Chapter 1

Introduction

This chapter explains the motivation for conducting the research and introduces the works presented in the thesis. In this chapter, we provide the basic idea of the foundation and formulation of cosmological problems of general relativity. We also present the brief highlights of various concepts, space-time, and theories of gravitations discussed in the following chapters of the thesis. Above all, we also present the literature review of articles to get a better understanding of similar works from the past and present. The review is helpful in assisting to identifying the knowledge gap as well as potential research ideas.

1.1 Motivation

The Earth looks like a beautiful blue marble due to its 70% water content. Even though the dark under the deep blue ocean remains unexplored to a large extent, we at least know water, and its stretch since ocean exploration started a long time ago. But, what if we are visiting an ocean for the first time. The first experience of the amazing view, the sound of the endless cold waves, and the beautiful shoreline will draw us into the magnificent ocean. This will be a magical experience, possibly igniting a spark of interest leading to the birth of a new scientific study. Presently, cosmologists are at the dawn of such a new scientific study since the discovery of the 70% content of the universe called dark energy (DE). The term "dark energy" was coined by Michael S. Turner in 1998. DE is the dominant component of the immense universe, according to literature and observations. This classifies DE as the perfect humour in that the dominating part of the universe is also the least studied. The late distinguished professor of IUCAA and a renowned astrophysicist, Thanu Padmanabhan labelled this dark component as the "Mystery of the Millennium" (Padmanabhan 2006). This dark entity is believed to be the driving force behind the late-time accelerated expansion of the universe. Cosmologists and theorological physicists all over the map, despite investing tremendous scientific efforts, details of its origin, nature, and application to modern cosmology are still up for grabs. Similarly, understanding precisely the origin of the universe, its evolution, and the ultimate fate are no less challenging for modern cosmology. As an effort to broaden our knowledge about the enigmatic DE and the dynamics of the mysterious universe, we have considered an investigation using a 5D spherically symmetric metric paired with some modified theories of gravity which is presented in this thesis titled "**Dark Energy in Higher Dimensional Spherically Symmetric Space-time**".

1.2 Dark energy

Gazing toward the night sky uncovers a little piece of the universe. Despite the fact that the universe seems static to the unaided eye, it is expanding at an expedited rate. Researchers attribute this expanding paradigm to a hypothetical form of energy called dark energy (DE). DE is a natural property of space with a constant energy density and a large negative pressure exerting a gravitationally repulsive effect driving the late-time accelerated expansion of the universe. It is the dominant component making up 70% of the universe. DE isn't straightforwardly noticed but instead deduced from perceptions of gravitational interactions between cosmic objects.

The four meetings of the Prussian Academy of Science during November 1915 can be set apart as the most memorable minutes in the life of the famous Einstein. On the fourth, eleventh, eighteenth, and twenty-fifth of the month, he introduced four of his outstanding communications (Einstein 1915a, 1915b, 1915c, 1915d) at the meetings, which prompted the establishment of the Theory of General Relativity, or simply General Relativity (GR). In 1917, he introduced the cosmological constant into his theory as a repulsive force to act against the attractive gravity to maintain a static universe (Einstein 1917). The cosmological constant is denoted by the Greek alphabet Λ. In 1929, Hubble made ground-breaking discoveries that showed the universe is expanding (Hubble 1929), defying the concept of a static universe. Einstein considered the introduction of Λ to his theory as the "greatest blunder". Finally, he dropped the constant from his work (Einstein 1931). Over the years, Λ went in and out of favour as new observational findings seemed to necessitate it time and again. There were suggestions in the early '90s that Λ might be needed once more.

In 1998, the astronomical observations of distant Type Ia supernovae by two independent teams of astronomers discovered that the rate of expansion of the universe was accelerating, rather than slowing down (Riess et al. 1998; Perlmutter et al. 1999). This discovery was based on the observation that the supernovae appear fainter than expected for a universe decelerating under gravity. For this, the supernovae must be farther away, and the expansion rate should be slower in the past.

Finally, researchers concocted three ways to explain the accelerated expansion. Perhaps it is a consequence of the abandoned Λ term. Perhaps there is some bizarre sort of energy-fluid filling up space. Possibly there is a mistake with the relativity theory, and another optimized theory could incorporate some sort of field that leads to the expedited expansion. Researchers don't have the foggiest idea of the right explanation, yet they termed the possible answer as "dark energy". DE exerts a gravitationally repulsive effect that pushes rather than pulls, driving the miraculous expanding phenomenon. The cosmological constant Λ is considered to be the most natural candidate for DE.

Cosmologists consider the equation of state (EoS) parameter ω a good choice to classify DE into specific categories. ω is defined as the ratio of the pressure of DE to its energy density. The value $\omega = -1$ represents the cosmological constant (CC), or in other words, vacuum energy (VE). Phantom energy has $\omega < -1$, whereas the range $-1 < \omega < \frac{-1}{3}$ signifies quintessence. To construct a cosmological model universe undergoing late-time expedited expansion, one should obtain the range $\omega < -\frac{1}{3}$ (Tripathi et al. 2017). According to the most recent Planck 2018 results (Collaboration et al. 2020), $\omega = -1.03 \pm 0.03$, which is an indication that the form of DE in the present universe is highly likely to be of phantom type.

As mentioned above, the cosmological constant Λ is the most natural candidate for DE. However, it falls short of explaining the enigma of the coincidence problem (Zlatev et al. 1999). After numerous attempts, researchers proposed different candidates of DE (Copeland et al. 2006). One such proposed candidate worth considering is the holographic dark energy (HDE) introduced by Gerard 't Hooft (Hooft 2009). As a result of the holographic principle (Bousso 2002) being applied to DE, HDE is formed. The work of Wang et al. (2017) provides a peek of HDE's fundamental nature and properties.

1.3 Higher dimensional cosmological model

According to our daily experience and observations, it is obvious that we are living in a 3D space with one time dimension i.e., 4D space-time. We can only move forward and backward, left and right, and upward and downward. We notice that our physical laws solely rely on just three spatial dimensions to explain the movements of living and non-living things around us. Then, why is all the fuss about this extra dimension? This question might appear valid, nevertheless, there is no solid logical justification that space-time should have no more four dimensions (Zumino 1986; Overduin & Wesson 1997; Rubakov 2001; Brax & Bruck 2003; Bruck & Longden 2019). The study on extra-dimensions began in order to explain some of the challenges that had arisen in physics, for instance, the cosmological constant problem (Zel'dovich 1967, 1968).

A cosmological model equipped with at least one extra dimension beyond the standard four dimensions is termed a higher dimensional cosmological model. According to Bahrehbakhsh et al. (2011), the first study that presented the construction of a unified theory based on extra dimensions can be found in the work by Nordstorm (1914). There was a time when the hunt for a unified explanation of gravity and particle interaction resulted in a large number of astrophysical works getting stuck. However, the problem was solved in 1921 when Kaluza extend GR from 4D to 5D, uniting gravity with electromagnetism (Kaluza 1921). In 1926, by considering the small size of the extra dimension, Klein modified the method to include quantum effects (Klein 1926). These led to the attribution of the introduction of the higher dimensional cosmological model in GR to Kaluza and Klein. In the following years, researchers have proposed numerous options for the possibility of having more than one extra dimension. As anyone might expect, a conspiracy seems to start to humiliate the advocates of extra dimension. For instance, the theoretical physicist Lee Smolin strongly criticizes string theory which employs extra dimensions, whereas Peter Woit claims that the theory is not even science (Woit 2006; Smolin 2006). Nonetheless, numerous studies have successfully developed compelling justifications for the existence and practical importance of employing extra dimensions.

The higher-dimensional model emerges as one of the good choices among cosmologists and theorological physicists. Such a model can explain both the early inflation and the late time expanding phenomenon of the universe (Farajollahi & Amiri 2010; Banik & Bhuyan K. 2017; Aly 2019). Marciano (1984) discusses a study to validate the existence of the extra dimension. Questions about the nature of DM and DE may find answers in theories involving extra dimensions (Bruck & Longden 2019). According to Zhang (2010), the employment of an extra dimension makes HDE models more complete and consistent. Extra dimensions help to solve the hierarchy problem in a natural way (Randall 2007). Wesson (2015) asserts that the fifth dimension has made a significant contribution to our understanding and the logical consistency of physics. Perhaps our lives would have been less interesting if we haven't been concerned with extra-dimensions.

1.4 Stabilization of extra dimensions

The study on the stabilization of extra dimensions is considered a phenomenological necessity in higher-dimensional models. Generally, we witness the discussion on stabilization in the field of particle physics, supersymmetry, supergravity, string theory, and braneworld models. We require a stabilization mechanism to prevent modification of gravity to an experimentally undesirable manner (Kribs 2006). The stabilization also makes sure the visible 4D universe with a long lifetime (Ketov 2019). Another benefit of stabilization is that we can ignore any unwanted outcomes of quantum gravity at Planck length distances (Hamed et al. 2002). One of the most classic solutions for stabilization is the Goldberger-Wise mechanism (Goldberger & Wise 1999), where stabilization is achieved in the presence of an additional scalar field. The works in this thesis are based on cosmological models in GR. In GR, generally, we cannot find conditions for stabilization, and all dimensions want to be dynamical (Bruck & Longden 2019). In an accelerating model with the cosmological constant, stabilization cannot be obtained (Rador 2007). Notwithstanding this stabilization problem in GR, we have presented, in Chapter 4, a trial to solve the issue in GR. Our work is most likely the first to establish the possible stability condition in GR.

1.5 Spherically symmetric space-time

Since the outset of GR in 1915, spherically symmetric (SS) space-time has garnered ample attention and praise. We can witness works on SS space-time as early as in the papers of 1916 and 1917 by renowned authors (Szenthe 2004b). With the progress of the research on SS space-time, the study on the relativistic theory of cosmology in GR has also been developed (Takeno 1952a). SS space-time can be considered as one of the important tools for studying GR owing to its comparative simplicity and useful applications to both astrophysics and cosmology. It simplifies the study of a system's dynamics by allowing the transformation of a 4D solution to 2D (Parry 2014). The space-time used in relativistic cosmology, including the space-time of the de-Sitter and the Einstein universes, is also SS (Takeno 1952a). The Robertson-Walker space-time model depicting the expanding cosmos is also SS (Karade 1980). To discuss a problem in GR, SS space-time is an excellent option to start with. Deriving non-trivial SS space-time as the exact solution of the Einstein equation is one of the first tasks taken up in GR, a crucial solution in terms of experimental verification of GR (Das & DeBenedictis 2012; Parry 2014). This led to the development of Schwarzschild space-time, which is perhaps the most significant SS solution, and then Birkhoff's theorem along with some of its generalizations (Birkhoff 1923; Wald 1984; Bronnikov & Melnikov 1995; Szenthe 2004a; Jebsen 2005). Some of the remarkable works with a great deal of information about SS space-time can be found in the articles of Takeno (1951, 1952a, 1952b, 1952c, 1952d, 1952e, 1952f, 1953, 1966), Takeno & Ikeda (1953), Kunzle (1967), Clark (1972), Foyster & Mclntosh (1973), Szenthe (2004a), Ferrando & Saez (2010), Tupper et al. (2012), Parry (2014) and Bagde et al. (2021). Since SS space-time is still noteworthy, and there is a lot of content about it spread across the literature, a discussion on it within the framework of GR to understand DE and the accelerating universe would be valuable.

