Chapter 4

Vacuum energy in Saez-Ballester theory and stabilization of extra dimensions

The work presented in this chapter is published in Universe 8 (2022) 60, DOI: 10.3390/universe8020060 (IF-2.278)

4.1 Introduction

Since the discovery of dark energy (DE) (Riess et al. 1998; Perlmutter et al. 1999), it has gained a reputation as one of the topics of paramount importance among the cosmological forums. Despite investing tremendous scientific efforts to explore it, its origin, bizarre nature, and future aspects to modern cosmology are still up for grabs. It is characterized by the distinctive feature of possessing a huge negative pressure opposing gravity resulting in the enigmatic phenomenon of the universe expanding at an expedited rate at late times. This cryptic dark entity is considered to be uniformly distributed and varies slowly or nearly unchanged with time (Carroll 2001a, 2001b; Chan 2015b; Peebles & Ratra 2003). Some worth mentioning studies on this mystic dark component that have not escaped our attention in the last few years are briefly presented below.

Recently, Singh & Singh (2021a) study a higher dimensional cosmological model to find the origin of DE. They further predict an $f(R,T)$ gravity model as a DE source (Singh & Singh 2021b). A presentation on the evolution of DE considering recent findings can be seen in the work presented by Wang et al. (2018). Collaboration et al. (2016) investigate the future of this dark entity beyond the bound of cosmological aspects. The estimation of DE density is presented by Dikshit (2019). Moradpour et al. (2014) put forward arguments for the need for DE. Gutierre (2015) analyses the status of the experimental data on DE. A fascinating comparison of the speed of DE with that of a photon can be found in work of Hecht (2013). The atom-interferometry constraints on DE are studied by Hamilton et al. (2015). In the publication of Josset et al. (2017), DE is obtained from the violation of energy conservation. Clery (2017) predicts that galaxies cluster as a result of stirring effect of DE. Lastly, Chan (2015a) claims that particles with imaginary energy density can lead us to the root of the ambiguous dark component.

Cosmologists have witnessed numerous theoretical attempts to obtain hints as to exactly predict the underlying physics of the miraculous expanding phenomenon of the universe at late times. Two well-appreciated methods have been adapted to explain this mystic phenomenon. First, several types of DE are constructed. Second, modifying ETG (Ahmed & Pradhan A 2020b; Clifton et al. 2012). Other than these two, recently, cosmologists and theoretical physicists have been successful in developing other interesting and convincing approaches. In the work of Gorji (2016), the phenomenon is explained by the infrared corrections. Narain & Li (2018) predict that an Ultraviolet Complete Theory leads to the expansion. A fascinating illustration is presented by Berezhiani (2017) where the expedited expansion occurs in the absence of DE.

To figure out the ambiguous nature of DE in as much detail as possible, the equation of state (EoS) parameter ω is studied with utmost importance. The most recent Planck 2018 results (Collaboration et al. 2020), estimates its value to be $\omega = -1.03 \pm 0.03$. The late time expedited expansion of the universe is obtained when $\omega < -\frac{1}{3}$ (Tripathi et al. 2017). $\omega = -1$ corresponds to the natural candidate of DE, the cosmological constant (CC), or in other words, vacuum energy (VE). However, CC or VE comes up short to explain the mystery of the coincidence problem (CP) (Zlatev et al. 1999). After multiple efforts, many other well-appreciated forms of DE are developed (Copeland et al. 2006). One such candidate that has not escaped our notice is the holographic dark energy (HDE). As a result of the holographic principle (Bousso 2002) being applied to DE, HDE is formed. The work of Wang et al. (2017) provides a peek of HDE's fundamental nature and properties. Recent works on some of the different forms of HDE can be seen in the publications by Korunur (2019), Pradhan et al. (2021), Prasanthi & Aditya (2020) and Srivastava et al. (2019). Construction of interacting HDE and dark matter (DM) models in spherically symmetric space-time settings is studied by Reddy D R K et al. (2016, 2016a), Singh & Singh (2019b) and many others. Interacting models can successfully represent modified gravity in the Einstein frame (Cai et al. 2016; Felice & Tsujikawa 2010; He et al. 2011; Kofinas et al. 2016; Zumalacarregui et al. 2013). According to the works of Amendola & Tocchini-Valentini (2001), Cai & Wang (2005), Zimdahl et al. (2001) and Zimdahl & Pavon (2004), we can find that such interacting models are effective in mollifying the CP.

Due to the fascinating natures of the HDE and VE, a spark of interest has been ignited among cosmologists so that they have started to examine HDE paired with VE. Singh & Kumar (2015) predict that their HDE model evolved from ΛCDM in early time and approaches to the same ΛCDM in the late time. They further mention that for a fixed value of a coupling parameter involved, their HDE model remains fixed in the ΛCDM model all through. Sadri et al. (2018) present an accelerating HDE model behaving similarly to the ΛCDM model. An explanation is presented by Dubey & Sharma (2020) in which the HDE model can't be discriminated from ΛCDM in the high-redshift region. Lee et al. (2007) assert that the vacuum entanglement energy is the probable candidate for HDE, where entanglement energy is the disturbed vacuum energy due to the presence of a boundary (Mukohyama et al. 1997). Hu et al. (2015) develop a heterotic DE model where the DE has two parts, the cosmological constant and HDE. A study of an HDE model where $\omega = -1$ is obtained is presented by Myung (2007). Lastly, Mathew et al. (2013) study a model where HDE ends at ΛCDM in the future.

