Fuzzy interior and fuzzy closure of fuzzy set with extended definition of fuzzy set

Bhimraj Basumatary

Department of Mathematical Sciences,
Bodoland University,
Kokrajhar, BTC, Assam, 783370, India
Email: brbasumatary14@gmail.com

Abstract: The main purpose of this work is to construct a basic concept of fuzzy topology. In this article, we will try to give definition of fuzzy topology, fuzzy open set, fuzzy closed set, fuzzy interior of a fuzzy set, fuzzy closure of a fuzzy set in fuzzy topology and also, we will try to prove some theorems on fuzzy interior and fuzzy closure of a fuzzy topology with the help of extended definition of fuzzy sets and complement of a fuzzy sets.

Keywords: fuzzy membership function; fuzzy reference function; fuzzy membership value; fuzzy open set; fuzzy closed set; fuzzy interior; fuzzy closure.

Reference to this paper should be made as follows: Basumatary, B. (2015) 'Fuzzy interior and fuzzy closure of fuzzy set with extended definition of fuzzy set', *Int. J. Computational Systems Engineering*, Vol. 2, No. 2, pp.94–97.

Biographical notes: Bhimraj Basumatary is a research scholar of Department of Mathematical Sciences, Bodoland University, BTC, Assam, India. He received his MSc degree from Tezpur University, Tezpur, Assam, India, in 2012. He has cracked state level eligibility test (SET) for lectureship, exam held in September 2012. His research interest is topology and fuzzy topology. He has published five articles in international journals.

1 Introduction

Fuzzy set theory was discovered by Zadeh in 1965. Chang (1968) introduced fuzzy topology. After the introduction of fuzzy sets and fuzzy topology, several researches were conducted on the generalisations of the notions of fuzzy sets and fuzzy topology. The theory of fuzzy sets actually has been a generalisation of the classical theory of sets in the sense that the theory of sets should have been a special case of the theory of fuzzy sets. But unfortunately, it has been accepted that for fuzzy set A and its complement A^C, neither $A \cap A^C$ is empty nor $A \cup A^C$ is the universal set. Whereas the operations of union and intersection of crisp sets are indeed special cases of the corresponding operation of two fuzzy sets, they end up giving peculiar results while defining $A \cap A^C$ and $A \cup A^C$. In this regard, Baruah (1999, 2011a, 2011b) has forwarded an extended definition of fuzzy sets which enable us to define complement of fuzzy sets in a way that give us $A \cap A^C$ is empty and $A \cup A^C$ is universal set.

In this article, we introduce definition of fuzzy topology, fuzzy open set, fuzzy closed set, fuzzy interior, fuzzy closure of fuzzy set in fuzzy topology with the help of extended definition of fuzzy sets. Also, we try to prove some theorems on fuzzy interior and fuzzy closure of fuzzy topology with the help of extended definition of fuzzy sets and complement of a fuzzy set.

2 Preliminaries

2.1 *Extended definition of fuzzy set*

Baruah (1999, 2011a, 2011b) gave an extended definition of fuzzy set. According to Baruah (1999, 2011a, 2011b), to define a fuzzy set, two functions namely fuzzy membership function and fuzzy reference function are necessary. Fuzzy membership value is the difference between fuzzy membership function and fuzzy reference function.

Let $\mu_1(x)$ and $\mu_2(x)$ be two functions such that $0 \le \mu_1(x) \le \mu_2(x) \le 1$. For fuzzy number denoted by $\{x, \mu_2(x), \mu_1(x); x \in U\}$, we call $\mu_2(x)$ as fuzzy membership function and $\mu_1(x)$ a reference function such that $(\mu_2(x) - \mu_1(x))$ is the fuzzy membership value for any x.

2.2 *Basic operations*

Let $A = \{x, \mu_1(x), \mu_2(x); x \in U\}$ and $B = \{x, \mu_3(x), \mu_4(x); x \in U\}$ be two fuzzy sets defined over the same universe U.

1. $A \subseteq B$ iff $\mu_1(x) \leq \mu_3(x)$ and $\mu_4(x) \leq \mu_2(x)$ for all $x \in U$.
2. $A \cup B = \{x, \max(\mu_1(x), \mu_3(x)), \min(\mu_2(x), \mu_4(x))\}$ for all $x \in U$.
3. $A \cap B = \{x, \min(\mu_1(x), \mu_3(x)), \max(\mu_2(x), \mu_4(x))\}$ for all $x \in U$. If for some $x \in U$, $\min(\mu_1(x), \mu_3(x)) \leq \max(\mu_2(x), \mu_4(x))\}$, then our conclusion will be $A \cap B = \phi$.
4. $A^C = \{x, \mu_1(x), \mu_2(x); x \in U\}^C = \{x, \mu_2(x), 0; x \in U\} \cup \{x, 1, \mu_1(x); x \in U\}$.
5. If $D = \{x, \mu(x), 0; x \in U\}$ then $D^C = \{x, 1, \mu(x); x \in U\}$ for all $x \in U$.

