CHAPTER 3

FUZZY SET AND FUZZY TOPOLOGY

3.1 Introduction

Since after the discovery of fuzzy set by Zadeh [86], it has been developed in theory and application. After the discovery of fuzzy set, fuzzy topology has been introduced by Chang [20]. Topology belongs to most important and elementary characteristic of a set. After the introduction of fuzzy sets and fuzzy topology, several researches were conducted on the generalizations of the notions of fuzzy sets and fuzzy topology. In fuzzy topology also many results are not same as general topology. To avoid such difficulty of fuzzy topology many new ideas were developed by several authors. The concept of intuitionistic fuzzy set was introduced by Atnassov [6] as a generalization of fuzzy set. By observing this idea in 1997 Coker [22] introduced the concept of intuitionistic fuzzy topology.

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1. The results discussed in this chapter have accepted for publication in *German Journal of Advanced Mathematical Sciences (GJAMS).* Singh K. P. & Basumatary B.,"A New View on Fuzzy Set and Fuzzy Topological Spaces with Reference to Extended definition of Fuzzy Set" in *German Journal of Advanced Mathematical Sciences (GJAMS).*
2. Basumatary B, "A note on Fuzzy complement and fuzzy function on the Basis of Fuzzy Complement", International seminar presented, held at NIT Manipur on 18-20 Nov. 2016.

Strong and Weak forms of continuous maps have been introduced and investigated by several researchers like (Arockiarani, Balachandran et al. [3], Dontchev [29], Ganster and Reilly [33], Levine [45, 46], Maheshwari and Thakur [52, 53], Mashour et al. [60, 61].The strong forms of continuous maps have been discussed by Noiri [64]. Levine [45, 46], Arya and Gupta [4] and Reilly and Vamanamurthy [75, 76], they have introduced strong continuous maps, strongly θ-continuous map, super continuous map and clopen continuous maps respectively. Biswas [18], Ganster and Reilly [33], Noiri [62],Mashour et al. [60, 61], Tong [81, 82, 83] and Devi et al. [24, 25, 26] have introduced and studied simple continuity, almost continuity, weak continuity,α-continuity, β-continuity, semi-weak continuity, weak almost continuity, α-generalized continuity and generalized-α-continuity respectively.

Topology of a crisp set is the collection of intersection and union of subsets of a set. The idea of fuzzy topology of a fuzzy set received a lot of attention from the researcher from the beginning of fuzzy set theory.

Since Fuzzy Mathematics is just a kind of Mathematics developed so fuzzy topology is a just kind of topology developed on fuzzy sets. Fuzzy topology is a generalization of topology in classical mathematics, but it also has its own marked characteristics. Also it cal deepen the understanding of basic structure of classical mathematic, offer new methods and results and obtained significant results of classical mathematics. Moreover, it also has applications in some important respects of science and technology.

Many authors have suggested several approaches to fuzzy topology. These approaches has discussed in the following.

To know the term “Topology”, it is helpful first to take a brief look at all abstract mathematical discipline. With this as a background we try to define topology. Generally topology is divided into two main branches; general topology and algebraic topology. Also, it is known as point-set topology and combinatorial topology respectively.

From our opinion we can define topology in two ways as under. Topology is the mathematical discipline concerned with the precise definition for the concept of spatial structure, comparing the various definitions of spatial structure that have been or may be given and investigations relations between properties that can be introduced into topological system.

Secondly topology is the study of all properties of a space that is to investigate the relationship among topological structures that have been defined by various means. This is marked contrast to other mathematical disciplines, where the structure to be studied is usually well determined so that there might be at least some study of the most to be elegant way to formulate the requirements to be imposed on the entities in terms of which the structure is defined.

Now let us discuss on some definition and properties of classical function and continuous function of a crisp set of classical topology.

3.2 FUNCTION ON CLASSICAL SET

Let X, Y be two non empty sets. A subset of $X \times Y$ is called a function f from X to Y if and only if to each $x \epsilon X$ there exists y in Y such that $(x, y) \epsilon f$.