If the isometry group of a space-time contains a subgroup that is isomorphic to the rotation group $SO(3)$, then the space-time is referred to as SS. By expressing space-time in terms of scalars and vectors, Takeno (1951) put forward the definition of a SS space-time in 4D. However, because the thesis mainly looks at higher-dimensional models, we won't go into great length concerning 4D. Takeno (1952e) further defines an n ($n \geq 5$) dimensional SS space-time with the metric tensor g_{ij} as an n dimensional Riemannian space with the following properties:

i. Its curvature tensor satisfies

$$K_{ijlm} = \rho^1 \alpha_{[i}\alpha_{[l}\beta_{j]}\beta_{m]} + \rho^2 g_{[i[l}\alpha_{j]}\alpha_{m]} + \rho^3 g_{[i[l}\beta_{j]}\beta_{m]} + \rho^4 g_{[i[l}g_{j]m]} \tag{1.5.1}$$

where $i, j, ... = 1, ..., n, \alpha_i$ and β_i are mutually orthogonal unit vectors satisfying

$$\nabla_i \alpha_j = \sigma \alpha_i \beta_j + \kappa \left(g_{ij} - \alpha_i \alpha_j - \beta_i \beta_j\right) - \overline{\sigma} \beta_i \beta_j \tag{1.5.2}$$

$$\nabla_i \beta_j = \overline{\sigma} \beta_i \alpha_j + \overline{\kappa} \left(g_{ij} - \alpha_i \alpha_j - \beta_i \beta_j -\right) \sigma \alpha_i \alpha_j \tag{1.5.3}$$

and $\rho^a, (a = 1, ..., 4); \sigma, \overline{\sigma}; \kappa, \overline{\kappa}$ are scalars determined from these equations.

ii. One of the five scalars $\rho^a, (a = 1, ..., 4)$ and $K \equiv K_{ij}^{ji}$ is such that its gradient vector is a linear combination of α_i and β_i.

iii.

$$\rho^4 + 2\left(\kappa^2 + \overline{\kappa}^2\right) \neq 0 \tag{1.5.4}$$

iv. Moreover, for the sake of simplicity and symmetry, the fundamental form is taken positive and $\alpha_s \alpha^s = \beta_s \beta^s = 1$.

In the above properties, ∇ is the usual Riemannian covariant derivative, whereas the ( ) and [ ] respectively denote the usual symmetric and antisymmetric relations. Corresponding theory based on g_{ij} of the type $(- - ... +)$ and $-\alpha_s \alpha^s = \beta_s \beta^s = 1$ can be derived with minor modifications.

Abolghasem et al. (1998) present a general form of the 5D SS metric provided in Eq. (1.5.5) and obtain solutions with potential applications to astrophysics and cosmology.

$$ds^2 = -P^2(t,r,y)\,dt^2 + Q^2(t,r,y)\left(dr^2 + r^2A^2(t,y)\,d\overline{\Omega}^2\right) + R^2(t,r,y)\,dy^2 \qquad (1.5.5)$$

where y is the coordinate corresponding to the extra dimension and $d\overline{\Omega}^2$ is the metric of the 2-sphere given by

$$d\overline{\Omega}^2 = d\Theta^2 + sin^2\Theta d\phi^2 \qquad (1.5.6)$$

In this thesis, we pair some modified theories of gravity with a 5D SS metric (Samanta & Dhal 2013) provided in Eq. (1.5.7) in order to broaden our knowledge about the enigmatic DE and the dynamics of the mysterious universe.

$$ds^2 = dt^2 - e^{\mu}\left(dr^2 + r^2d\Theta^2 + r^2\sin^2\Theta d\phi^2\right) - e^{\delta}dy^2 \qquad (1.5.7)$$

where $\mu = \mu(t)$ and $\delta = \delta(t)$ are cosmic scale factors.

1.6 Cosmology

Cosmology, the scientific investigation of the beginning of the universe and its progression by and large, is as old as humankind. A bone fragment depicting a lunar calendar unearthed in Sub-Saharan Africa circa 20,000 BC is the earliest known proof for cosmological reasoning among humans, and the Nebra sky disc, which dates back to roughly 1,600 BC, is the oldest record of cosmic observation (Corneanu & Corneanu 2016). The term cosmology is derived from the Greek words "kosmos" and "logia" which mean "world" and "study of" respectively. The term first appeared in English in 1656 and was later adopted in Latin by German philosopher C. Wolff in 1731, but it was after WWII that it became scientifically mainstream (Kragh 2007; Hetherington 2014).

Humans have gazed at the stars for centuries, pondering about the mysteries of the universe's dynamics and evolution. However, it wasn't until the '90s when scientists developed modern observational tools and theory, which revolutionizes cosmology forever. This marked the era of the birth of modern cosmology (Topper 2013; Nussbaumer 2014). The present-day modern cosmology has entered a beautiful era - The Golden Age of Cosmological Physics, thanks to the raw data derived from the accurate calculations of different cosmological parameters from several experiments (Garcia-Bellido 2000). The enormous increase in observational approach explicitly committed to cosmological problems demonstrates that modern cosmology is becoming a mature physical science with its own subject and method (Baryshev et al. 2008).

The hot Big Bang model, which explains the universe's evolution from the first fraction of a second to the current era, about 13.8 billion years later, is the foundation of our current knowledge of the cosmos. This model was developed in 1931 by Lemaitre, in which he assumed that the universe expanded from an initial point - "primeval atom" (Lemaitre 1931). It is the most widely accepted theory about how the universe began. It states that the universe began with an infinitely hot and dense singularity, which subsequently inflated, initially, at an extremely high speed, then at a more quantifiable rate throughout the following years to become the universe we see today. The model is homogeneous and isotropic, with matter and radiation fluids as its major components, and kinematic properties agree with those measured in the actual universe. Furthermore, the radiation component of the energy density is considered to be of cosmological origin, which is why the model is referred to as "hot" (Coles & Lucchin 2002). Undoubtedly, our actual universe isn't perfectly homogeneous and isotropic, so that this model has some flaws. However, this standard model offers us a platform to explore the creation of objects like galaxies and their clusters from minor alterations in the density of the early cosmos.

The Big Bang cosmology is based upon four solid foundations, a GR-based theoretical basis, presented by Einstein (1917, 1922) and Friedmann (1922), as well as three fundamental observational facts. The first is Hubble's discovery of the universe's expansion (Hubble 1929). Second, the elucidation of the relative abundance of light elements in the '40s by Gamow (1946, 1948). The third is the cosmic microwave background (CMB), discovered by Penzias & Wilson (1965) as the afterglow of the Big Bang. These findings contributed to the hot Big Bang becoming the most favoured model, and they have been verified to near-perfect accuracy (Garcia-Bellido 2005).

Despite its widespread acceptance, the hot Big Bang model is not free from drawbacks. In the Big Bang scenario, the universe is homogeneous and isotropic. However, these conditions, which appear to be self-evident at first glance, are not fully addressed by the theory. This is a major flaw in this theory, or perhaps, numerous flaws that are interconnected. The inflationary theory is one viable solution. The inflationary theory, in particular, proposes a method to give rise to cosmic perturbations. The Big Bang lacks such a method, which is also a major flaw of the theory. This is one of the reasons that the inflation theory is so appealing. Guth (1981) is credited for introducing the theory of cosmic inflation. The scenario that triggered inflation involves a scalar field in a local (but not global) minimum of its potential energy function (Guth 2004). Starobinsky (1979, 1980) introduced a similar concept a little earlier as a (failed) effort to address the initial singularity problem. Later, the Norwegian Academy of Science and Letters awarded the 2014 Kavli Prize in Astrophysics to Alan Guth, Andrei Linde, and Alexei Starobinsky for pioneering the theory of cosmic inflation. According to the inflationary theory, there was a brief period of extremely rapid cosmological expansion preceding the more gradual Big Bang expansion. During the period, the universe's energy density was dominated by a CC type of VE, which then decayed, resulting in the formation of matter and radiation. An in-depth explanation of inflationary theory as a possible solution to shortcomings of Big Bang theory, viz. horizon problem, flatness problem, entropy problem, and primordial perturbation problem, is presented in the book authored by Gorbunov & Rubakov (2011).

1.7 Theory of general relativity

The four meetings of the Prussian Academy of Science in November 1915 are among Einstein's most memorable moments. At the sessions, he gave four remarkable presentations (Einstein 1915a, 1915b, 1915c, 1915d) that led to the establishment of GR. His intellect reimagined space and time, foreshadowing a universe so strange and vast that it defied human imagination. GR is a fundamental concept in modern physical science. It correlates gravity to curvature of space-time geometry, or, to put it another way, it explains gravity in the context of bending space. Einstein came up with GR after a decade he put forward the special theory of relativity (Einstein 1905), which asserts that space and time are inextricably linked but did not address the presence of gravity. To be specific, GR, as the name implies, is the generalized form of the special theory. The mathematical equations of GR, which have been confirmed repeatedly, are by far the most accurate tool to describe gravitational interactions, effectively replacing those proposed by Newton (Newton 1687) hundreds of years ago. GR is a remarkable achievement. It is now commonly regarded as one of the two foundations of modern physics, alongside quantum field theory. Despite the benefits of GR, due to certain incompatibilities, we don't yet have a quantum field theory counterpart of GR. Harmonizing GR with quantum physics is still a work in progress in modern physics (Kiefer & Weber 2005; Alfonso-Faus 2007; Mamedov 2015; Jakobsen 2020).

In 1918, Einstein put forward the following three principles on which the establishment of GR rests (Einstein 1918).

(a) ***Principle of relativity:*** The laws of nature are only assertions of time-space coincidences; therefore they find their unique, natural expression in generally covariant equations.

(b) ***Principle of equivalence:*** Inertia and weight are identical in essence. From this and from the results of the special theory of relativity, it follows necessarily that the symmetric "fundamental tensor" (g_{ij}) determines the metric properties of space, the inertial relations of bodies in it, as well as gravitational effects. We will call the condition of space, described by the fundamental tensor, the "G-field".