Saez-Ballester Theory (SBT), introduced by Saez and Ballester (Saez & Ballester 1986), can be regarded a viable approach for studying DE and the expanding cosmos. It belongs to the Scalar Tensor Theory (STT) class of gravity theories. In SBT, the metric potentials are associated with a scalar field φ. Scalar fields are regarded to be important in gravitation and cosmology since they may depict phenomena such as DE, DM, and so on (Aditya et al. 2021). They can be considered a probable contributor to the universe's late-time acceleration (Kim 2005). STT is a straightforward extension and generalization of GR (Panotopoulos & Rincon 2018). STT appears to be an ideal contender for DE (Mandal et al. 2018). Linde (1982) and Guth (1981) assert that the inflation at the start of the evolution might have been caused by a scaler field. Pradhan et al. (2013) and Sharma et al. (2019) discuss Bianchi Type-V cosmology in SBT obtaining a transit from decelerating universe to accelerating phase. Currently, SBT and general relativity are held to align with observation.

The higher-dimensional model has become one of the good choices among cosmologists and theorological physicists. The idea of such a model was put forward by Kaluza and Klein (Kaluza 1921; Klein 1926). Aly (2019) and Banik & Bhuyan (2017) claim that such a model can explain the late time expanding phenomenon. In the work of Farajollahi & Amiri (2010), it is mentioned that extra-dimensional theories of gravity might explain the early inflation and late-time acceleration of the universe. There is a remarkable improvement in our knowledge and the logical consistency of physics by the introduction of the fifth dimension (Wesson 2015). A study to validate the existence of the extra dimension is presented by Marciano (1984). There is a chance that the unknown fifth dimension might be related to two the ambiguous and unseen dark components - dark energy and dark matter (Chakraborty & Debnath 2010). According to Zhang (2010), the employment of an extra dimension makes HDE models more complete and consistent. Some recent worth mentioning studies on higher dimension can be seen in the works of Ahmed & Pradhan (2020a), Astefanesei et al. (2020), Ghaffarnejad et al. (2020), Mishra et al. (2019), Montefalcone et al. (2020) and Saha & Ghose (2020).

Taking into consideration the above noteworthy related studies, we consider a minimal DE-DM interaction within the framework of SBT using a 5D spherically symmetric (SS) space-time. In this chapter, we present an in-depth discussion on every cosmological parameter obtained. The definition of shear scalar and its physical significance are provided. We discuss the initial and future singularity of the model universe. Additionally, we calculate the present values of the overall density parameter, deceleration parameter, and the dark energy EoS parameter. We also discuss the conditions to solve the stabilization problem of extra dimensions in general relativity. The chapter is divided into sections. After the introduction, in Sect. 4.2, we present the formulation of the problem with solutions to the parameters. In Sect. 4.3, the solutions are discussed with graphical representations. In Sect. 4.4, we present the explanation on the solution to stabilization problem of extra dimensions in GR. Lastly, to sum up the observations, a concluding note is provided in Sect. 4.5.

4.2 Formulation of problem and solutions

In our universe, the five-dimensional SS metric (Samanta & Dhal 2013) of following the form is considered

$$ds^2 = dt^2 - e^{\mu}\left(dr^2 + r^2 d\Theta^2 + r^2 \sin^2\Theta d\phi^2\right) - e^{\delta} dy^2 \tag{4.2.1}$$

where μ and δ are cosmic scale factors which are functions of time only.

We consider the following Saez-Ballester field equations

$$R_{ij} - \frac{1}{2} g_{ij} R - \omega_{SB}\varphi^n \left(\varphi_{,i}\varphi_{,j} - \frac{1}{2} g_{ij}\varphi_{,k}\varphi^{,k}\right) = -\left(T_{ij} + S_{ij}\right) \tag{4.2.2}$$

where T_{ij} and S_{ij} are the energy momentum tensors for matter and HDE respectively, R and R_{ij} are respectively the Ricci scalar and tensors, whereas the scalar field φ satisfies

$$2\varphi^n \varphi^{;i}_{;i} + n\varphi^{n-1}\varphi_{,k}\varphi^{,k} = 0 \tag{4.2.3}$$

where n is an arbitrary constant.

We define $T_{i\,j}$ and $S_{i\,j}$ as

$$T_{i\,j} = \rho_m u_i u_j \tag{4.2.4}$$

$$S_{i\,j} = (\rho_d + p_d)\, u_i u_j - g_{i\,j} p_d \tag{4.2.5}$$

where ρ_m and ρ_d represent the energy densities of matter and HDE respectively and p_d represents the pressure of the HDE.

Here, the energy is conserved and obviously, we have

$$T^{i\,j}_{;j} + S^{i\,j}_{;j} = 0 \tag{4.2.6}$$

By using the co-moving coordinate system, the surviving field equations are obtained as follows

$$\frac{3}{4}\left(\dot{\mu}^2 + \dot{\mu}\dot{\delta}\right) + \frac{\omega_{_{SB}}}{2}\varphi^n\dot{\varphi}^2 = \rho \tag{4.2.7}$$

$$\ddot{\mu} + \frac{3}{4}\dot{\mu}^2 + \frac{\ddot{\delta}}{2} + \frac{\dot{\delta}^2}{4} + \frac{\dot{\mu}\dot{\delta}}{2} - \frac{\omega_{_{SB}}}{2}\varphi^n\dot{\varphi}^2 = -p_d \tag{4.2.8}$$

$$\frac{3}{2}\left(\ddot{\mu} + \dot{\mu}^2\right) - \frac{\omega_{_{SB}}}{2}\varphi^n\dot{\varphi}^2 = -p_d \tag{4.2.9}$$

and from Eq. (4.2.6), we have

$$\ddot{\varphi} + \dot{\varphi}\left(\frac{3\dot{\mu} + \dot{\delta}}{2}\right) + \frac{n}{2}\dot{\varphi}^2\varphi^{-1} = 0 \tag{4.2.10}$$

where an overhead dot represents differentiation w.r.t. t.