The range of f is the set of all images under f and is denoted by $f(X)$ and is defined as

f(X)={yϵY: y=f(x) }. If A is subset of X then the set {f(x): xϵA} is called image of A and is denoted by f(A). If B is subset of Y, then the set {xϵX: f(x)ϵB } is called inverse image of B under f and is denoted by f^{-1}(B).

3.2.1 EQUALITY OF TWO CLASSICAL FUNCTIONS

Two functions f: X$\rightarrow$Y and g: X$\rightarrow$Y are said to be equal if and only if f(x) = g(x), for every x in X and we write f=g.

3.2.2 CONSTANT FUNCTION

A function f: X$\rightarrow$Y is said to be constant function if for some $y_0\epsilon$Y, we have f(x)=y_0, for every x in X.

3.2.3 IDENTITY FUNCTION

Let a function I: X$\rightarrow$X be defined as I(x)=x, for every x in X, then the function I is called identity function on X.

3.2.4 CHARACTERISTIC FUNCTION

Let A be a subset of X. The characteristic function χ_A of the set A is defined by

$$\chi_A(x) = \begin{cases} 1 \; if \; x \epsilon X \\ 0 \; if \; x \notin X \end{cases}$$

3.2.5 THEOREMS ON FUNCTIONS

Let f: $X \rightarrow Y$ be a function. Then

(i) $B_1 \subseteq B_2 \Rightarrow f^{-1}[B_1] \subseteq f^{-1}[B_2]$, B_1 and B_2 are subsets in Y.

(ii) $A_1 \subseteq A_2 \Rightarrow f[A_1] \subseteq f[A_2]$, A_1 and A_2 are subsets in X.

(iii) $B \supseteq f[f^{-1}[B]]$ for any subset B in Y.

(iv) $A \subseteq f^{-1}[f[A]]$ for any subset A in X.

(v) $f^{-1}[\bigcup B] = \bigcup f^{-1}[B]$

(vi) $f^{-1}[\bigcap B] = \bigcap f^{-1}[B]$

(vii) $f[\bigcup A] = \bigcup f[A]$

(viii) $f[\bigcap B] = \bigcap f[B]$

3.3 CLASSICAL TOPOLOGICAL SPACE

Let X be a set and τ be a collection of subsets of X satisfying the following three conditions

(T1) $0_X, 1_X \in \tau$

(T2) $G_1 \bigcap G_2 \in \tau$, for any $G_1, G_2 \in \tau$

(T3) $\bigcup G_i \in \tau$, for any arbitrary family $\{G_i : G_i \in \tau, i \in I\}$.

Then τ is called topology for X and the fair (X, τ) is called topological space.

Example:

Let X={a, b, c} and consider the collection of the subsets of X τ={ϕ, {a}, {b}, {a, b}, X}, then τ is topology on X.

3.4 CONTINUOUS FUNCTION ON TWO TOPOLOGICAL SPACES

Let X and Y be two topological spaces. A function *f*: X→Y is continuous if and only if the inverse image under *f* of every open set in Y is open in X.

Example:

Let X={a, b, c} and τ={ϕ, {a}, {b, c}, X} be topology on X. Also let Y={p, q, r} and ∂={ϕ, {r}, {p, q}, Y} be topology on Y.

Now let us define a function *f*: X→Y such that *f*(a)=r, *f*(b)=p, *f*(c)=q.

Then clearly f^{-1}[{r}]={a}, f^{-1}[{p, q}]={b, c}, since {a} and {b, c} are open sets in τ, it follows that *f* is continuous.

Now before going to fuzzy topology, let us discuss on fuzzy set and fuzzy function on the basis of reference function.

3.5 A NEW APPROACH TO FUZZY SET AND FUZZY FUNCTION

3.5.1 NEW DEFINITION OF FUZZY SET ON THE BASIS OF REFERENCE FUNCTION

Not everything is counted from zero level. For example if we consider the watering with pipe in the flower garden and if we stop watering then we can see that inside a pipe due to the presence of air, the containing of water is displaced part by part. In that case if we try to measure the contain of water inside the pipe, then we cannot measure from initial point of the pipe to end point of the pipe that is to say we cannot measure from zero level to end point of the pipe.