3 Propositions on extended definition of fuzzy set

1. For fuzzy sets A, B, and C over the same universe X, we have the following propositions:
   a. $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$
   b. $A \cap B \subseteq A, A \cap B \subseteq B$
   c. $A \subseteq A \cup B, B \subseteq A \cup B$
   d. $A \subseteq B \Rightarrow A \cap B = A$
   e. $A \subseteq B \Rightarrow A \cup B = B$.
2. Let $\tau = \{A_r: r \in I\}$ be a collection of fuzzy sets over the same universe U. Then
   a. $\bigcup_i A_i = \{x, \max(\mu_{1i}), \min(\mu_{2i}); x \in U\}$
   b. $\bigcup_i A_i = \{x, \min(\mu_{1i}), \max(\mu_{2i}); x \in U\}$
   c. $\left\{\bigcup_i A_i\right\}^C = \bigcap_i A_i^C$
   d. $\left\{\bigcap_i A_i\right\}^C = \bigcup_i \{A_i\}^C$.

4 Fuzzy topology

4.1 Definition of fuzzy topology

A fuzzy topology on a non-empty set X is a family τ of fuzzy set in X satisfying the following axioms

A fuzzy topology on a non-empty set X is a family τ of fuzzy set in X satisfying the following axioms:

T1 $0_X, 1_X \in \tau$

T2 $G_1 \cap G_2 \in \tau$, for any $G_1, G_2 \in \tau$

T3 $\cup G_i \in \tau$, for any arbitrary family $\{G_i: G_i \in \tau, i \in I\}$.

In this case, the pair (X, τ) is called a fuzzy topological space and any fuzzy set in τ is known as fuzzy open set in X and clearly every element of τ^C is said to be closed set.

Here, $1_x = \{x, 1, 0; x \in X\}$ and $0_X = \{x, \mu(x), \mu(x); x \in X\}$.

Example:

Let X = {a, b}

Let A = {(a, 0.5, 0), (b, 0.6, 0)}, B = {(a, 0.8, 0), (b, 0.7, 0)}.

Then the family = $\{0_X, 1_X, A, B\}$ is fuzzy topology, here $1_x = \{x, 1, 0; x \in X\}$ and $0_X = \{x, 0, 0; x \in X\}$.

4.2 Definition of fuzzy co topology

A fuzzy co topology on a non-empty set X is a family δ of fuzzy set in X satisfying the following axioms

T1 $0_X, 1_X \in \delta$

T2 $G_1 \cup G_2 \in \delta$, for any $G_1, G_2 \in \delta$

T3 $\cap G_i \in \delta$, for any arbitrary family $\{G_i: G_i \in \delta, i \in I\}$.

In this case, the pair (X, δ) is called a fuzzy topological space and any fuzzy set in δ is known as fuzzy closed set in X and clearly every element of δ^C is said to be open set.

5 Fuzzy interior and fuzzy closure of a fuzzy set

5.1 Definition of fuzzy interior

Let (X, τ) be fuzzy topology also $A=\{x, \mu_A(x), \gamma_A(x); x \in X\}$ be fuzzy set on X.

Then interior of a fuzzy set A is defined as union of all open subsets contained in A, denoted it as int(A) and is defined as follows:

$$\text{Int}(A) = \bigcup\{P : P \text{ is open set in X and } P \subseteq A\} = \left\{x, \max\left(\mu_{iA}\right), \min\left(\gamma_{iA}\right); x \in X\right\}.$$

5.2 Definition of fuzzy closure

Let (X, τ) be fuzzy topology also $A = \{x, \mu_A(x), \gamma_A(x); x \in X\}$ be fuzzy set on X.