Considering ω as the EoS parameter of the dark energy so that we have

$$p_d = \omega\rho_d \tag{4.2.11}$$

Now, the conservation equation is given by

$$\dot{\rho}_d + (1+\omega)\left(\frac{3\dot{\mu} + \dot{\delta}}{2}\right)\rho_d + \dot{\rho}_m + \rho_m\left(\frac{3\dot{\mu} + \dot{\delta}}{2}\right) = 0 \tag{4.2.12}$$

Due to the minimal interaction of HDE and matter, according to Sarkar (2014a, 2014b), both the components conserve separately thereby obtaining

$$\dot{\rho}_m + \rho_m\left(\frac{3\dot{\mu} + \dot{\delta}}{2}\right) = 0 \tag{4.2.13}$$

$$\dot{\rho}_d + (1+\omega)\,\rho_d \left(\frac{3\dot{\mu}+\dot{\delta}}{2}\right) = 0 \tag{4.2.14}$$

Also, we have

$$\dot{\rho} + (\rho + p)\left(\frac{3\dot{\mu}+\dot{\delta}}{2}\right) = 0 \tag{4.2.15}$$

From Eqs. (4.2.13) and (4.2.14), we have

$$\rho_m = a_0 e^{-\left(\frac{3\mu+\delta}{2}\right)} \tag{4.2.16}$$

$$\rho_d = b_0 e^{-(1+\omega)\left(\frac{3\mu+\delta}{2}\right)} \tag{4.2.17}$$

where a_0 and b_0 are arbitrary constants.

From Eqs. (4.2.8) and (4.2.9), we obtain the expression for cosmic scale factors as

$$\mu = c_1 + \log\,(v\,t - uc_2)^{\frac{u}{v}} \tag{4.2.18}$$

$$\delta = kc_1 + \log\,(v\,t - uc_2)^{\frac{ku}{v}} \tag{4.2.19}$$

where $c_1,\ c_2,\ u,\ v$ and $k \neq 0$ are arbitrary constants.

From Eqs. (4.2.16)-(4.2.19), the energy densities of matter and DE are respectively obtained as

$$\rho_m = a_0 e^{-\frac{(k+3)c_1}{2}}\,(vt - uc_2)^{-\frac{(k+3)u}{2v}} \tag{4.2.20}$$

$$\rho_d = b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}}\,(vt - uc_2)^{-\frac{(1+\omega)(k+3)u}{2v}} \tag{4.2.21}$$

Using Eqs. (4.2.18) and (4.2.19) in Eq.(4.2.10), the expression for scalar field is obtain as

$$\varphi = c_2 e^{\frac{2\log\left(e^{\frac{u}{2}(k+3)\left(\frac{t}{v^2 t - uvc_2} - 2c_1\right)} - (n+2)\left(uvc_2 - v^2 t\right)\right) - \frac{(k+3)u\,t}{v^2 - uvc_2}}{n+2}} \tag{4.2.22}$$

From Eqs. (4.2.20) and (4.2.21), the expression for energy density of the model universe is obtained as

$$\rho = a_0 e^{-\frac{(k+3)c_1}{2}}\,(vt - uc_2)^{-\frac{(k+3)u}{2v}} + b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}}\,(vt - uc_2)^{-\frac{(1+\omega)(k+3)u}{2v}} \tag{4.2.23}$$

Using Eqs. (4.2.18), (4.2.19) and (4.2.23) in Eq. (4.2.15), the expression for pressure of the model universe is obtained as

$$p = -\left(a_0 e^{-\frac{(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(k+3)u}{2v}} + b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(1+\omega)(k+3)u}{2v}}\right) \tag{4.2.24}$$

From Eqs. (4.2.11) and (4.2.21), the pressure of dark energy is obtained as

$$p_d = \omega\ b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(1+\omega)(k+3)u}{2v}} \tag{4.2.25}$$

At any time $t = t_0$, we can assume that $p = p_d$ so that

$$\left(a_0 e^{\frac{\omega(k+3)c_1}{2}} (vt - uc_2)^{\frac{\omega(k+3)u}{2v}} + b_0 (1+\omega)\right) e^{-\frac{(1+\omega)(k+3)c_1}{2}} (vt - uc_2)^{-\frac{(1+\omega)(k+3)u}{2v}} = 0 \tag{4.2.26}$$

The expression for ω will be given by Eq. (4.2.26).

Now, the expressions for the different cosmological parameters are obtained as given below

Spatial volume:

$$V = e^{\frac{(k+3)c_1}{2}} (vt - uc_2)^{\frac{(k+3)u}{2v}} \tag{4.2.27}$$

Scalar expansion:

$$\theta = \frac{(k+3)\,u}{2\,(vt - uc_2)} \tag{4.2.28}$$

Hubble parameter:

$$H = \frac{(k+3)\,u}{8\,(vt - uc_2)} \tag{4.2.29}$$

Deceleration parameter:

$$q = \frac{8v}{(k+3)\,u} - 1 \tag{4.2.30}$$

Shear scalar:

$$\sigma^2 = \frac{1}{72}\left(\frac{16vt^2 - 4\,(3k + 8c_2 + 9)\,uvt + 3\,(3k + 4kc_2 + 12c_2 + 9)\,u^2 + 16uc_2{}^2}{(vt - uc_2)^2}\right) \tag{4.2.31}$$