(c) ***Mach's principle:*** The G-field is determined without residue by the masses of bodies. Since mass and energy are equivalent according to the results of the special theory of relativity and since energy is described formally by the symmetric energy tensor (T_{ij}), this means that the G-field is conditioned and determined by the energy tensor.

Further, in the footnote of his work (Einstein 1918), he wrote that he was introducing the term "Mach's principle" for the first time. The principle (a) can also be referred to as the principle of general covariance since the latter is a generalization of principle (a) (Norton 1993). Ellis & Williams (1988) extended principle (b) by stating that "the laws of physics are the same for all observers, no matter what their state of motion".

Carmeli (1982) defined three versions of the general covariance principle as given below, which, he mentioned, were "not quite equivalent".

- All coordinate systems are equally good for stating the laws of physics. Hence, all coordinate systems should be treated on the same footing, too.
- The equations that describe the laws of physics should have tensorial forms and be expressed in a four-dimensional Riemannian space-time.
- The equations describing the laws of physics should have the same form in all coordinate systems.

In 1907, Einstein observed that an object in a free fall doesn't feel its weight, later established as the principle of equivalence (Samaroo 2020). The principle of equivalence incorporates gravity's effects into the formation of GR. It establishes the equivalence of the forces exerted by gravity and acceleration. Accordingly, a physical experiment cannot differentiate between gravitational and acceleration forces. As presented by Pauli (1958), the equivalence principle would be satisfied even though the coordinate system is not physically realizable. In addition, the principle is satisfied if and only if a manifold is physically realizable.

Mach's proposal that inertial motion is regulated by the whole of masses in the universe, rather than Newton's absolute space and time (Mach 1872, 1883) was one of the main influences to Einstein's formulation of GR as it hinted to a relationship between geometry and matter. Later in 1918, Einstein presented a specific statement of it in the framework of GR (Einstein 1918). Einstein later dropped the principle (Einstein 1949) when he established that inertia is implicit in the geodesic equation of motion and doesn't rely on the presence of matter somewhere else in the cosmos. In the literature, there are various ways in which Mach's principle is formalized, particularly in the framework of GR (Barbour & Pfister 1995; Barbour 2010; Putz 2019).

The 4D line element in special theory of relativity is given by

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2 dt^2 \tag{1.7.1}$$

where x, y and z are Cartesian coordinates.

The space-time in GR is described by the pseudo-Riemannian metric given by

$$ds^2 = g_{ij} dx^i dx^j, \ i, j = 1, ..., 4 \tag{1.7.2}$$

This is the generalization of the 4D space-time in special relativity. The symmetric metric tensor g_{ij} acts as gravitational potential. In GR, the space-time is 4D and the gravitational orbits are geodesics. Einstein's field equations (EFE) which describe the behaviour of space and time are given by

$$G_{ij} = R_{ij} - \frac{1}{2} R g_{ij} = -\frac{8\pi G}{c^4} T_{ij} \tag{1.7.3}$$

where G_{ij} is the Einstein tensor, R_{ij} is the Ricci tensor, R is the Ricci scalar (Scalar curvature), c is speed of light in vacuum, G is the Newtonian constant of gravitation and T_{ij} is the energy-momentum tensor due to matter.

In order to maintain a static universe (Einstein 1917), Einstein modified the field equations by introducing the Λ term as follows.

$$G_{ij} = R_{ij} - \frac{1}{2} R g_{ij} + \Lambda g_{ij} = -8\pi T_{ij} \tag{1.7.4}$$

In 1929, Hubble provided breakthrough discoveries to indicate that the universe is expanding, contradicting the idea of a static universe (Hubble 1929). Einstein considered the introduction of the Λ term as the "greatest blunder". Finally, he dropped the constant term from his work (Einstein 1931). The left-hand sides of both the Eqs. (1.7.3) and (1.7.4) represent the geometry of space-time determined by the metric, and the right-hand sides, the matter distribution/energy content of the space-time.

1.8 Modified theory of gravity

Newton proposed the mathematical formalization of gravity in 1687 and presented one of the most important results in physics (Newton 1687), as given below.

$$F = G\frac{m_1 m_2}{r^2} \tag{1.8.1}$$

Newton's work, however, ended as just half of the picture. The second half is GR, postulated by Einstein more than two centuries later. Einstein's GR is regarded as one of the greatest achievements of twentieth-century physics. It has far-reaching implications for many cosmological phenomena. Besides describing the anomalous precession of planetary orbits, it also explains the origin and evolution of the universe, the physics of black holes, and gravitational lensing. In recent years, scientists and engineers have created modern methods and technologies that accurately depict the effects of GR, for instance, the effect of GR ensures Global Positioning System (GPS) gadget detect a location precisely within a few meters (Ashby 1995, 2003).

However, in recent decades, researchers have raised issues that cannot be effectively addressed by GR alone. GR, although its accomplishment in characterising the universe and the solar system, falls short of being the ultimate theory of gravity. With the introduction of the dark universe scenario, the constraints of GR have come into prominence. There has been indications for nearly three decades that if gravity is governed by EFE, then the cosmos should contain a significant quantity of DM, and DE has just been discovered to be required to explain the universe's purported accelerated expansion. If GR is accurate, it appears that approximately 96 percent of the cosmos is made up of energy densities that do not interact electromagnetically (Clifton et al. 2012). Because of such an unusual content, researchers have suggested that GR may not be the right gravity theory to address the universe. The emergence of a dark universe could be another clue that we need to explore outside the scope GR. We may also witness a discussion about the limitations of GR in Krogdahl's work (Krogdahl 2007). As a result, researchers devised alternate theories to extend GR, employing various approaches to generate different field equations and cosmic consequences. Such theories are known as modified theories of gravity or alternatives to GR.

Over the last decade, the concept of modifying gravity theory has exploded in popularity. Such modification has been partly motivated by the introduction of higher-dimensional cosmological models and the advancement in the formulation of renormalizable gravity theories (Clifton et al. 2012). Modifying GR, in general, brings in additional degrees of freedom, which must be effectively filtered on terrestrial and cosmic scales for the modified theory to be credible (Sbisa 2014). Furthermore, one can evaluate the reliability of such theories by reviewing the theory's results with solar system tests and observational findings (Nojiri & Odintsov 2007; Clifton et al. 2012). Notwithstanding the strict limitations of the solar system tests, there are many forms of modified theory that could challenge GR. However, to reconcile such theories with a range of observational data and solar system experiments, a more detailed analysis is required.

Modified theories are a generalization of GR in which a set of curvature invariants substitutes or is introduced to the classical Einstein–Hilbert action. Thus, in this perspective, the early and late-time acceleration of the universe may be induced. In applications for late-time accelerating universe and DE, the modified gravity technique is highly appealing. Moreover, the mathematical framework of modified theories, as well as their features, is a fascinating area of study. A few of the worth mentioning benefits of the modified theory of gravity are listed below.

- A modified theory provides a natural gravitational substitute for DE.
- Such a theory unifies the early inflation and late-time acceleration in a very natural way.
- The transition from decelerating to accelerating universe is well explained by modified theory.
- It could be the foundation for a unified theory of DE and DM. It can also describe some cosmic phenomena, such as galaxy rotation curve.
- Without the need to add any exotic matter, such a theory might naturally characterize the shift from non-phantom to phantom phase. Generally, with modified theory, the cosmic doomsday can be avoided in the phantom type DE model (Nojiri & Odintsov 2007).
- Modified gravity is found to help address the coincidence problem.
- Such a modified theory can explain the source of DE.
- Modified gravity theory also serves as a helpful tool in the field of high-energy physics.

In 1922, Whitehead proposed the Whitehead's theory of gravity, a simpler alternative to GR that does not require any arbitrary parameters (Whitehead 1922) and probably, the first modified theory of gravity. Notwithstanding the arguments revealed in 1971 by Will that Whitehead's theory contradicts experimental results (Will 1971), academics remained interested in it. Further in 2008, Will teamed up with Gibbons and pointed out that the theory falls short in explaining its validity in five different experimental tests, finding that Whitehead's theory is essentially a failure, despite its solid intellectual roots (Gibbons & Will 2008). Since then, many researchers have proposed various fascinating modified theories of gravity that have effectively and convincingly captured the attention of cosmologists. A handful of such theories that have not escaped our attention are Brans-Dicke theory, Saez-Ballester theory, Lyra manifold, scale covariant theory, and $f(R,T)$ gravity. In the upcoming sub-sections, we'll go over these modified theories in more detail.

1.8.1 Scale covariant theory

Many cosmologists have successfully proposed many well-appreciated optimised modified theories of gravity throughout the years, which firmly match with current cosmic trends. One such modification that has caught our interest is the scale covariant theory (SCT) introduced by Canuto et al. (1977a) and Canuto et al. (1977b). They formulated the theory by applying the mathematical operation of scale transformation with the physics of using different dynamical systems to measure space-time distances Canuto et al. (1977a). According to them, the generalized Einstein's field equations are invariant under scale transformation and they successfully investigated many astrophysical tests with SCT Canuto et al. (1977b). In this theory, EFE are valid in gravitational units whereas atomic units are used for physical quantities. The metric tensors associated with these two systems of units are connected by a conformal or scale transformation $\overline{g_{ij}} = \varphi^2(x^k)g_{ij}$, where bar denotes gravitational units and the unbar denotes atomic quantities whereas φ is a gauge function which is a homogeneous function of all space-time coordinates satisfying $0 < \varphi < \infty$, without possessing any wave equation. Using this transformation, Canuto et al. (1977a) and Canuto et al. (1977b) transform the usual Einstein equations into

$$R_{ij} - \frac{1}{2}g_{ij}R + f_{ij}\left(\varphi\right) = -8\pi G(\varphi)T_{ij} + \Lambda\left(\varphi\right)g_{ij} \tag{1.8.2}$$

such that

$$\varphi^2 f_{ij} = 2\varphi\varphi_{i;j} - 4\varphi_{,i}\varphi_{,j} - g_{ij}\left(\varphi\varphi^{,k}_{;k} - \varphi^{,k}\varphi_{,k}\right) \tag{1.8.3}$$

where all the symbols have their usual meanings.

According to Katore et al. (2014), SCT is one of the best alternatives to ETG. This theory permits the variation of the gravitational constant G (Wesson 1980; Will 1984). The ambiguous DE and the mysterious expanding phenomenon have been successfully studied by many authors within the framework of SCT. In the recent study by Singh et al. (2020), it is asserted that SCT might be one of the probable contributors to the late time accelerated expanding phenomenon. Zeyauddin et al. (2020) present a cosmological model in SCT that decelerates during the initial phase and accelerates during the present evolution. Ram et al. (2015) present a forever expanding DE-dominated universe in SCT which tends to the de sitter universe in the future. Naidu et al. (2015) present a DE model with early inflation and late-time acceleration. Katore et al. (2014) investigate three Bianchi space-times involving magnetized anisotropic DE. Zeyauddin & Saha (2013) study an endlessly expanding and shearing model with an initial singularity within the theory. Reddy et al. (2012) construct an expanding DE model in SCT, which doesn't evolve from a singularity in the initial epoch. In the present scenario, SCT paired with DE is considered to align with cosmological observations.