Then, closure of a fuzzy set A is defined as the intersection of all the closed subsets containing A, denoted it as cl(A) and is defined as follows:

$$\text{Cl}(A) = \bigcap\{Q : Q \text{ is open set in X and } A \subseteq Q\} = \left\{x, \min\left(\mu_{iA}\right), \max\left(\gamma_{iA}\right); x \in X\right\}.$$

6 Theorem on fuzzy interior and fuzzy closure

Let (X, τ) be fuzzy topology, then

7.1 $\text{Int}(1_X) = 1_X, \text{int}(0_X) = 0_X$

7.2 $\text{cl}(1_X) = 1_X, \text{cl}(0_X) = 0_X$

7.3 $\text{int}(A) \subseteq A$

7.4 $A \subseteq \text{cl}(A)$

7.5 $\text{Int}(\text{int}(A)) = \text{int}(A)$

7.6 $\text{cl}(\text{cl}(A)) = \text{cl}(A)$

7.7 $A \subseteq B \Rightarrow \text{int}(A) \subseteq \text{int}(B)$

7.8 $A \subseteq B \Rightarrow \text{cl}(A) \subseteq \text{cl}(B)$

7.9 $\text{Int}(A \cap B) = \text{Int}(A) \cap \text{int}(B)$

7.10 $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$

Proof: The proof of the Theorems 7.1, 7.2, 7.3, 7.4, 7.5, 7.6, 7.7 and 7.8 are comes from the definition of interior and closure of a fuzzy topology.

Let $A = \{x, \mu_A(x), \gamma_A(x); x \in X\}$ and $B = \{x, \mu_B(x), \gamma_B(x); x \in X\}$ be two fuzzy sets.

Proof of Theorem 7.9: We have $\text{int}(A) = \{x, \max(\mu_{Ai}(x)), \min(\gamma_{Ai}(x)); x \in X\}$ and $\text{int}(B) = \{x, \max(\mu_{Bi}(x)), \min(\gamma_{Bi}(x)); x \in X\}$.

$$A \cap B = \{x, \min(\mu_A(x), \mu_B(x)), \max(\gamma_A(x), \gamma_B(x)); x \in X\}$$

So we have

$$\text{Int}(A \cap B) = \bigcup \left\{ \begin{array}{c} x, \min(\mu_{Ai}(x), \mu_{Bi}(x)), \\ \max(\gamma_{Ai}(x), \gamma_{Bi}(x)); x \in X \end{array} \right\}.$$

Now clearly $\text{int}(A \cap B) \subseteq \text{int}(A)$ and $\text{int}(A \cap B) \subseteq \text{int}(B)$ so $\text{int}(A \cap B) \subseteq \text{int}(A) \cap \text{int}(B)$.

Also from Theorem 7.3, we have $\text{Int}(A) \cap \text{int}(B) \subseteq A \cap B$.

Also from Theorem 7.5, we have $\text{Int}(A) \cap \text{int}(B) = \text{int}(\text{Int}(A) \cap \text{int}(B)) \subseteq \text{int}(A \cap B)$.

Hence, $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$.

Proof of Theorem 7.10: Similarly, we can prove following prove of the Theorem 7.9.

7 Theorems on relation between fuzzy interior and fuzzy closure

Let (X, τ) be fuzzy topology and $A = \{x, \mu_i(x), \gamma_i(x); x \in X\}$ be any fuzzy set, then

a $(\text{int}(A))^C = \text{cl}(A^C)$

b $\text{Int}(A^C) = (\text{cl}(A))^C$

c $\text{Int}(A) = (\text{cl}(A^C))^C$

d $\text{Cl}(A) = (\text{int}(A^C))^C$.

Proof:

a Given fuzzy set is $A = \{x, \mu_i(x), \gamma_i(x); x \in X\}$, so $\text{int}(A) = \{x, \max(\mu_i(x)), \min(\gamma_i(x)); x \in X\}$.

Now

$$\begin{aligned}(\text{int}(A))^C &= \{x, \max(\mu_i(x)), \min(\gamma_i(x)); x \in X\}^C \\ &= \{x, \min(\gamma_i(x)), 0; x \in X\} \\ &\cup \{x, 1, \max(\mu_i(x)); x \in X\}\end{aligned}$$

And $A^C = \{x, \gamma_i(x), 0; x \in X\} \cup \{x, 1, \mu_i(x); x \in X\}$.

So $\text{cl}(A^C) = \{x, \min(\gamma_i(x)), 0; x \in X\} \cup \{x, 1, \max(\mu_i(x)); x \in X\}$.

Thus, $(\text{int}(A))^C = \text{cl}(A^C)$.

b We have $A^C = \{x, \gamma_i(x), 0; x \in X\} \cup \{x, 1, \mu_i(x); x \in X\}$.

So $\text{int}(A^C) = \{x, \max(\gamma_i(x)), 0; x \in X\} \cup \{x, 1, \min(\mu_i(x)); x \in X\}$.