Anisotropic parameter:

$$A_h = 3\left(\frac{k-1}{k+3}\right)^2 \tag{4.2.32}$$

Dark energy density parameter:

$$\Omega_d = \frac{\rho_d}{3H^2} = \frac{64}{3}\left(\frac{b_0 e^{-\frac{(1+\omega)(k+3)c_1}{2}} (vt - uc_2)^{2-\frac{(1+\omega)(k+3)u}{2v}}}{3\,(k+3)^2\,u^2}\right) \tag{4.2.33}$$

Matter density parameter:

$$\Omega_m = \frac{\rho_m}{3H^2} = \frac{64}{3}\left(\frac{a_0 e^{-\frac{(k+3)c_1}{2}}(vt-uc_2)^{2-\frac{(k+3)u}{2v}}}{3\left(k+3\right)^2 u^2}\right) \tag{4.2.34}$$

Overall density parameter:

$$\Omega = \frac{64}{3}\left(\frac{\left(a_0 + b_0 e^{-\frac{\omega(k+3)c_1}{2}}(vt-uc_2)^{-\frac{\omega(k+3)u}{2v}}\right)e^{-\frac{(k+3)c_1}{2}}(vt-uc_2)^{2-\frac{(k+3)u}{2v}}}{3\left(k+3\right)^2 u^2}\right) \tag{4.2.35}$$

According to Ghaffari et al. (2015), the expression for the state finder diagnostic pair $\{r, s\}$ is given by

$$r = 1 + \frac{3\dot{H}}{H^2} + \frac{\ddot{H}}{H^3} \tag{4.2.36}$$

$$s = \frac{r-1}{3\left(q-\frac{1}{2}\right)} \tag{4.2.37}$$

From Eqs. (4.2.29), (4.2.36) and (4.2.37), we have

$$\{r,\, s\} = \{1,\, 0\} \tag{4.2.38}$$

4.3 Discussion

In this section, for convenience sake, we opt to choose $a_0 = b_0 = c_1 = c_2 = k = 1, u = 2.78$ and $v = \frac{1}{2}$. The discussion on the nature of the parameters with respect to cosmic time t are presented in details with graphs as follows.

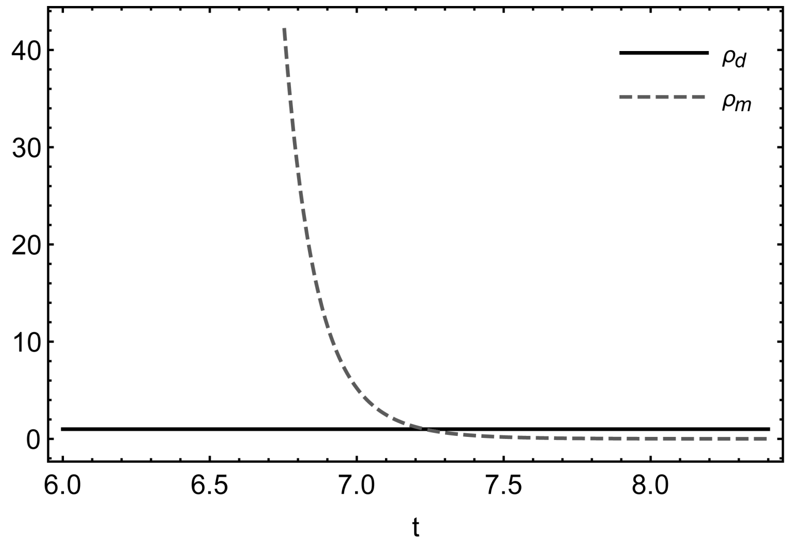

Figure 4.1: Variation of the energy densities of DE ρ_d and DM ρ_m with t when $a_0 = b_0 = c_1 = c_2 = k = 1, u = 2.78, v = \frac{1}{2}$

From Eqs. (4.2.20) and (4.2.21), it is obvious that ρ_d and ρ_m are functions of t. Fig. 4.1 shows that ρ_d is almost consistent throughout whereas ρ_m decreases in the entire course of evolution, which are acceptable scenarios as the ambiguous DE varies slowly or is unchanged with time (Carroll 2001a, 2001b; Chan 2015b; Peebles & Ratra 2003), on the other hand, DM diminishes continuously as a result of the galaxies scattering away from one another during expansion (Carroll 2001b). Moreover, when $t \to \infty$, $\rho_m \to 0$. From these, it would be appropriate to conclude that the universe will be progressively dominated by this cryptic DE. Similar increasing dominant nature of DE can also be seen in the papers of Singh & Singh (2019b), Singh & Samanta (2019) and Caldwell et al. (2003).