1.8.2 $f(R,T)$ gravity theory

The modified theory of gravity, $f(R,T)$ gravity, introduced by Harko et al. (2011) has the gravitational Lagrangian expressed by an arbitrary function of the Ricci scalar R and the trace T of the energy-momentum tensor. The action of $f(R,T)$ gravity theory is given by

$$S = \int \left(\frac{1}{16\pi} f(R,T) + \mathcal{L}_m \right) \sqrt{-g} d^4x \tag{1.8.4}$$

where $g \equiv det(g_{ij})$, f is an arbitrary function of the Ricci scalar $R = R(g)$ and the trace $T = g^{ij}T_{ij}$ of the energy-momentum tensor of matter T_{ij} defined by Koivisto (2006) as

$$T_{ij} = -\frac{2}{\sqrt{-g}} \frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta\, g^{ij}} \tag{1.8.5}$$

The matter Lagrangian density $\mathcal{L}_m$ is assumed to rely solely on g_{ij}, and hence

$$T_{ij} = g_{ij}\mathcal{L}_m - 2\frac{\partial \mathcal{L}_m}{\partial g^{ij}} \tag{1.8.6}$$

The action S is varied w.r.t. the metric tensor g^{ij}, so that the field equations of $f(R,T)$ gravity is given by

$$f_R\left(R,T\right) R_{ij} - \frac{1}{2} f\left(R,T\right) g_{ij} + \left(g_{ij}\Box - \nabla_i \nabla_j\right) f_R\left(R,T\right) = 8\pi T_{ij} - f_T\left(R,T\right) T_{ij} - f_T\left(R,T\right) \theta_{ij} \tag{1.8.7}$$

where

$$\theta_{ij} = -2\,T_{ij} + g_{ij}\mathcal{L}_m - 2g^{lk}\frac{\partial^2 \mathcal{L}_m}{\partial g^{ij}\partial g^{lk}} \tag{1.8.8}$$

The subscripts appearing in f represent the partial derivative w.r.t. R or T and $\Box \equiv \nabla^i\nabla_i, \nabla_i$ being the covariant derivative.

Taking ρ and p respectively as the energy density and pressure such that the five velocity u^i satisfies $u^i u_i = 1$ and $u^i\nabla_j u_i = 0$, we opt to consider the perfect fluid energy-momentum tensor of the following form

$$T_{ij} = (p+\rho)\,u_i u_j - p g_{ij} \tag{1.8.9}$$

We let $\mathcal{L}_m = -p$ so that Eq. (1.8.8) becomes

$$\theta_{ij} = -2T_{ij} - p g_{ij} \tag{1.8.10}$$

The field equations of $f(R,T)$ gravity, in general, rely on the physical aspect of the matter field too, and therefore there are three types of field equations given by.

$$f(R,T) = \begin{cases} R + 2f(T) \\ f_1(R) + f_2(T) \\ f_1(R) + f_2(R) f_3(T) \end{cases} \tag{1.8.11}$$

Our research will be focused on the type $f(R,T) = R + 2f(T)$, with $f(T)$ as an arbitrary function. Now, the field equations of the theory is reduced to

$$R_{ij} - \frac{1}{2}R g_{ij} = 8\pi\,T_{ij} + 2f'(T)\,T_{ij} + \left\{\,2p\,f'(T) + f(T)\,\right\}\,g_{ij} \tag{1.8.12}$$

where the prime indicates differentiation w.r.t. T, and we consider that $f(T) = \lambda T$, where λ is an arbitrary constant.

The $f(R,T)$ gravity theory has fascinated many cosmologists in recent years since it proposes natural gravitational alternatives for DE (Chirde & Shekh 2019). Myrzakulov (2020) has recently looked into the theory and predicted the requirements for an expanding universe without DE. Mishra et al. (2016b) and Singh & Kumar (2016) study the link of $f(R,T)$ gravity theory with DE. The discussion of model within the theory with DE driven acceleration can be found in the work of Mishra et al. (2016a). Sun & Huang (2016) explore expanding models within the framework of the theory in the absence of DE. Sahoo et al. (2020) investigate a mixture of barotropic fluid and DE within the theory where the universe starts from the Einstein static era and attains ΛCDM. Zia et al. (2018) discuss the theory studying future singularities in a DE model. Fayaz et al. (2016) present a discussion of a DE model with phantom or quintessence scenario. Houndjo and Piattella (2012) redevelop the theory from HDE. It won't be a bad assumption to conclude that the combination of DE and $f(R,T)$ gravity must have some form of hidden relationship.

1.8.3 Brans-Dicke theory

As an alternative to GR, scalar-tensor theories of gravitation are intensively researched. Brans-Dicke theory (BDT) is one such theory that has effectively challenged ETG. BDT was originally formulated as a simple modified theory by Brans & Dicke (1961) with regard to an action developed from a metric g_{ij} and a scalar field φ, exclusively relying on dimensional assertions, where the matter Lagrangian is minimally coupled. In the theory, φ describes the dynamics of gravity, whereas g_{ij} depicts the space-time geometry. The gravity interacts with φ through a dimensionless parameter ω_{BD}, known as the BD coupling parameter, and $G \sim \frac{1}{\varphi}$.

The action for the BDT is given by

$$S = \int d^5x\sqrt{-g}\left[\varphi R - \frac{\omega_{BD}}{\varphi} g^{ij}\varphi_{,i}\varphi_{,j}\right] + \frac{16\pi}{c^4}\int d^5x\sqrt{-g}L_m \tag{1.8.13}$$

where φ is the scalar field, R is the curvature scalar corresponding to the 5D metric g_{ij}, ω_{BD} is the BD coupling parameter, and L_m is the 5D Lagrangian of matter fields. The field equations of g_{ij} from Eq. (1.8.13) is given by

$$R_{ij} - \frac{1}{2}g_{ij}R + \omega_{BD}\varphi^{-2}\left(\varphi_{,i}\varphi_{,j} - \frac{1}{2}g_{ij}\varphi_{,k}\varphi^{,k}\right) + \varphi^{-1}\left(\varphi_{i;j} - g_{ij}\varphi^{k}_{;k}\right) = -8\pi\varphi^{-1}T \tag{1.8.14}$$

where T is the energy momentum tensors for matter field, and R_{ij} is the Ricci tensor. Here, we consider $G = 1 = c$.

The BDT appears to be an intriguing approach to constructing a much more accurate account of the universe, one that provides an account as to why the accelerated expansion is observed only in the present era (Hrycyna & Szydlowski 2013a). BDT is a viable alternative to GR for explaining the accelerated expansion of the universe, and it also passes the solar system tests (Dubey et al. 2021). Among all the known modified theories, the BDT is perhaps the most favourable, since it has effectively handled the difficulties of inflation as well as the early and late time dynamics of the cosmos (Kumar et al. 2020). BD scalar field can be considered as a DE candidate (Zia & Maurya 2018). Higher-dimensional BDTs are being considered as possible options for studying cosmic acceleration (Qiang et al. 2005). According to Tripathy et al. (2015), BDT has proven to be a preferable option to study GR, thus it's worth discussing DE models within the theory.

1.8.4 Saez-Ballester theory

The Saez-Ballester theory (SBT) is also one of the scalar-tensor theories of gravitation that many authors prefer to investigate GR. The theory was introduced by Saez & Ballester (1986). Its applications to cosmology yield reasonable findings. In the SBT, unlike in BDT, the scalar field does not serve the part of varying G. Rather, it's regarded as a dimensionless field that doesn't have to adhere to any restrictions imposed by observations. The intensity of the coupling between gravity and the scalar field is defined by a dimensionless parameter ω_{SB}, known as the SB coupling parameter. Weak fields are adequately described by this coupling (Singh & Shriram 2003).

The action for the SBT is given by

$$S = \int d^5x\sqrt{-g}\left[\varphi R - \omega_{SB}\varphi^n g^{ij}\varphi_{,i}\varphi_{,j}\right] + 8\pi L_m \tag{1.8.15}$$

where φ is the scalar field, R is the curvature scalar corresponding to the 5D metric g_{ij}, ω_{SB} is the SB coupling parameter, and L_m is the 5D Lagrangian of matter fields. The field equations of g_{ij} from Eq. (1.8.15) is given by

$$R_{ij} - \frac{1}{2}g_{ij}R - \omega_{SB}\varphi^n\left(\varphi_{,i}\varphi_{,j} - \frac{1}{2}g_{ij}\varphi_{,k}\varphi^{,k}\right) = -T \tag{1.8.16}$$

where T is the energy momentum tensors for matter field, and R_{ij} is the Ricci tensor. Here, φ satisfies

$$2\varphi^n\varphi^{;i}_{;i} + n\varphi^{n-1}\varphi_{,k}\varphi^{,k} = 0 \tag{1.8.17}$$

where n is an arbitrary constant.

The dimensionless scalar field in SBT can lead to the emergence of an anti-gravity phase (Singh & Shriram 2003), which can be related to the anti-gravity DE. Rao et al. (2012) present a DE model in SBT, obtaining results agreeing with recent observations. It has been proved that the missing matter problem in cosmology can be solved by SBT (Rasouli & Moniz 201). The investigation of SBT draws the attention of numerous researchers because of its importance in explaining the initial phases of evolution (Mohanty & Sahu 2003). The SBT scalar field is crucial when considering DE models and the initial phases of the evolution (Naidu et al. 2012). Within cosmology, different consequences from the SBT are frequently used to derive solutions in either 4D or 5D by adopting different line elements (Mohanty et al. 2007; Naidu et al. 2012; Pimentel 1987; Rao et al. 2012; Rao et al. 2015; Singh & Agrawal 1991; Singh & Shriram 2003; Yadav 2013). Owing to its useful applications in cosmology, SBT has recently been investigated by several authors to study DE and the accelerating universe, both in 4D and 5D (Aditya & Reddy 2018; Mishra & Chand 2020; Naidu et al. 2012, 2021; Pradhan et al. 2013; Raju et al. 2016; Ramesh & Umadevi 2016; Rao et al. 2015, 2018a, 2018b; Reddy 2017; Reddy et al. 2016b; Santhi & Sobhanbabu 2020; Shaikh et al. 2019; Sharma et al. 2019; Vinutha et al. 2019).