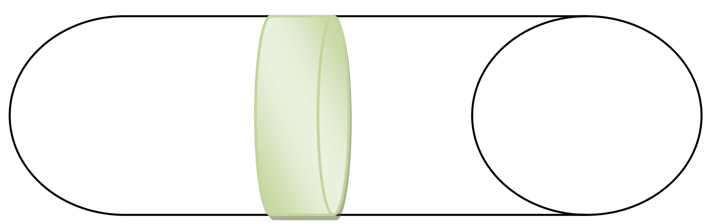

Fig 2-A pipe containing water

Baruah [5, 6] gave an extended definition of complementation of fuzzy set. According to Baruah [5, 6] to define a fuzzy set, two functions namely fuzzy membership function and fuzzy reference function are necessary. Fuzzy membership value is the difference between fuzzy membership function and fuzzy reference function.

Baruah [5, 6] gave an extended definition of complementation of fuzzy set. According to Baruah [5, 6] to define a fuzzy set, two functions namely fuzzy membership function and fuzzy reference function are necessary. Fuzzy membership value is the difference between fuzzy membership function and fuzzy reference function.

Let $\mu_1(x)$ and $\mu_2(x)$ be two functions such that $0 \leq \mu_2(x) \leq \mu_1(x) \leq 1$. For fuzzy number denoted by $\{x, \mu_1(x), \mu_2(x); x \epsilon U\}$, we call $\mu_1(x)$ as fuzzy membership function and $\mu_2(x)$ a reference function such that $(\mu_1(x) - \mu_2(x))$ is the fuzzy membership value.

3.5.2 Union and intersection of fuzzy sets on the basis of reference function

If we consider the fuzzy sets $A=\{x, \mu_1(x), \mu_2(x) : x \epsilon X\}$ and $B=\{x, \mu_3(x), \mu_4(x) : x \epsilon X\}$

Case-1:

When reference function is zero that is when $\mu_2(x)=0$ and $\mu_4(x)=0$, then on the basis of Baruah's [5, 6] definition of fuzzy sets we can represent the union of two fuzzy sets A and B as following figure and

$A \cup B=\{x, \mu_1(x), 0: x \epsilon X\} \cup \{x, \mu_3(x), 0: x \epsilon X\}$

$=\{x, \max\{\mu_1(x), \mu_3(x)\}, \min\{0, 0\}: x \epsilon X\}$

$=\{x, \mu_1\{x\}, 0: x \epsilon X\} =A$

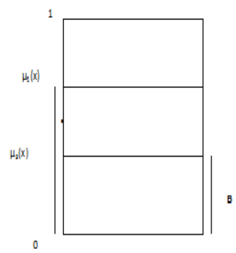

Fig-3 Union of two fuzzy set when reference function is zero

Case-2:

When reference function is not zero, then on the basis of Baruah's [5. 6] definition of fuzzy sets we can represent the union of two fuzzy sets A and B as following figure

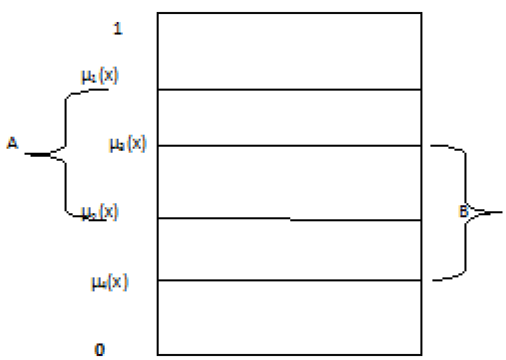

Fig-4 Union of two fuzzy set when reference function is not zero

$A \cup B = \{x, \mu_1(x), \mu_2(x): x \epsilon X\} \cup \{x, \mu_3(x), \mu_4(x): x \epsilon X\}$

$= \{x, \max\{\mu_1(x), \mu_3(x)\}, \min\{\mu_2(x), \mu_4(x)\}: x \epsilon X\}$

$= \{x, \mu_1\{x\}, \mu_4(x): x \epsilon X\}$

Similarly following fig- 2 and fig-3 we can show the intersection of two fuzzy sets A and B.