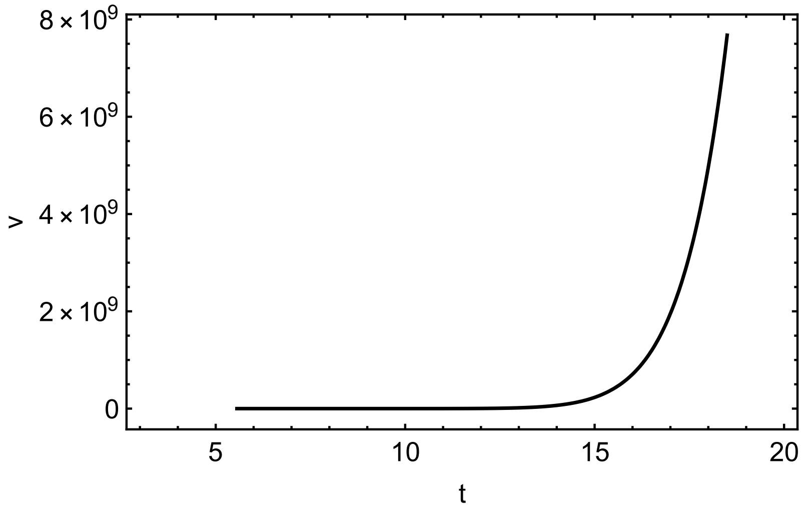

Figure 4.2: Variation of the spatial volume V with t when $c_1 = c_2 = k = 1, u = 2.78, v = \frac{1}{2}$

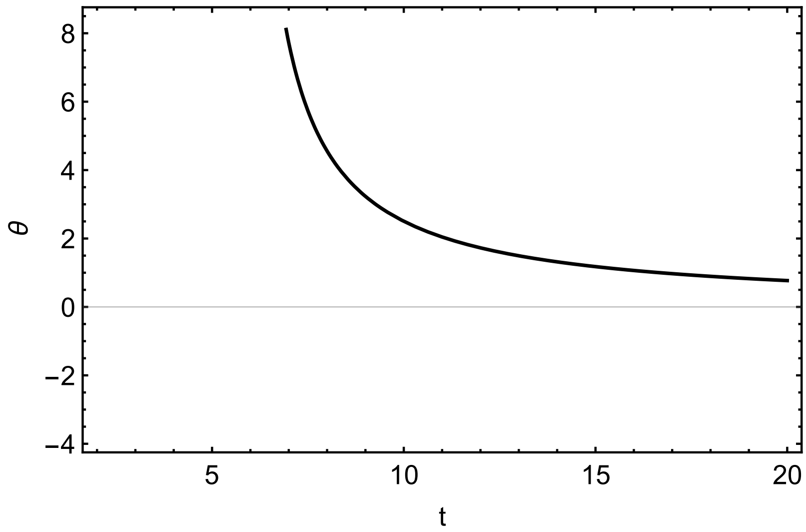

Figure 4.3: Variation of the expansion scalar θ with t when $c_2 = k = 1, u = 2.78, v = \frac{1}{2}$

Fig. 4.2 can be regarded as perfect supporting evidence for the present observation of the spatial expansion of the universe. However, at the initial epoch when $t = 0$, $V = 0$. Also, from Fig. 4.3, we can see that θ initially emerges with a large value, decreases with evolution, and finally, tends to become constant after some finite time which is the indication of the Big-Bang scenario (Mollah et al. 2018). The prediction of a similar scenario with similar cosmological settings can also be seen in a research of Aditya & Reddy (2018). On considering $a_0 = b_0 = c_1 = c_2 = k = 1, u = 2.78$, $v = \frac{1}{2}$ and assuming the present age of the universe to be $t_0 = 13.8$ Gyr which align with the estimated present age by the most recent Plack 2018 results (Collaboration et al. 2020), from Eq. (4.2.26), the value for EoS parameter is measured to be $\omega = -1$. The Planck 2018 results estimates its value to be $\omega = -1.03 \pm 0.03$ (Collaboration et al. 2020). So, the dark energy candidate we are dealing with is the vacuum energy or the cosmological constant. Moreover, from Fig. 4.1, it can be seen that the dark energy density ρ_d remains almost constant throughout evolution, and from Eq. (4.2.27), $v \to \infty$ when $t \to \infty$. So, it would be a pertinent fact that the universe has no end; expanding forever, ultimately, leading to the Big Freeze singularity in the far future. In a thermodynamic sense, the model universe will enter a point of minimum temperature and maximum entropy. It will be almost as though all astrophysical process is being smothered, as the fuel for growth and reproduction gets so diffuse that it can't be used (Skibba 2020). It will be an ending point characterized by increasing isolation, inexorable decay, and an eons-long fade into darkness (Mack 2020).

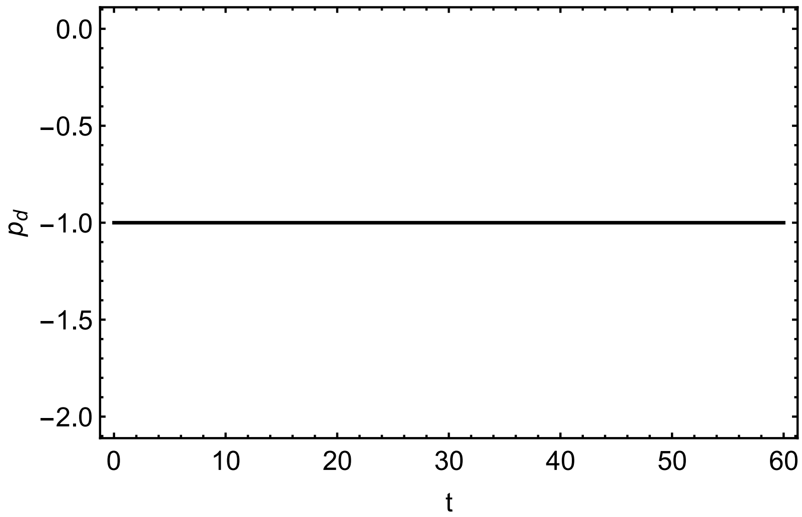

Figure 4.4: Variation of the DE pressure p_d with t when $b_0 = c_1 = c_2 = k = 1, \omega = -1, u = 2.78, v = \frac{1}{2}$

The pressure of DE p_d varies in the negative plane throughout, as seen in Fig. 4.4, which is consistent with the enigmatic feature of DE accountable for the universe's rapid expansion.

u	v	k	q
2.78	$\frac{1}{2}$	1	-0.64
2.78	$\frac{1}{1.6}$	1	-0.55
2.25	$\frac{1}{2}$	1	-0.55
2.25	$\frac{1}{1.6}$	1.9	-0.54

Table 4.1: Values of deceleration parameter q for different values of u, v and k.