1.8.5 Lyra manifold

The sessions of the Prussian Academy of Science held during November 1915 might be considered as the most memorable moments of Albert Einstein's life. During the event, he revealed four of his most important works (Einstein 1915a, 1915b, 1915c, 1915d), which contributed to the formation of GR. Since then, different authors have investigated gravity in various contexts. Weyl (1918) was the first to try to extend GR to combine gravity and electromagnetic forces geometrically. Lyra's modification (Lyra 1951), similar to Weyl's, introduces a gauge function into the structureless manifold, resulting in one of the well-known modified theories of gravity. The modified EFE based on Lyra manifold (LM) were obtained by Sen (1957) and Sen & Dunn (1971).

The field equations are derived from the Lagrangian density

$$L = K\sqrt{-g}\left(x^0\right)^4 \tag{1.8.18}$$

where K is the contracted curvature scalar (Sen 1957). The simplification of Eq. (1.8.18), with the consideration of the natural gauge $x^0 = 1$ yields the field equations given by

$$R_{ij} - \frac{1}{2}g_{ij}R + \frac{3}{2}\varphi_i\varphi_j - \frac{3}{4}g_{ij}\varphi_k\varphi^k = -T_{ij} \tag{1.8.19}$$

where φ_i is the displacement vector and other symbols have their usual meaning as in Riemannian geometry. The displacement vector φ_i takes the time dependent form

$$\varphi_i = (\beta(t),\ 0,\ 0,\ 0,\ 0) \tag{1.8.20}$$

The notion that φ_i is time-independent, i.e. constant, is ambiguous because no particular scientific justification exists for how a constant displacement vector aids to the late-time expansion of the universe at an expedited rate. (Yadav 2020). Above everything, considering a constant displacement vector field is purely for simplicity's sake and has no scientific basis (Singh & Desikan 1997).

In recent years, cosmologists have become increasingly interested in studying the enigmatic DE coupled with the LM. We can see a DE universe in LM in an article by Hova (2013), which demonstrates that the expansion can be attained without any negative pressure energy element. Khurshudyan et al. (2014) present a paper on the examination of a two-component DE model in LM. Bhardwaj & Rana (2020) look at the presence of LM in the context of normal matter and DE interaction. In agreement with the current observation, Ram et al. (2020) propose a DE cosmological model coupled with LM. The study of Patra et al. (2019) discusses the influence of DE on models with linearly changing deceleration parameters in LM. Aditya et al. (2019) investigate a KK DE model in LM, resulting in an exponentially expanding cosmos. According to Singh & Sharma (2014a), a DE model in LM with a constant deceleration parameter may be developed. We may conclude from these noteworthy research works that the LM may be one of the viable candidates for studying DE and the expansion of the universe.

1.9 Literature review

A literature review is an examination of scholarly sources on a particular subject. It gives us a broad perspective of current knowledge, helping us spot pertinent ideas, methodologies, and future research. It is a vital chapter of the thesis, with the objective of providing context and rationale for the study conducted (Bruce 1994). A literature review, in principle, recognizes, analyses, and reconstructs significant literature in a particular area of study. It elucidates how knowledge has progressed in the domain, the previously performed research, the widely recognized concept, the new focus of interest, and the present state of knowledge on the subject. It is critical for researchers to be able to determine what is known about a specific topic and, by extension, what is unknown. A substantive, thorough, and sophisticated literature review is a precondition for doing substantive, thorough, and sophisticated research (Boote & Beile 2005).

1.9.1 Literature review on related works

In this section, a total of 108 research articles have been examined, with the key methodology and findings of the investigations highlighted. Most of them are articles published in the last few years.

Ali et al. (2015) present a comprehensive categorization of SS space-time through Noether symmetries. According to the authors, SS solutions to EFE are crucial in GR. The study of SS space-times is intriguing as it aids in expanding our knowledge on gravitational collapse and black holes. The quest for SS space-times is a crucial undertaking, given their importance in comprehending the universe.

Adhav et al. (2015) investigate a Bianchi type-V universe, which is spatially homogeneous and anisotropic, where DM and HDE interaction occurs. The authors use a particular form of deceleration parameter and special law of variation of Hubble parameter to obtain exact solutions. There is no coincidence problem when the interaction between DM and DE is suitably defined. In addition, the anisotropy fades fast and is replaced by isotropy within a short period. The authors obtain the largest value of the Hubble parameter and the fastest expansion of the universe.

Adhav et al. (2014) investigate a Bianchi type-I universe, which is anisotropic and homogeneous, where DM and HDE interaction occurs. The authors study models with two forms of deceleration parameter, one is of a fixed value and the other is of a particular form. There is no coincidence problem when the interaction between DM and DE is suitably defined. The anisotropy fades fast and is replaced by isotropy within a short period. The authors use the statefinder parameters to differentiate their DE models from the models developed by other authors.

Aditya et al. (2019) study the behaviour of a KK DE model in the presence of a large scalar field in the LM. The DE model corresponds to ΛCDM. The energy density of the model is positive and decreases during evolution. The values of Hubble parameter, scalar expansion, and shear scalar are finite at $t = 0$ and tend to infinity at $t \to \infty$. The enormous scalar field impacts all of the parameters at the minimum scale.

Aditya et al. (2021) study a Bianchi type-VI_0 DE cosmological model in the presence of a large scalar field. The model is non-singular and undergoes early inflation. The cosmological parameters H, θ, and σ^2 are finite at $t = 0$ and tend to infinity at $t \to \infty$. The anisotropy fades and is replaced by isotropy at late time. The authors obtain a phantom DE model which approaches ΛCDM at late time.

Aditya & Reddy (2018) investigate anisotropic HDE models in SBT. The model universe undergoes a transition from decelerating to accelerating phase during evolution. The EoS parameter of the interacting model crosses the phantom divide, whereas the EoS parameter of the non-interacting model tends to -1. The DE density parameter is obtained to be $\Omega = 0.73$.

Adler & Overduin (2005) investigate the shape of the universe. They claim that the universe is nearly flat (not exactly flat). They also provide three interpretations of a nearly flat universe. They claim that all three interpretations are equivalent and are based on a particular constant.

Agarwal (2011) discusses a Bianchi Type-II cosmological model in LM. The authors obtain models which evolved from an initial singularity, which are expanding and shearing. The displacement vector corresponds to the cosmological constant. The authors also discuss the entropy of the universe.

Ahmed & Pradhan (2020a) explore the accelerated expansion of an FRW universe and the evolution of DE across the cosmological constant boundary in universal extra dimensions. The model is homogeneous and anisotropic. The model universe undergoes a transition from decelerating to accelerating phase during evolution. The authors assert that in the present epoch, the DE of the model is of phantom type. During evolution, the DE EoS parameter crosses the phantom divide.

Ahmed et al. (2016) investigate a Bianchi type-V model universe within the framework of $f(R,T)$ theory, with field equation of the class $f(R,T) = f_1(R) + f_2(T)$. The model universe is expanding, shearing, and non-rotating. The cosmological constant of the model decreases and tends to a small positive value in the far future.

Alcaniz (2006) states that the so-called DE, a pretty absurd content of the universe, offers one of the biggest struggles cosmologists have ever faced. The authors proposed three different possible forms of DE. They explored all of these possible forms and appear to be capable to describe some of the present cosmological observations, but no definite judgement on the present characteristics of DE can be reached.

Aly (2019) investigate a HDE model in a $n + 1$ dimensional FRW universe. The model accelerates driven by DE of phantom nature. A higher dimensional model is a good choice to explain the late time expanding phenomenon. The constructed model doesn't show any ΛCDM character.

Amirhashchi (2017) investigates the behaviour of DE in the context of an anisotropic Bianchi type-V universe. The DE EoS parameter is compared for viscous and non-viscous cases, and a correlation of DE with quintessence is established. Finally, the author looks at the circumstances within which the Bianchi type-V universe may be transformed to the FRW universe.

Arapoğlu et al. (2018) study the dynamics of a 5D universe in the context of dynamical system analysis. The authors predict that with EoS $\omega < -\frac{1}{3}$, one can obtain a flat universe, however, stabilization of the extra dimension is not achieved. With $\omega > -\frac{1}{3}$, the stabilization problem can be solved.

Araujo (2005) presents a discussion on the dynamics of a DE-dominated universe. The author examines and analyses cosmological models with a DE component, with a unique property of unending accelerating. It is mentioned that the universe might start evolving with and ends at an inflationary epoch. Further, if the DE is of CC type, the universe will ultimately go through an exponential expansion scenario.

Baushev (2010) states that the DE component that pervades the present universe is of phantom type and its density increases as the universe expand. This will the density to tend to an infinite quantity, ultimately leading to the cosmic doomsday, the big rip. However, with certain arguments and explanations, the author claims that the universe is free from any type of singularity.

Benvenuto et al. (2004) present the cosmological constraints on the variation of G. It is believed that G is a function of cosmic time. Here, in the study, the authors predicts two possible bounds on the variation. According to the authors, if $\dot{G} < 0$, the allowed bound on the variation is $-2.5 \times 10^{-10} \leq \frac{\dot{G}}{G} \leq 0$ yr^{-1}, whereas for $\dot{G} > 0$, we have $0 \leq \frac{\dot{G}}{G} \leq 4 \times 10^{-10}$ yr^{-1}.

Berezhiani et al. (2017) attempt to explain the mechanism behind the accelerated expansion of the universe. It is believed that the expanding phenomenon can be explained by two approaches - the dark energy approach, and the other is the modified gravity approach. However, the authors present an approach that explains the expanding paradigm by DM-baryon interactions, in the absence of DE.

Biswas et al. (2019) discuss the dynamics of a generalized ghost DE model in the FRW universe. The cosmological parameter ρ_d is assumed to be in the form $\rho_d = aH + bH^2$, where a and b are constants. It is observed that ρ_d decreases with expansion. The value of the Hubble parameter is obtained to be $H \approx 68.9$. The DE EoS parameter lies with the phantom region and tends to -1 in the future. The model undergoes a transition from decelerating to accelerating phase.

Bruck & Longden (2019) investigate a theory of modified gravity with extra dimensions. The study on the stabilization of extra dimensions is considered a phenomenological necessity in higher-dimensional models. However, according to the authors, in GR, generally, we cannot find conditions for stabilization, and all dimensions want to be dynamical.

Calder & Lahav (2008) state that one of science's greatest puzzles is DE. The origin and the idea of DE are tracked historically to Newton and Hooke of the seventeenth century in the work. The authors also discuss a hypothetical relationship between the CC and the total mass of the universe.

Capolupo (2018) investigate the dynamics of vacuum condensates, which describe a wide range of physical processes. The author mentions that many attempts have been undertaken to learn more about the properties and origins of DE. According to the author, vacuum condensate may lead the way to the DE source.