Case 1

when reference function is zero

$A \cap B = \{x, \mu_1(x), 0: x \in X\} \cap \{x, \mu_3(x), 0: x \in X\}$

$= \{x, \min\{\mu_1(x), \mu_3(x)\}, \max\{0, 0\}: x \in X\}$

$= \{x, \mu_3\{x\}, 0: x \in X\}$

$= B$

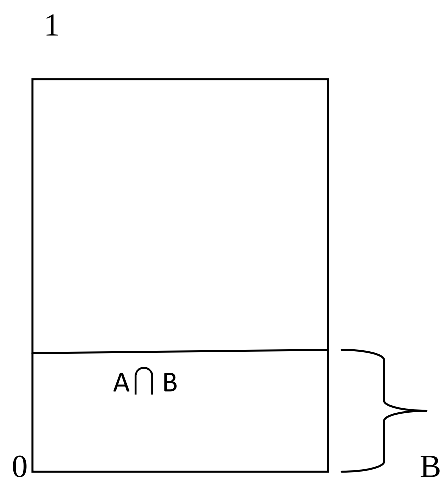

Fig-5 Intersection of two fuzzy set when reference function is zero

Case 2

when reference function is not zero

$A \cap B = \{x, \mu_1(x), \mu_2(x): x \in X\} \cap \{x, \mu_3(x), \mu_4(x): x \in X\}$

$=\{x, \min\{\mu_1(x), \mu_3(x)\}, \text{miax}\{\mu_2(x) , \mu_4(x)\}: x\epsilon X\}$

$=\{x, \mu_3\{x\}, \mu_2(x): x\epsilon X\}$.

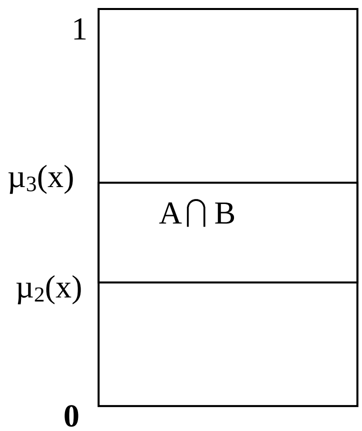

Fig-6 Intersection of two fuzzy set when reference function is not zero

3.5.3 Complement of fuzzy sets on the basis of reference function

Baruah put forward the notion of complement of usual fuzzy sets reference function 0 in the following way-

Let $A=\{x, \mu(x), 0; x\epsilon X\}$ and $B=\{x, 1, \mu(x); x\epsilon X\}$ be two fuzzy sets defined over the same universe X.

Now on the basis of reference function, we represent these two fuzzy sets A and B in the following figure.

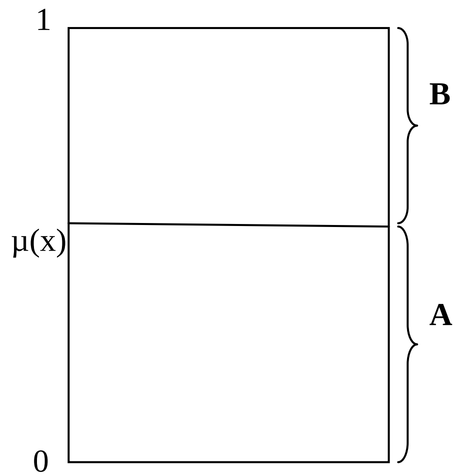

Fig-7 Complement of fuzzy sets when reference function is zero

Now we have

A⋂B={x, μ(x), 0; xϵX}⋂{x, 1, μ(x), xϵX}

={x, min{μ(x), 1}, max{0, μ(x)}: xϵX}

={x, μ(x), μ(x) : xϵX}

=ϕ

Also

$A \cup B = \{x, \mu(x), 0: x\epsilon X\} \cup \{x, 1, \mu(x): x\epsilon X\}$

$= \{x, \max\{\mu(x), 1\}, \min\{0, \mu(x)\}: x\epsilon X\}$

$= \{x, 1, 0; x\epsilon X\}$

When reference is not zero then let $A=\{x, \mu_1(x), \mu_2(x); x\epsilon X\}$ and