From Eq. (4.2.30), the deceleration parametr q depends on u, v and k. In Table 4.1, we present different values of q for different values of u, v and k. Recently, Camarena & Marra (2020) predict its value as $q = -0.55$, whereas Capozziello et al. (2019) estimate the value as $q = -0.644 \pm 0.223$ and $q = -0.6401 \pm 0.187$. With all the values of the constants in Table 4.1, we obtain the EoS paramter of CC. Since q lies in the range $-1 < q < 0$, the accelerating model universe undergoes exponential expansion (Singh & Bishi B K 2017), in agreement with the present cosmology.

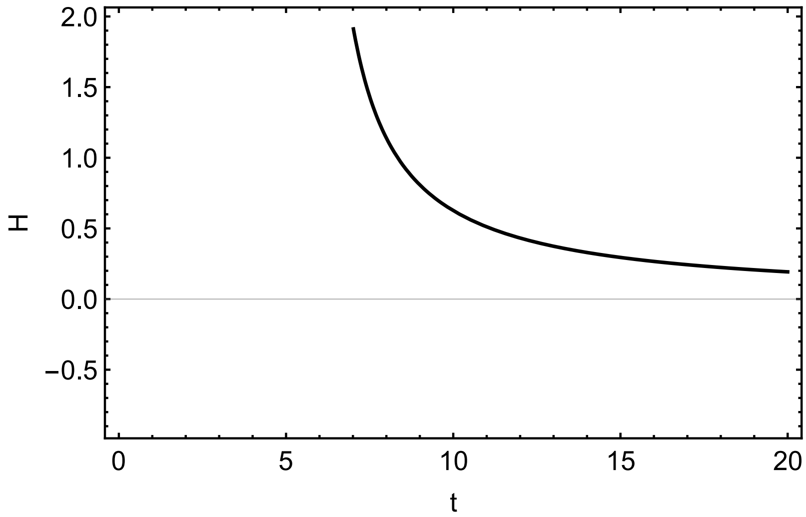

Figure 4.5: Variation of the Hubble parameter H with t when $c_2 = k = 1, u = 2.78, v = \frac{1}{2}$

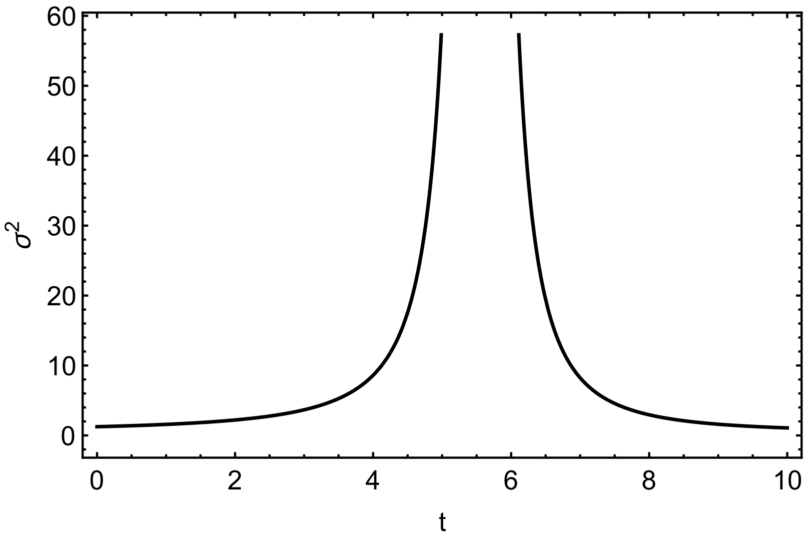

Figure 4.6: Variation of the shear scalar σ^2 with t when $c_2 = k = 1, u = 2.78$ and $v = \frac{1}{2}$

Fig. 4.5 shows the decreasing nature of Hubble parameter H which is within the limit of the present cosmological scenario (Biswas et al. 2019; Mishra & Chand 2020). Shear scalar σ^2 shows us the rate of deformation of the matter flow within the massive cosmos (Ellis & Elst 1999). The evolution of σ^2 can be seen in Fig. 4.6. It evolves with a constant value, then diverges after some finite time, and again converges to become constant. It vanishes for a finite period during evolution. From these, we can summarize that in the initial epoch, the model universe expands with a very slow and uniform change of shape, but after some finite time, the change becomes faster. Then, it again tends to become very slow and uniform after expanding without any deformation for a finite period. From

Eq. (4.2.32), the anisotropic parameter $A_h = 0$ for $k = 1$ so that the constructed model is isotropic.