Carroll (2001a) put forward a review of cosmology of the existence of CC and the physics of a minimal VE. It is an obvious fact that the universe is dominated by the cryptic DE with negative pressure and positive energy density. However, the author asserts that NED is possible only if the DE is in the form of VE.

Chakraborty & Debnath (2010) construct a 4+d dimensional EFE in a 4D space-time with a FRW metric. The model is anisotropic. The authors claim that the unknown extra dimensions might be related to two unseen DE and DM.

Chan (2015a) presents an interesting study to highlight the DE problem and discuss a natural approach to solve it. According to the author, recent data suggest that the expansion rate of our universe is decreasing, casting doubt on the standard ΛCDM model. The author further claims that the presence of particles with imaginary energy densities can explain the decreasing rate and give a comprehensive answer to the root of DE.

Collaboration et al. (2020) discuss and present the values of various cosmological parameters. This is the most recent Planck results and considered to be one of the most standard results of astrophysics. The values of some of the cosmological parameters predicted in the work are $\Omega_d = 0.679 \pm 0.013$, $\Omega_m = 0.315 \pm 0.007$, $H_0 = 67.4 \pm 0.5$ km^{-1} Mpc^{-1} and $\omega = -1.03 \pm 0.03$.

Copeland et al. (2006) address the dynamics of DE. The authors present the explanation in favour of DE, as well as current advancements in determining its characteristics. They examine the observable information for the universe's current rapid expansion and propose a variety of DE models. They present different aspects of the possible future singularities and approaches to avoid cosmic doomsday. They also propose methods of modifying gravity that can induce accelerated expansion in the absence of DE.

Dasunaidu et al. (2018) study cosmic string in $f(R,T)$ gravity theory with the consideration of a 5D SS space-time. The model doesn't evolve from an initial singularity and is anisotropic all through. All the cosmological parameters, except the increasing volume, vanish at $t \to \infty$. The value of the deceleration parameter is obtained to be $q = -0.73$.

Dikshit (2019) presents a study discussing different aspects of the universe. The author put forward a pure quantum mechanical approach to explain DE. Additionally, the work also discusses the shape, size, and age of the universe.

Dubey et al. (2021) construct an interacting HDE model in BDT in FRW universe. The model constructed can induce an early decelerating phase, followed by an expedited expansion of the universe at late time. The DE component of the model starts from the quintessence region and crossed the phantom divide line. The value of the parameter H attains the value 70 in the far future. According to the authors, BDT is a viable alternative to GR for explaining the accelerated expansion of the universe, and it also passes the solar system tests.

Dubey & Sharma (2021) consider studying different HDE models in their work. They compare and contrast their newly defined DE models using $r - s$ and $r - q$ trajectories. Some of the DE models agree with the standard ΛCDM model. They also discuss the stability of the newly defined model with squared sound speed.

Farajollahi & Amiri (2010) study a 5D cosmology within the framework of KK cosmology. The 4D part of the model is taken to be FRW, while the fifth dimensional part consists of DE density. The authors use the model to describe the early inflation and late time acceleration of the universe.

Gontijo (2012) tries to find out a possible DE source. In the study, the author presents a physical mechanism as one of the origins of DE. The author asserts that the study could offer up new possibilities in cosmology by reinterpreting the dark entities as a scalar field contained in the space-time metric.

Gutierrez (2015) presents a discussion on the evolution of DE. The author explains that among the most significant unresolved problems among the cosmological society is the expansion of the universe at an expedited, which is induced by the component DE. This enigmatic DE makes up around 70% of the universe. The author goes through the present state of DE experimental results and gives a brief overview of the future studies that will make us understand in detail about this dark component.

Hrycyna & Szydłowski (2013a) discuss BDT using a scalar field potential function. They show that emergence of the ΛCDM scenario from BDT. According to them, BDT appears to be an intriguing approach to go toward constructing a much more accurate account of the universe, one that provides an account as to why the accelerated expansion is observed only in the present era.

Huterer & Shafer (2018) briefly recap the events that revealed the presence of DE. The parametric representations of DE and the cosmological tests that assist us in familiarizing ourselves with its characteristics are discussed. The cosmic investigations of dark energy are also presented. The authors also discuss the underlying mechanism of each investigation. Finally, they go through the present state of DE research.

Joyce et al. (2016) address the comparison of DE with modified gravity theory. Knowing the cause of the apparent expansion of the universe at an expedited rate is one most basic unanswered issues of science. According to the authors, a distinction of DE from modified gravity has been developed among physical theories for this expedition. They present a summary of models in both cases, and also about their behaviour and nature. They also make a clear difference between DE and modified gravity.

Knop et al. (2003) measure the values of the cosmological parameter Ω_d, Ω_m and ω with WFPC2 on the Hubble Space Telescope. Under two conditions, they measure two different bound on -1.61$< \omega <$-0.78 and -1.67 $< \omega <$ -0.62. The values of Ω_d and Ω_m are measured to be $\Omega_d = 0.75^{+0.06}_{-0.07}$ (statistical) ± 0.04 (identified systematics) and $\Omega_m = 0.25^{+0.07}_{-0.06}$ (statistical) ±0.04 (identified systematics), respectively.

Kolb et al. (2006) address the universe's late-time acceleration at an expedited rate. The expanding phenomena is caused by the DE component of the cosmos, which is widely acknowledged. In the absence of DE, however, the authors anticipate a circumstance that will result in acceleration. The back-reaction of cosmic perturbations, they say, is responsible for the rapid expanding phenomena.

Korunur (2019) discusses a Bianchi type-III universe with a DE component as the Tsallis HDE. It is found that under two conditions, the DE component corresponds to quintessence nature and phantom nature. The one with phantom nature attains the standard ΛCDM during evolution. The author also establishes a link between the model and few popularly used scalar fields.

Kumar et al. (2020) investigate anisotropic DE models in BDT considering an LRS metric. The assert that among all the known modified theories, the BDT is perhaps the most favourable, since it has effectively handled the difficulties of inflation as well as the early and late time dynamics of the cosmos.

Kumar & Suresh (2005) study a fascinating topic to discuss the validity of a higher-dimensional universe. The authors present a quick run-down of the theories incorporating the concept of extra dimensions, spanning earlier periods to the current day. The work ends with some visualizing examples and a brief explanation of the astrophysical consequences and probable existence of extra dimensions.

Macorrav & German (2004) discuss the cosmology of scalar fields. In the cosmological society, it is an accepted fact that the universe is dominated by the DE with negative pressure and positive energy density. However, the authors present an explanation of energy density with negative value with equation of state parameter (EoS) $\omega < -1$.

Mishra et al. (2016a) study an anisotropic universe in $f(R,T)$ gravity theory. They consider a Bianchi type-VI_h space-time, where $h = -1, 0, 1$. When $h = -1, 0$, the CC starts with a divergent nature and tends to become small at late time. The EoS parameters of these two models are also found to be negative. Both the models show quintessence under certain conditions. However, $h = 1$ doesn't yield a reliable model.

Mishra & Chand (2020) investigate a Bianchi type-I universe with perfect fluid in SBT.

The model universe undergoes a transition from accelerating to decelerating phase. The parameter ρ decrease with cosmic time. It is seen that Ω_m and Ω_d have the same values during the early evolution, whereas Ω increases from negative to positive and tends to a constant value. The model is also found to be very close to the standard ΛCDM model.

Mohanty et al. (2008) discuss 5D models in LM. In one model, it is seen that the LM perishes, and the metric coefficients tend to remain unchanged. Additionally, the gauge function also tends to become constant at $t = 0$, and vanishes at $t \to \infty$. In another model, the extra dimension shrinks with the increase of cosmic time. Further, the LM scenario will fade away quickly.

Mollah et al. (2018) consider a Bianchi type-II model universe in LM. In the study, the authors use a quadratic EoS. The deceleration parameter of the model tends to -1 at $t \to \infty$. The model is anisotropic all through. The parameter ρ is always positive, and σ^2 vanished at $t \to \infty$. The expansion scalar θ evolved with a very large value and becomes constant with the increase of cosmic time.

Moradpour et al. (2013) emphasize the presence of DE in the universe. Thermodynamic reasons are used to support their assertion. They assert that the universe with require a DE component with the EoS parameter $\omega < -\frac{1}{3}$. The presence of a DE component in the universe would cause it to achieve a thermal equilibrium.

Muley & Nagpure (2016) attempt to study the dynamics of a homogeneous cosmological model in LM considering a SS space-time. The expanding model evolves from an initial singularity. At $t \to \infty$, the cosmic parameters V, θ, and σ^2 tend to vanish, which is an indication that the model universe will end at the big crunch singularity. The anisotropy of the model fades and is replaced by isotropy at infinite time.

Narain & Li (2018) investigate the late-time acceleration of the universe at an expedited rate. They believe DE is the driving force behind the expanding phenomenon. However, the authors predict a condition to obtain acceleration, not because of DE. They present an interesting work to obtain acceleration from an Ultraviolet Complete Theory.

Neiser (2020) develops a cosmological model associated with an antineutrino star to search the origin of DE. The author asserts that the degenerate remains of an antineutrino star might have a mass density that is comparable to the DE density in the standard ΛCDM. Further, it is mentioned that the developed model could explain to us the root of

DE.

Parry (2014) presents a survey on SS space-times. SS space-time can be considered as one of the important tools for studying GR owing to its comparative simplicity and useful applications to both astrophysics and cosmology. It simplifies the study of a system's dynamics by allowing the transformation of a 4D solution to 2D. As a result, it's a good idea to start exploring GR with SS space-time.

Rao et al. (2018a) study an anisotropic cosmological model filled with matter and holographic Ricci DE in SBT. To obtain exact solutions, the authors consider two cosmological assumptions. The model universe undergoes accelerated expansion. The cosmological parameters H, θ, ρ_m, ρ_d and σ^2 diverge at $t = 0$, and tend to become constant at $t \to \infty$. The authors conclude by mentioning that scalar field φ is indeed an important parameter in DE cosmology.

Rao et al. (2018b) investigate a plane-symmetric model universe in the presence of matter and DE in SBT. The model universe undergoes a transition from decelerating to accelerating phase during evolution. The $r - s$ plane of the model corresponds to ΛCDM limit. The EoS parameter of the model implies a quintom scenario.

Rao & Jaysudha (2015) consider a 5D SS space-time in BDT of gravitation. The exact solutions are obtained under the assumption of two certain cosmological conditions. The expanding model universe is found to be isotropic. The expanding model evolved from an initial singularity. The cosmological parameters H, p and ρ diverge and vanish at $t \to \infty$.