Also, $\text{cl}(A) = \{x, \min(\mu_i(x)), \max(\gamma_i(x)); x \in X\}$.

$$\begin{aligned}(\text{cl}(A))^C &= \{\{x, \max(\gamma_i(x)), 0, x \in X\} \\ &\cup \{x, 1, \min(\mu_i(x)); x \in X\}\}^C\end{aligned}$$

Hence, $\text{Int}(A^C) = (\text{cl}(A))^C$.

c We have $A = \{x, \mu_i(x), \gamma_i(x); x \in X\}$, so $\text{int}(A) = \{x, \max(\mu_i(x)), \min(\gamma_i(x)); x \in X\}$.

Also, $A^C = \{x, \gamma_i(x), 0; x \in X\} \cup \{x, 1, \mu_i(x); x \in X\}$.

So $\text{cl}(A^C) = \{x, \min(\gamma_i(x)), 0; x \in X\} \cup \{x, 1, \max(\mu_i(x)); x \in X\}$.

$$\begin{aligned}\{\text{cl}(A^C)\}^C &= \{\{x, \min(\gamma_i(x)), 0; x \in X\} \\ &\cup \{x, 1, \max(\mu_i(x)), x \in X\}\}^C \\ &= \{x, \min(\gamma_i(x)), 0; x \in X\}^C \\ &\cap \{x, 1, \max(\mu_i(x)); x \in X\}^C \\ &= \{x, 1, \min(\gamma_i(x)); x \in X\} \\ &\cap \{x, \max(\mu_i(x)), 0; x \in X\} \\ &= \{x, \max(\mu_i(x)), \min(\gamma_i(x)); x \in X\}\end{aligned}$$

Hence, $\text{Int}(A) = (d(A^C))^C$.

d $\text{cl}(A) = \{x, \min(\mu_i(x)), \max(\gamma_i(x)); x \in X\}$.

Also, $A^C = \{x, \gamma_i(x), 0; x \in X\} \cup \{x, 1, \mu_i(x); x \in X\}$.

And $\text{int}(A^C) = \{x, \max(\gamma_i(x)), 0; x \in X\} \cup \{x, 1, \min(\mu_i(x)); x \in X\}$.

$$\begin{aligned}(\text{int}(A^C))^C &= \{\{x, \max(\gamma_i(x)), 0; x \in X\} \\ &\cup \{x, 1, \min(\mu_i(x)); x \in X\}\}^C \\ &= \{x, \max(\gamma_i(x)), 0; x \in X\}^C \\ &\cap \{x, 1, \min(\mu_i(x)), x \in X\}^C \\ &= \{x, 1, \max(\gamma_i(x)), x \in X\} \\ &\cap \{x, \min(\mu_i(x)), 0; x \in X\} \\ &= \{x, \min(\mu_i(x)), \max(\gamma_i(x)); x \in x\}.\end{aligned}$$

Hence, $\text{Cl}(A) = (\text{int}(A^C))^C$.

8 Conclusions

We have seen that in articles (Baruah, 2011b) that if a fuzzy set is characterised with respect to a reference function, we can define the complement of a fuzzy set in its actual perspective. As for the definition using a reference function, we are clear that the original definition is sufficient for fuzzy arithmetic. However, from original definition $A \cap A^C = \phi$ does not follow. From Baruah's (1999, 2011a, 2011b) definition of complement of a fuzzy set, we can remove this difficulty. So in this article, we tried to remove this difficulty by applying Baruah's (1999, 2011a, 2011b) definition of fuzzy set and Baruah's definition of complement of a fuzzy set to give the definitions of fuzzy topology, interior and closure of fuzzy topology and to prove some theorems on interior and closure of fuzzy topology.

References

Baruah, H.K. (1999) 'Fuzzy membership with respect to a reference function', *Jr. of Assam Science Society*, Vol. 40, No. 3, pp.65–73.

Baruah, H.K. (2011a) 'The theory of fuzzy sets: belief and realities', *Int. Jr. of Energy, Information and Communications*, May, Vol. 2, No. 2, pp.1–22.

Baruah, H.K. (2011b) 'Towards forming a field of fuzzy sets', *Int. Jr. of Energy, Information and Communications*, February, Vol. 2, No. 1, pp.16–20.

Chang, C.L. (1968) 'Fuzzy topological space', *Journal of Mathematical Analysis and Application*, Vol. 24, No. 1, pp.182–190.

Zadeh, L.A. (1965) 'Fuzzy sets', *Information and Control*, Vol. 8, No. 3, pp.338–353.