$B=\{x, 1, \mu_1(x); x\epsilon X\} \cup \{x, \mu_2(x), 0; x\epsilon X\}$ be two fuzzy sets on same universe X.

Now for two fuzzy sets A and B we represents the following figure-

$A \cap B=\{x, \mu_1(x), \mu_2(x); x\epsilon X\} \cap \{\{x, 1, \mu_1(x); x\epsilon X\} \cup \{x, \mu_2(x), 0; x\epsilon X\}\}$

$=\{\{x, \mu_1(x), \mu_2(x); x\epsilon X\} \cap \{x, 1, \mu_1(x); x\epsilon X\}\} \cup \{\{x, \mu_1(x), \mu_2(x); x\epsilon X\} \cap \{x, \mu_2(x), 0; x\epsilon X\}\}$

$=\{x, \min\{\mu_1(x), 1\}, \max\{\mu_2(x), \mu_1(x)\}; x\epsilon X\} \cup \{x, \min\{\mu_1(x), \mu_2(x)\}, \max\{\mu_2(x), 0\}; x\epsilon X\}$

$=\{x, \mu_1(x), \mu_1(x); x\epsilon X\} \cup \{x, \mu_2(x), \mu_2(x); x\epsilon X\}$

$=\phi \cup \phi$

$=\phi$

Also

$A \cup B=\{x, \mu_1(x), \mu_2(x); x\epsilon X\} \cup \{\{x, 1, \mu_1(x); x\epsilon X\} \cup \{x, \mu_2(x), 0; x\epsilon X\}\}$

$=\{\{x, \mu_1(x), \mu_2(x); x\epsilon X\} \cup \{x, 1, \mu_1(x); x\epsilon X\}\} \cup \{x, \mu_2(x), 0; x\epsilon X\}$

$=\{x, \max\{\mu_1(x), 1\}, \min\{\mu_2(x), \mu_1(x)\}; x\epsilon X\} \cup \{x, \mu_2(x), 0; x\epsilon X\}$

$=\{x, 1, \mu_2(x); x\epsilon X\} \cup \{x, \mu_2(x), 0; x\epsilon X\}$

$=\{x, \max\{1, \mu_2(x)\}, \min\{\mu_2(x), 0\}; x\epsilon X\}$

$=\{x, 1, 0; x\epsilon X\}$, which is universal set.

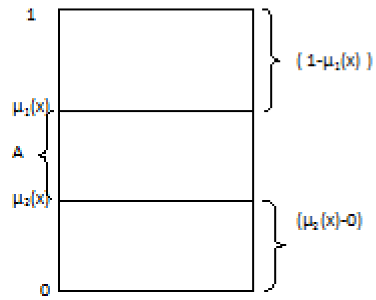

Fig-8 Complement of fuzzy sets when reference function is not zero

Therefore it is clear that if we express the complement of a fuzzy set A={x, μ(x), 0; x϶X} as A^C={x, 1, μ(x); x϶X }, when reference function is zero

Then we have

1. $A \cap A^C$= the null set ф, and
2. $A \cup A^C$= the universal set X.

Also for non zero reference function we have the complement for a fuzzy set A ={x, $\mu_1(x)$, $\mu_2(x)$; x϶X} as

A^C={x, 1, $\mu_1(x)$; x϶X} ⋃ {x, $\mu_2(x)$, 0; x϶X}

Then we have

1. $A \cap A^C$= the null set ф, and
2. $A \cup A^C$= the universal set X.