Rao & Rao (2015) study a 5D anisotropic DE model in $f(R,T)$ gravity. The model undergoes early inflation and late-time acceleration. The model universe is found to be expanding, shearing, and non-rotating. The anisotropy fades and is replaced by isotropy at late time. The DE EoS parameter is obtained to be $\omega = -1$.

Rasouli & Moniz (2017) attempt to construct a 4D modified SBT from 5D SBT with the application of an intrinsic dimensional reduction. On contrary to usual SBT, the constructed 4D modified SBT is found to have significant new characteristics. According to the authors, the extra dimensions shrink with cosmic time.

Reddy (2017) studies a spatially homogeneous and anisotropic Bianchi type-V model universe in the presence of matter and DE in SBT. The author obtains three cosmological models that undergo a transition from decelerating to accelerating phase during evolution. The models evolved from an initial singularity. One of the models shows a constant value of H, implying a continuously expanding universe at a constant rate throughout evolution.

Reddy (2018) investigate a cosmological model considering a 5D SS metric within the framework of LM. The model doesn't come across an initial singularity during evolution. The displacement vector of the model is found to diverge. The constructed model universe is found to be isotropic experiencing expansion at an expedited rate.

Reddy et al. (2016a) present an expanding 5D model universe within the framework of DBT, where DE-DM interactions occur. To obtain exact solutions, the authors apply two reasonable cosmological assumptions. It is found that the H, θ and σ^2 diverge at $t = 0$, and tend to vanish at $t \to \infty$. The model universe is anisotropic throughout evolution. The DE EoS parameter corresponds to the phantom scenario.

Reddy et al. (2016b) study a Bianchi type-VI_0 model universe in the presence of interacting matter and DE in SBT. The authors apply a hybrid expansion law to obtain exact solutions. The authors obtain an expanding universe, evolving from an initial singularity. The DE EoS parameter crosses the phantom divide. The model undergoes a transition from decelerating to accelerating phase during evolution.

Reddy et al. (2012) discuss a Bianchi type-I DE cosmological model universe in SCT. The model doesn't evolve from an initial singularity. It is found that the H, θ and σ^2 diverge at $t = 0$, and tend to vanish at $t \to \infty$. The model universe is anisotropic throughout evolution.

Reddy et al. (2016) construct a 5D universe in the presence of interacting matter and DE within the framework of BDT. The authors consider a 5D SS space-time in their work. The model doesn't evolve from an initial singularity. It is found that H, θ, ρ_m, ρ_d, p_d and σ^2 diverge at $t = 0$ and vanish at $t \to \infty$. Under certain conditions, the model reduces to the standard ΛCDM model.

Sadjadi & Vadood (2008) present a note on an interacting HDE model in the FRW universe and study the nature of DE density in an expanding scenario. They discuss the characteristics and dynamics of HDE. They investigate the EoS of the model crossing the phantom divide. They predict some conditions that will lead to a transition from quintessence to the phantom scenario. These conditions might also help in alleviating the coincidence problem.

Sadri & Khurshudyan (2019) study both interacting and non-interacting new NHDE models within the framework of a spatially flat FRW universe. The EoS parameter and the parameter q describe an accelerating universe. The $r-s$ diagnosis reveals that the DE component of the interacting and non-interacting models correspond to quintessence and phantom nature, respectively.

Saha & Ghose (2020) explore the Tsallis HDE model experiencing accelerated expansion in 5D. In the context of Compact KK gravity, an interacting DE is presented using Generalized Chaplygin gas. The authors point out that the DE dominating the model universe might have evolved from the phantom phase during the early evolution.

Sahoo & Singh (2003) investigate a homogeneous and isotropic cosmological model within the framework of a generalized BDT. The BD scalar field decreases with cosmic time. The authors also found that the variational of gravitation constant G is safely below 4×10^{-10} yr^{-1}.

Sahoo & Mishra (2014a) study an anisotropic DE cosmological model considering a 4D space-time. It is predicted that the model universe can attain isotropy during evolution. The authors obtain the largest value of the Hubble parameter and the fastest expansion of the universe. The cosmological parameters ρ and ω diverge when $t \to \infty$ and remain constant at $t = 0$.

Sahoo & Mishra (2014b) construct an accelerated expanding 5D KK space-time with wet dark fluid in $f(R,T)$ gravity theory. The authors use a new DE EoS in the form of wet dark fluid. The model undergoes the early inflation and late-time acceleration. The authors claim that accelerated expansion depends on geometric contribution and matter content. The model attains isotropy during evolution. The cosmological parameter θ is constant, whereas σ^2 is finite and vanishes at $t \to \infty$.

Sahoo et al. (2020) discuss a model in $f(R,T)$ filled with barotropic fluid and DE. The authors claim that accelerated expansion depends on geometric contribution and matter content. The model evolved with large positive ρ and large negative p. However, these two parameters vanish at $t \to \infty$. The model universe is anisotropic throughout evolution. During evolution, the model universe attains ΛCDM in the future.

Samanta & Dhal (2013) present a 5D expanding cosmological model in $f(R,T)$ gravity theory, considering a 5D SS space-time. The model universe is isotropic throughout evolution. According to the authors, the extra dimensions shrink with cosmic time. The model evolved with large positive ρ and large negative p. However, these two parameters vanish at $t \to \infty$.

Samanta et al. (2014) investigate an accelerated expanding 5D space-time with DE in the form of wet dark fluid in $f(R,T)$ gravity theory. The model universe doesn't attain the standard ΛCDM model during evolution. The model universe is anisotropic all through. The extra dimensions shrink with cosmic time. The value of $j(t)$ coincides with that of flat ΛCDM model.

Santhi et al. (2016) an interacting HDE Model with generalized Chaplygin gas in SBT. The model stars evolving from a point-type singularity. At $t \to \infty$, the comic parameters V tends to infinity, whereas the other parameters θ and H vanish. The DE component of the model shows a quintessence nature. The model undergoes a transition from deceleration to acceleration phase.

Santhi et al. (2019) study a Bianchi type-I universe in $f(R,T)$ gravity theory. The model is isotropic and non-shearing universe. The model evolves from an initial singularity. The authors claim that their model might approach de Sitter expansion under a certain condition. The model universe undergoes a transition from decelerating to accelerating phase during evolution. The model evolved with a large negative p.

Sarkar (2014a) considers work on Bianchi type-I universe with interacting DM and HDE. The anisotropic parameter of the model vanishes at $t \to \infty$. The model universe ends at the cosmic doomsday. The ratio $\frac{\rho_d}{\rho_m}$ diverges with cosmic time. The author considers an equivalence between the energy density of DE and that of Chaplygin gas DE.

Sarkar (2014b) investigate an expanding Bianchi type-V universe with interacting DM and HDE. At $t \to \infty$, the anisotropic parameter of the model vanishes, and the shape of the model universe becomes flat. The ratio $\frac{\rho_d}{\rho_m}$ diverges with cosmic time. In the far future, the DE EoS parameter tends to -1. Lastly, the author explains the evolution of black holes, interacting with a mixture of DE and DM.

Sarkar (2015) presents an FRW model universe with interacting HDE. The ratio $\frac{\rho_m}{\rho_d}$ of the model decreases with time. The model universe undergoes a transition from decelerating to accelerating phase during evolution. The DE of the universe is of phantom type, leading the model to the cosmic doomsday. Before the occurrence of the cosmic doomsday, during evolution, the model encounters a phase where ρ_m and ρ_d are almost equal.

Satheeshkumar & Suresh (2011) explain the dynamics of gravity and consider extra dimensions in their study. The authors explain the ways human knowledge of gravity is rapidly evolving, and how prior theories have impacted contemporary advancements in the area such as superstrings and braneworlds. The authors assert that with an infinite-volume extra dimension, one doesn't need stabilization.

Sharif & Nawazish (2018) present interacting and non-interacting DE cosmological models in $f(R)$ gravity theory. They discuss the evolution and the expansion of the cosmos. They claim that at the observational scale, we can find proof confirming the existence of interaction between DE and DM or cold DM.

Sharif & Ikram (2019) study the dynamics of HDE in an accelerated expanding FRW universe within the framework of $f(G,T)$ gravity. The authors mention that accelerated expansion depends on geometric contribution and matter content. The DE EoS parameter of the model corresponds to phantom energy, whereas the $r-s$ plane corresponds to the Chaplygin gas model.

Sharma et al. (2019) investigate a homogeneous and anisotropic Bianchi-V universe considering SBT. The model universe undergoes a transition from decelerating to accelerating phase during evolution. The value of the deceleration parameter is obtained to be $q=-0.63$. At $t\to\infty$, the anisotropic parameter of the model vanishes. The cosmological parameters H and θ decreases with cosmic time.

Singh & Kumar (2015) discuss HDE models in a homogeneous and isotropic FRW universe within the framework of $f(R,T)$ gravity. The authors mention that an interacting HDE model can explain the accelerated expansion of the universe. The authors also discuss the models with the consideration of $r-s$ and $r-q$ trajectories.

Singh & Kumar (2016) present non-viscous and viscous HDE models in an FRW universe within the framework of $f(R,T)$ gravity. The authors mention that an interacting HDE model can explain the accelerated expansion of the universe. The authors try to find out if infrared cut-off could describe the expansion of the universe at an expedited rate. In the case of the non-viscous model, during evolution, the author obtains the fixed ΛCDM point under certain conditions, whereas in the viscous case, the model remains fixed in ΛCDM.

Singh & Kar (2019) try out to predict a source of DE. DE is the component of the universe that is responsible to drive the expansion of the universe at an expedited rate. The authors claim that an emergent D-instanton could be a possible source of DE.

Singh & Bishi (2017) present FRW models with modified Chaplygin gas considering BDT. The exact solutions are obtained by applying a particular form of deceleration parameter. For particular choices of the values of the constants involved, the cosmological parameters of the models obtained are found to align with the previous cosmological findings.

Singh et al. (2004) study a spatially flat 5D universe in LM. The authors consider a time G in their study. They claim that the extra dimensions either shrink or expand slowly with cosmic time. The authors also briefly discuss the variation of the gravitational constant. They mention that G can be either decreasing or increasing.

Singh & Sharma (2014a) consider a spatially homogeneous and anisotropic Bianchi Type-II models universe in LM. The models undergo accelerated expansion at an expedited rate. The authors predicted that their models evolved from zero volume. In the power-law model, the DE EoS parameter is negative, whereas, in the exponential model, the DE EoS parameter tends to 1 for a small value of the cosmic time.

Singh et al. (2020) investigate FRW models in SCT and discuss the accelerated expansion of the universe. The models also discuss the past as well as the present of the universe. The DE component of the models is of CDM and quintessence nature. One of the models ends at the cosmic doomsday in the far future. The authors predict that the interaction of DE and DM is boosted by gauge function.