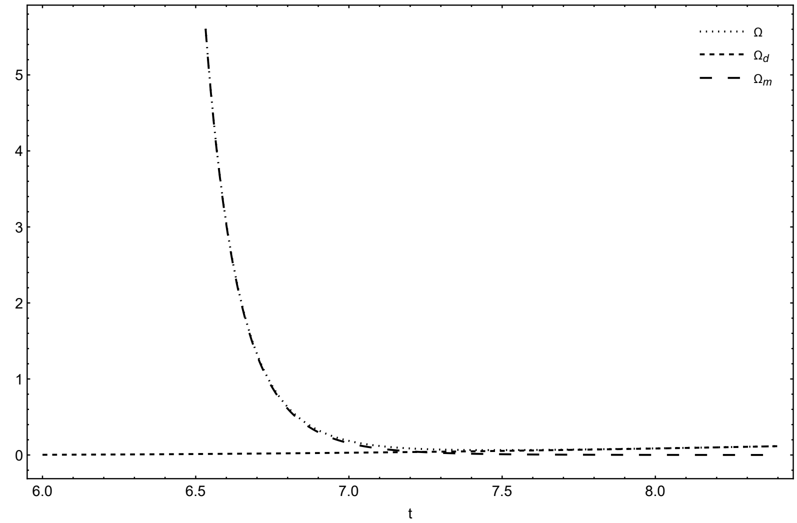

Figure 4.7: Variation of the overall density parameter Ω, DE density parameter Ω_d and DM density parameter Ω_m with t when $a_0 = b_0 = c_1 = c_2 = k = 1, u = 2.78$ and $v = \frac{1}{2}$

Fig. 4.7 shows us the variation of Ω, Ω_d and Ω_m with t. Here, since DE varies slowly or is unchanged with time (Carroll S M 2001a, 2001b; Chan 2015b; Peebles & Ratra 2003), we can see that Ω_d tends to remain constant or increases very slowly. However, Ω_m decreases in the entire course of evolution as a result of the galaxies scattering away from one another leading DM to diminish continuously (Carroll 2001b). Above all, with $k = 1, u = 2.78$ and $v = \frac{1}{2}$, from Eq. (4.2.35), the overall density parameter is obtained to be $\Omega = 0.97(\approx 1)$. For an exactly flat universe, $\Omega = 1$ (Holman 2018; Khodadi et al. 2015; Levin & Freese 1994). Recently, many authors advocate against the belief of an exactly flat universe (Khodadi et al. 2015; Valentino et al. 2020; Javed et al. 2020; Nashed & Hanafy 2014). It will be a right conclusion to say that the universe is close to or nearly flat, but not exactly flat (Khodadi et al. 2015; Nashed & Hanafy 2014; Adler & Overduin 2005). Above all, the most recent Planck 2018 results (Collaboration et al. 2020) obtaining Ω ranging close to unity can be treated as a perfect piece of evidence for a nearly flat universe. Hence, our model obtaining Ω not exactly equal to 1 is justified.

Lastly, from Eq. (4.2.38), we can see that the value of the state finder diagnostic pair $\{r, s\} = \{1, 0\}$ which corresponds to the ΛCDM scenario so that the model universe we are considering is a ΛCDM model. Hence, our interacting HDE model can be considered as an alternate cosmological model to the standard ΛCDM model.

4.4 Stabilization of extra dimensions

The study on the stabilization of extra dimensions can be considered as a phenomenological necessity in higher-dimensional models. The discussion on stabilization is mostly confined to particle physics, supersymmetry, supergravity, string theory, and braneworld models. We require a stabilization mechanism to prevent modification of gravity to an experimentally undesirable manner (Kribs 2006). The stabilization also makes sure the visible 4D universe with a long lifetime (Ketov 2019). Another benefit of stabilization is that we can ignore any unwanted outcomes of quantum gravity at Planck length distances (Hamed et al. 2002). One of the most classic solutions for stabilization is the Goldberger-Wise mechanism (Goldberger & Wise 1999), where stabilization is achieved in the presence of an additional scalar field. Carroll et al. (2002) claim that stabilization can be achieved by introducing a potential of the dilaton field. Chung & Freese (1999) present a study of an isotropic 3-brane model where stabilization is achieved with the only value of the EoS $\omega(t) = -\frac{2}{3}$. Another observation of stabilization in an isotropic perfect fluid model in 5D with the value of EoS $\omega > -\frac{1}{3}$ is presented by Arapoglu et al. (2018). Bronnikov & Rubinn (2006) show that the issue of stabilization can be overcome in a theory of gravity involving high-order curvature invariants. Sundrum (2005) obtains stabilization by quantum corrections from massive matter. In the investigation of Kainulainen & Sunhede (2006), we can find the investigation of a class of dilatonic STT where stabilization is achieved by quantum corrections to the effective 4D Ricci scalar. Mazumdar (1999) presents an argument calming that stabilization is attained as soon as inflation ends, on the contrary, Ferrer & Rasanen (2007) assert that inflation ends if stabilization is attained. According to Chirkov & Pavluchenko (2021), to achieve a realistic theoretical model, we should assume that the visible three dimensions are expanding isotropically, whereas the extra dimensions are contracting (or contracted for a period during the evolution). Similarly, Rasouli & Moniz (2017) predict that the extra dimension contracts with the cosmic time. According to Moraes & Correa (2019), the hidden extra dimension is related to scalar fields. Bruck & Longden (2019) also represent the size of the extra dimensions in terms of a scalar field. Hamed et al. (2001) investigate 4D gauge theories that dynamically generate a 5D, where stabilization is no longer needed. In his works (Tosa 1984, 1985), Tosa studies the Kaluza-Klein cosmology for a torus space with a cosmological constant and matter. He predicts that the number of the extra dimensions should be more than 1, and the extra dimensions should be of small size. However, during recent years, many authors have successfully predicted models with just one extra dimension, where stabilization is obtained (Das et al. 2018; Dudas & Quiros 2005; Egorov & Volobuev 2017; Kanti et al. 2002; Ponton & Poppitz 2001; Wongjun 2015). Additionally, we can also witness large extra dimensions the works by Gong et al. (2008), Wu et al. (2008) and Wang (2002), and infinite-volume extra dimensions in the fourth paragraph of this section.