This would enable us to establish that fuzzy sets do form a field if we define the complementation in Baruah's ([8, 9, 10, 11, 12, 13, 14]) definition of complement of a fuzzy set.

3.5.4 Some propositions

1. $A \bigcup B = B \bigcup A$, $A \bigcap B = B \bigcap A$

2. $A \bigcup (B \bigcup C) B = (A \bigcup B) \bigcup C$, $A \bigcap (B \bigcap C) = (A \bigcap B) \bigcap C$

3. $(A \cup B)^C = A^C \cap B^C$

4. $(\cup A_i)^C = \cap A_i^C$

It can be verified that the above propositions are satisfied if the complementation is defined on the basis of reference function.

It is very important to mention that the Baruah's [5, 6] definition of complementation of a fuzzy sets does satisfies almost all the properties of fuzzy sets. As we see that this new definition of fuzzy sets performs precisely as the corresponding operation on crisp sets.

3.4 FUZZY FUNCTION ON THE BASIS OF REFERENCE FUNCTION

Definition**:** Let X and Y be two non empty sets and f: $X \rightarrow Y$ be a function

Let $B=\{y, \mu_B(y), \gamma_B(y): y \epsilon Y\}$ be fuzzy set on Y, then preimage of B under f denoted by

$f^{-1}(B)$, is fuzzy set in X defined by

$f^{-1}(B) = \{x, f^{-1}(\mu_B)(x), f^{-1}(\gamma_B)(x): x \epsilon X\}$,

where $f^{-1}(\mu_B)(x) = \mu_B(f(x))$ and $f^{-1}(\gamma_B)(x) = \gamma_B(f(x))$.

If $A=\{x, \mu_A(x), \gamma_A(x): x \epsilon X\}$ be fuzzy set in X, then image of A under f is denoted by f(A) and defined as

$f(A)(y) = \bigcup \{ x \epsilon X, f(x)=y, \mu_A(x), \gamma_A(x)\}$.

3.4.1 THEOREMS ON FUZZY FUNCTION ON THE BASIS OF REFERENCE FUNCTION

Let f be a function from X to Y. Then

1. $B_1 \subseteq B_2 \Rightarrow f^{-1}[B_1] \subseteq f^{-1}[B_2]$, B_1 and B_2 are fuzzy sets in Y.
2. $A_1 \subseteq A_2 \Rightarrow f[A_1] \subseteq f[A_2]$, A_1 and A_2 are fuzzy sets in X.
3. $B \supseteq f[f^{-1}[B]]$ for any fuzzy subset B in Y.
4. $A \subseteq f^{-1}[f[A]]$ for any fuzzy subset A in X.
5. $f^{-1}[\bigcup B] = \bigcup f^{-1}[B]$
6. $f^{-1}[\bigcap B] = \bigcap f^{-1}[B]$
7. $f[\bigcup A] = \bigcup f[A]$
8. $f[\bigcap B] = \bigcap f[B]$

Proof: Prove of the theorems are true if complement is defined on the basis of reference function.

3.4.2 THEOREMS ON FUZZY FUNCTIONS

Let f be a function from X to Y. Then

1. $f^{-1}[1_U] = 1_U$.

2. $f^{-1}[0_U]=0_U$.

3. $f^{-1}[B^C]=\{f^{-1}[B]\}^C$ for any fuzzy set B in Y.

4. $\{f[A]\}^C \subseteq f[A^C]$ for any fuzzy set A in X.

Proof:

1. **Case1:**

When reference function's zero. We can write $1_U=B\cup B^C$, where B is fuzzy set in Y.

So $f^{-1}(1_U)=f^{-1}(B\cup B^C)$

$= f^{-1}(B)\cup f^{-1}(B^C)$

$=\{x, \mu_B(f(x)), 0: x\epsilon X\}\cup\{x, 1, \mu_B(f(x)): x\epsilon X\}$

$=\{x, 1, 0: x\epsilon X\}$

$=1_U$

Case2:

When reference function is not zero. Let

$f^{-1}(B)=\{x, \mu_B(f(x)), \gamma_B(f(x)): x\epsilon X\}$.

