5, it is clear that the deceleration parameter q of the model universe is constant all the time with $q = -4$ which shows that our universe undergoes super exponential expansion throughout the evolution.

Fig. 6 shows that shear scalar (σ^2) converges initially and diverges with the increase of cosmic time. Shear scalar provides us the rate of distortion of the matter flow of the large scale structure of cosmology [76]. Here, in our model universe, the universe is expanding with a uniform change of shape.

From Eq. (37), it is clear that anisotropic parameter $A_h = 0$ all the time which indicates that our model universe is isotropic throughout the evolution.

The variations of Ω, Ω_d and Ω_m with cosmic time t are shown in Fig. 7. Here, Ω and Ω_d are decreasing with the increase of cosmic time t and tend to become constant whereas Ω_m decreases but with a greater extent which might be supported by the fact that the matter density is diminishing with the accelerated expansion of the universe [4]. Here, it may be predicted that our model universe will become increasingly dark energy dominated in the far future. Moreover, on assuming that $a_0 = b_0 = a_1 = b_1 = 1$, $c_1 = 14.301443981790266$ and taking EoS parameter $\omega = -1.047$ which is in agreement with the latest observational value of ω [18–20], we find that the expression for EoS given by Eq. (31) is satisfied by time $t_0 = 13.8$ which is age of the universe at the present epoch. Also, under these assumptions, Eq. (40) gives us the value of the overall density parameter $\Omega = 0.905988$ at $t = 13.8$ which is very close to unity and is consistent with the present cosmological belief. Above all, at $t = 13.8$, Eq. (34) gives us the value of Hubble's parameter $H = 68$ which is very close to $H_0 = 67.36 \pm 0.54\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, the value of Hubble's parameter from the latest Planck 2018 results [20].

Lastly, in our study, the jerk parameter $j(t)$ is determined to be 28. Jerk parameter is used to describe the closeness of models to ΛCDM. Its value for flat ΛCDM model is 1. The universe transits from decelerating to accelerating phase when the jerk and deceleration parameters are positive and negative respectively which are satisfied in our case.

4. Conclusion

Here, we have studied a five dimensional spherically symmetric space-time accompanied by minimally interacting fields – matter

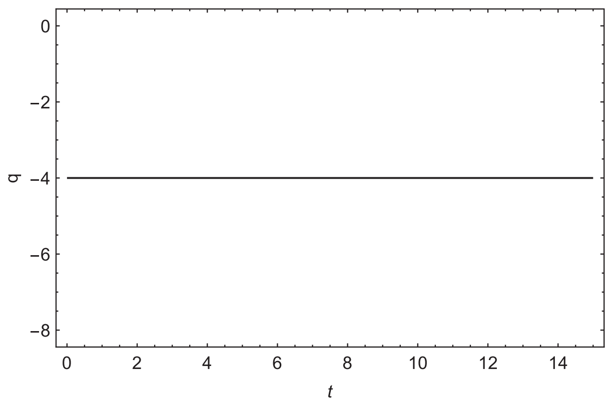

Fig. 5. Variation of deceleration parameter q with time t when $a_0 = b_0 = a_1 = b_1 = 1$, $c_1 = 14.301443981790266$ showing that $q = -4$ throughout evolution so that out universe experiences super exponential expansion as $q < -1$.

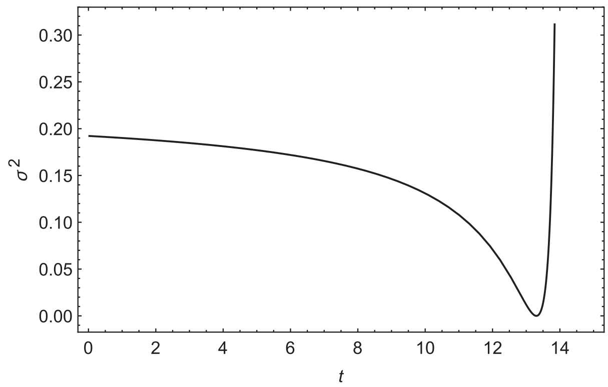

Fig. 6. Variation of shear scalar σ^2 with time t when $a_0 = b_0 = a_1 = b_1 = 1$, $c_1 = 14.301443981790266$ showing that σ^2 converges initially and diverges with the increase of cosmic time so that our universe is expanding with uniform change of shape.

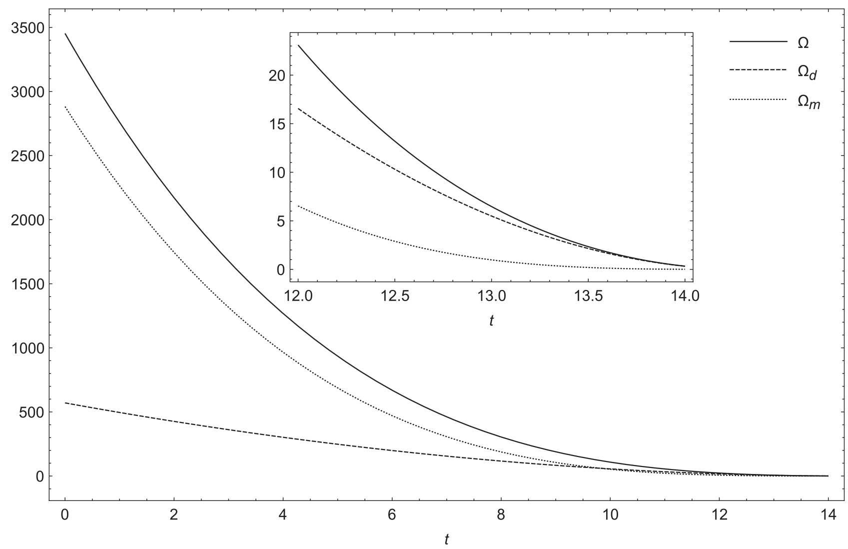

Fig. 7. Variation of overall density parameter Ω, dark energy density parameter Ω_d and matter density parameter Ω_m with time t when $a_0 = b_0 = a_1 = b_1 = 1$, $c_1 = 14.301443981790266$. Here, Ω and Ω_d decreases and tend to become constant whereas Ω_m decreases with a greater extent.

and dark energy components in Brans–Dicke scalar. It is predicted that our model universe will be increasingly dominated by dark energy in the far future. It is observed that the model universe is isotropic throughout the evolution. It is noticed that our universe undergoes super exponential expansion. Our model universe is free from initial singularity at $t = 0$ but may face the big crunch singularity in the far future. With reasonable assumptions of the values of the constants and $\omega = -1.047$ which is consistent with the value of ω from the latest Planck 2018 results [20], we obtain the value of overall density parameter $\Omega = 0.905988(\approx 1)$ which agrees with the present cosmological observation. Above all, at $t = 13.8$, we obtain the value of Hubble's parameter $H = 68$ which is very close to $H_0 = 67.36 \pm 0.54\,\text{km s}^{-1}\text{Mpc}^{-1}$, the value of Hubble's parameter from the latest Planck 2018 results [20].

Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.cjph.2019.05.003.

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