Singh & Samanta (2019) study two DE models in BDT considering a SS space-time. In one model, the DE component is phantom and the occurrence of negative time if possible. The DE model reduces to flat ΛCDM during evolution. In the other model, the DE component is of a quintessence nature. In this model, it is found that DE induces big bang, and it reduces to DM during evolution.

Singh et al. (2017a) emphasize the importance of DE outside the scope of astrophysics.

The work of the authors presents a fascinating explanation of the applicability of DE in solving the issue of global warming.

Singh et al. (2017b) attempt to predict a source of DE. DE is the component of the universe that is responsible to drive the expansion of the universe at an expedited rate. During the investigation of a 5D cosmological model in LM, the authors found that the LM behaves as a DE source.

Singh & Singh (2019a) emphasize the application of DE beyond the scope of astrophysics. The authors try to find the positive aspect of DE in the field of health sciences. They study a 5D universe in BDT. It is found that the DE component of the universe can aid in the treatment and healing of diseases.

Skibba (2020) presents an interesting study discussing the ultimate end of the universe. The authors explain in detail the big crunch, big rip, and big freeze singularities, one of which is considered as the possible end of the universe.

Srivastava & Singh (2018) investigate a new HDE model in $f(R, T)$ gravity theory. The authors discuss the possible future singularity of the model. The model reduces to the standard ΛCDM model in the future. It is claimed that bulk viscosity is an important aspect in the explanation of DE. Lastly, the thermodynamic aspects of the model are also studied in detail.

Srivastava et al. (2019) study a new HDE model in Bianchi type-III model universe. The model undergoes a transition from decelerating to accelerating phase during evolution. The model reduces to the flat ΛCDM during evolution. The DE of the model is made up of two components, i.e. CC and HDE.

Szenthe (2004a) presents a discussion on the global geometry of SS space-time. According to the author, ever since the inception of GR, SS space-times have been studied by many authors. Eventually, a comprehensive theory of SS space-times was developed, including basic findings and important results relating to their global geometry. The author further mentions that to this day, it appears that a broad global framework is missing. The author presents some basic details about the global geometry of SS space-times in the work.

Szenthe (2004b) asserts that ever since the inception of GR, SS space-times have been studied by many authors. In the work, the author provides a detailed compilation of the important topological aspects of SS space-time.

Takeno (1951) discusses in detail the characteristic system of SS space-times. Firstly, the author provides the definition of SS space-time. Secondly, the characterizing vectors and scalars are presented. Thirdly, it is shown the definition is equivalent to that of Einstein. Fourthly, some important properties of SS space-time are provided.

Takeno (1952a) claims that SS space-times serve a vital part in GR. Through using the idea of the characteristic system presented by the present author, a theory is constructed for assessing if a space-time described by a line element randomly defined in any coordinate system is SS.

Takeno (1952e) extends the definition of 4D SS space-time to dimensions greater than or equal to 5. The author claims that the characteristics of the 4D case apply to the later with minor adjustments. Further, the later case is simpler in certain ways than the 4D one.

Umadevi & Ramesh (2015) study an interacting HDE model in Bianchi type-III universe within the framework of BDT. The model undergoes accelerated expansion at an expedited rate. It is found that H, θ and σ^2 diverge at $t = 0$ and vanish at $t \to \infty$. The ratio $\frac{\rho_d}{\rho_m}$ tends to -1 at $t \to \infty$, whereas it tends to infinity at $t = 0$. The anisotropy fades and is replaced by isotropy at late time.

Valentino & Mena (2020) construct a cosmological model involving the interaction of the two dark components of the universe - DE and DM. The author assert that their interacting model can be helpful to alleviate the Hubble constant tension.

Yadav & Bhardwaj (2018) try to find if LM can be obtained in a hybrid universe with interacting DE in the Bianchi-V universe. The authors consider a particular form of $a(t)$ in the study. The model universe undergoes a transition from decelerating to accelerating phase during evolution. The DE dominating the universe is of quintessence type. The time-dependent displacement vector behaves as the time-dependent CC.

Wesson (2015) present a study to explain the necessity for the fifth dimension. The mention of a 5D theory can be seen that explain the origin of VE. There is a remarkable improvement in our knowledge and the logical consistency of physics by the introduction of the fifth dimension.

Zeyauddin & Saha (2013) discuss an anisotropic Bianchi type-VI universe in SCT. The authors consider certain reasonable cosmological assumptions to obtain exact solutions. The model evolves from an initial singularity. It is found that H, θ and σ^2 diverge at $t = 0$ and vanish at $t \to \infty$. The model undergoes accelerated expansion at an expedited rate.

Zeyauddin et al. (2020) investigate an anisotropic Bianchi type-V universe in SCT. The authors consider a particular form of $a(t)$ in the study. The deceleration parameter is time-dependent. The model agrees with the standard ΛCDM model. The cosmological parameters H, θ and σ^2 diverge at $t = 0$ and vanish at $t \to \infty$. The anisotropy fades and is replaced by isotropy at late time.

Zhang (2010) presents a study emphasizing the fate of the universe in the HDE scenario. In the present day, the DE component dominating the universe is considered to be of the phantom type, which will lead the universe to cosmic doomsday. However, HDE and the big rip scenario are incompatible with each other. This issue can be solved with the employment of extra dimensions. Such employment will avoid the cosmic doomsday, and ultimately lead the universe to the de Sitter phase. According to him, the employment of an extra dimension makes HDE models more complete and consistent.

Zimdahl (2012) investigate the model of interacting DE. The cosmic evolution is altered by the interaction of DE and DM. The expansion of the universe at an expedited rate is a pure manifestation of interaction. The interaction between these two dark components is a crucial aspect of the evolution of the universe. When compared to non-interacting models, interacting models provide better cosmic dynamics.

1.9.2 Summary of key findings

A total of 108 research articles have been examined, with the key methodology and findings of the investigations highlighted. Following are some findings that may be drawn after examining the works.

- DE constitutes 70% of the universe. It is the driving force behind the expansion of the universe at an expedited rate. The universe requires a DE component to achieve thermal equilibrium.
- One of the most difficult problems cosmologists have ever confronted is the so-called DE, a somewhat ridiculous content of the cosmos. They have yet to be able to accu-

rately understand the nature and properties of DE. Various authors have postulated various DE sources.

- The equation of state (EoS) parameter ω is a suitable tool for classifying DE into particular groups. The CC, often known as VE, is represented by $\omega = -1$. The range $-1 < \omega < \frac{-1}{3}$ represents quintessence, while phantom energy has $\omega < -1$. According to recent observation, phantom energy is the most probable type of DE in the current universe.

- Since the DE component dominating the universe is highly likely to be phantom type, the universe is believed to end at the cosmic doomsday, the big rip singularity. Some authors, on the other hand, provide conditions to avoid singularity.

- The mysterious DE, which has negative pressure and positive energy density, dominates the universe. Some authors, however, claim that negative energy density is possible under certain circumstances.

- Modified gravity can be presented in terms of the interaction of the two dark components in the Einstein frame. Such a model might also help in alleviating the coincidence problem and Hubble constant tension. The expansion of the universe at an expedited rate is a pure manifestation of interaction. The interaction between these two dark components is a crucial aspect of the evolution of the universe. We can find proof confirming the existence of interaction between DE and DM or cold DM. When compared to non-interacting models, interacting models provide better cosmic dynamics.

- There are two ways to explain the universe's accelerated expansion. First is the DE candidate method, and the second is the modified theories of gravity approach.

- SS space-time can be considered as one of the important tools for studying GR owing to its comparative simplicity and useful applications to both astrophysics and cosmology. It simplifies the study of a system's dynamics by allowing the transformation of a 4D solution to 2D. It serves a vital part in GR. It is so crucial in comprehending the universe. As a result, it's a good idea to start exploring GR with SS space-time.

- Numerous studies have successfully developed compelling justifications for the existence and practical importance of employing extra dimensions. The employment of an extra dimension also makes HDE models more complete and consistent. The unknown extra dimensions might be related to the two unseen DE and DM. Such a model is a good choice to explain the late time expanding phenomenon. There is a remarkable improvement in our knowledge and the logical consistency of physics by the introduction of 5D. A 5D model can describe the early inflation and late time acceleration of the universe.

- The study on the stabilization of extra dimensions is considered a phenomenological necessity in higher-dimensional models. In GR, generally, we cannot find conditions for stabilization, and all dimensions want to be dynamical. Generally, we witness the discussion on stabilization in the field of particle physics, supersymmetry, supergravity, string theory, and braneworld models.

1.9.3 Implications and ideas for further research

Following are some of the implications and ideas for further research after reviewing the research papers.

- As DE sources, many authors have proposed various theories. However, according to the literature we have come across, only the Lyra manifold is predicted to be a DE source among the modified theories of gravity in GR. It is worth seeing whether any of the other modified theories can act as DE sources.
- In most of the constructed DE models reviewed, we cannot find the calculation of the values of the cosmological parameters. We believe it is critical to test the model's reliability by calculating the cosmological parameter values and comparing them to observation data.
- We can't find criteria for stability in general in GR, and all dimensions want to be dynamical. However, we can at least attempt to find some kind of stabilizing conditions in GR.
- One of the hottest topics in cosmology is the finite-time future singularity or, in other words, the universe's ultimate fate. Because the dominant DE component is of phantom type, the big rip singularity is the most likely scenario. We feel it would be worthwhile if we could develop a reliable theory that would allow us to avoid such a terrible fate of the universe.
- Energy density should be to be positive to obtain a reliable cosmological model. On the other hand, negative energy density is claimed by some authors to be conceivable in certain conditions. We feel that constructing a reliable model involving negative energy density would be interesting.
- HDE models become more complete and consistent when an extra dimension gets involved. The two unseen DE and DM might be linked to the unknown extra dimensions. A model like this is a strong fit for explaining the universe's late-time

expansion, as well as early inflation and late-time acceleration. The introduction of 5D has made a significant advancement in our understanding and physics' logical consistency. With these points in mind, it would be a good idea to research DE and the universe's expanding phenomena in higher dimension.

- The expansion of the universe at an expedited rate is a pure manifestation of DE interaction. The interaction between DE and DM is a crucial aspect of the evolution of the universe. When compared to non-interacting models, interacting models provide better cosmic dynamics. We can find proof confirming the existence of interaction between DE and DM or cold DM. As a result, it would be more relevant to construct cosmological models that involves interaction.

- Because of its relative simplicity and practical applicability in both astrophysics and cosmology, SS space-time may be regarded one of the most significant instruments for investigating GR. It serves a vital part in GR. It is so crucial in comprehending the universe. It makes studying a system's dynamics easier. As a result, starting to explore GR with SS space-time would be a smart idea.