In our work, we have discussed a 5D SS cosmological model in general relativity (GR) with the cosmological constant (CC), or in other words, vacuum energy (VE) as the DE candidate. In GR, generally, we cannot find conditions for stabilization, and all dimensions want to be dynamical (Bruck & Longden 2019). In an accelerating model with CC, stabilization cannot be obtained (Rador 2007). Therefore, in a trial to solve the stabilization problem in GR, we consider two options. The first one is the Casimir energy and the second is the infinite-volume extra dimension, which is discussed below.

Casimir energy is a DE candidate with the ability to drive the late-time accelerated expansion and stabilize the extra dimensions automatically (Wongjun 2015; Greene & Levin 2007). Casimir energy is VE emerging from imposing boundary conditions on the quantum fluctuations of fields and the EoS's of both Casimir energy and CC are of the same form (Wongjun 2015). Further, Wongjun (2015) interpretes Casimir energy as CC. Additionally, Roberts (2000) equates VE with Casimir energy. Ichinose (2012) also identifies Casimir energy with CC. If the CC is to be created from the Casimir energy, then there will be only one extra dimension (Dupays et al. 2013). Coincidently, in our spherically symmetric cosmological model with the CC as the DE candidate, there is only one extra dimension.

The study on extra dimensions has been widely considered in brane world models (Dick 2001; Freese & Lewis 2002; Hogan 2001; Ichiki et al. 2002; Langlois 2003; Shiromizu et al. 2000; Zhu & Fujimoto 2002, 2003, 2004), one of which is the DGP model (Dvali et al. 2000), which presents an accelerating 5D scenario with an infinite-volume extra dimension. This infinite-volume extra dimension drives the expedited expansion of the universe at late times (Alcaniz 2006). Kumar & Suresh (2005) and Satheeshkumar & Suresh (2011) assert that with an infinite-volume extra dimension, one doesn't need stabilization. They further claim that the infinite-volume scenario can explain us the late time cosmology and the acceleration of the universe driven by DE, which are one of the core components of GR. According to Dvali & Michael (2003), infinite-volume extra dimensions might result to the emergence of DE. Hence, it would be appropriate to conclude that the extra dimension in our study on 5D spherically symmetric cosmological model is of infinite-volume.

One of the most classic solutions for stabilization is the Goldberger–Wise (GW) mechanism (Goldberger & Wise 1999). We can witness the application of the GW mechanism in the field of string theory, M-Theory, and Randall and Sundrum (RS) model in the noteworthy works by Wang (2010), Wu et al. (2009), Wang & Santos (2008, 2010), Devin et al. (2009) and Wang et al. (2008). In these works, the authors consider a 5D static metric with a 4D Poincare symmetry. To obtain stability, they introduce the proper distance and a massive scalar field and show that the effective radion potential has a minimum. Since the Casimir energy (force) provides a natural alternative to the GW mechanism (Garriga & Pomarol 2003), the stabilization mechanism applied by the aforementioned noteworthy works might have some sort of relationship with the Casimir energy stabilization approach which we have predicted above. Above all, one may consider it as an advantage above the GW mechanism that the introduction of an ad hoc classical interaction between the branes is not needed in the Casimir energy approach of stabilization (Garriga & Pomarol 2003). We may note the work by Garriga et al. (2001) predicting that the Casimir force will not lead to stabilization to the right value unless a tuning of parameters. Fortunately, the work by Garriga & Pomarol (2003) shows that this conclusion of Garriga et al. (2001) is not general, and proves that Casimir energy (force) provides a natural alternative to the GW mechanism in the RS model. There might be more advantages or relationships of our predicted stabilization approaches with the GW mechanism, which we would like to find out in our future works.

We have presented two conditions for the stabilization of extra dimensions. Probably, our work might be the first to predict such conditions in GR. Nevertheless, these two conditions are toy models which require further in-depth analysis considering different cosmological aspects. We need more investigation on the reliability of considering, within GR, the identification of Casimir energy with cosmological constant, or in other words, vacuum energy. We also need to verify all the possible outcomes of assuming the extra dimension is of infinite volume in a higher-dimensional vacuum energy model within GR.

4.5 Conclusions

In this chapter, we have analysed a cosmological model in SS space-time in a 5D setting with minimally interacting matter and HDE in SBT. We predict that the expanding isotropic universe will be progressively DE dominated. The pressure of DE is negative all through. We estimate few values of the deceleration parameter and the values are found very close to the recently predicted values. The Hubble's parameter H decreases which agree with the present cosmological scenario. In the initial epoch, the model universe expands with a very slow and uniform change of shape, but after some finite time, the change becomes faster. Then, it again tends to become very slow and uniform after expanding without any deformation for a finite period. The value of the DE EoS parameter is measured to be $\omega = -1$ indicating that the DE we are dealing with is the VE or CC. The value of the overall density parameter is obtained as $\Omega = 0.97(\approx 1)$, which is not exactly equal to 1, since the universe is close to or nearly flat, but not exactly flat. We observe that the model universe starts with the Big-Bang and ends at the Big Freeze singularity. The value of the state finder diagnostic pair obtained corresponds to the ΛCDM model so that our interacting HDE model can be considered as an alternate cosmological model to the standard ΛCDM model. Lastly, we present two conditions to solve the stabilization problem of extra dimension in GR, the first one is the identification of Casimir energy with CC, or in other words, VE and the second is assuming the extra dimension is of infinite volume.