So, $[f^{-1}(B)]^C=\{x, \gamma_B(f(x)), 0: x\epsilon X\}\cup\{x, 1, \mu_B(f(x)): x\epsilon X\}$.

Now, $1_U=B\cup B^C$.

So $f^{-1}(1_U)=f^{-1}(B\cup B^C)$

$= f^{-1}(B) \cup f^{-1}(B^C)$

$=\{x, \mu_B(f(x)), \gamma_B(f(x)): x\epsilon X\} \cup [\{x, \gamma_B(f(x)), 0: x\epsilon X\} \cup \{x, 1, \mu_B(f(x)): x\epsilon X\}]$

$=[\{x, \mu_B(f(x)), \gamma_B(f(x)): x\epsilon X\} \cup \{x, \gamma_B(f(x)), 0: x\epsilon X\}] \cup \{x, 1, \mu_B(f(x)): x\epsilon X\}$

$=\{x, \max(\mu_B(f(x)), \gamma_B(f(x))), \min(\gamma_B(f(x)), 0): x\epsilon X\} \cup \{x, 1, \mu_B(f(x)): x\epsilon X\}$

$=\{x, \mu_B(f(x)), 0: x\epsilon X\} \cup \{x, 1, \mu_B(f(x)): x\epsilon X\}$

$=\{x, \max(\mu_B(f(x)), 1), \min(0, \mu_B(f(x))): x\epsilon X\}$

$=\{x, 1, 0: x\epsilon X\}$

$=1_U$

2. Proof: Let 0_U be null set then clearly we can write $0_U=B \cap B^C$, for fuzzy set B in Y.

Case 1:

When reference function is zero.

Then

$f^{-1}(0_U)=f^{-1}(B \cap B^C)$

$= f^{-1}(B) \cap f^{-1}(B^C)$

$=\{x, \mu_B(f(x)), 0: x\epsilon X\} \cap \{x, 1, \mu_B(f(x)): x\epsilon X\}$

$=\{x, \mu_B(f(x)), \mu_B(f(x)): x\epsilon X\}$

$=\phi_U$

Case2:

When reference function is not zero.

Now $f^{-1}(0_U)=f^{-1}(B\cap B^C)$

$=\{x, \mu_B(f(x)), \gamma_B(f(x)): x\epsilon X\}\cap [\{x, \gamma_B(f(x)), 0: x\epsilon X\}\cup\{x, 1, \mu_B(f(x)): x\epsilon X\}]$

$=[\{x, \mu_B(f(x)), \gamma_B(f(x)): x\epsilon X\}\cap\{x, \gamma_B(f(x)), 0: x\epsilon X\}]\cup[\{x, \mu_B(f(x)), \gamma_B(f(x)): x\epsilon X\}\cap\{x, 1, \mu_B(f(x)): x\epsilon X\}]$

$=\{x, \gamma_B=\{x, \gamma_B(f(x)), \gamma_B(f(x)): x\epsilon X\}\cup\{x, \mu_B(f(x)), \mu_B(f(x)): x\epsilon X\}$

$=\phi_U\cup\phi_U$

$=\phi_U$

3. Proof: Let $B=\{y, \mu_B(y), \gamma_B(y): y\epsilon Y\}$ be fuzzy set on Y.

Case 1:

When reference function is zero. Then Let $B=\{y, \mu_B(y), 0: y\epsilon Y\}$.

So, $B^C=\{y, 1, \mu_B(y), : y\epsilon Y\}$

Then $f^{-1}(B^C)= \{x, 1, \mu_B(f(x)): x\epsilon X\}$

$=\{x, \mu_B(f(x)), 0: x\epsilon X\}^C$

$=[f^{-1}\{y, \mu_B(y), 0: y\epsilon Y\}]^C$

$=[f^{-1}(B)]^C$.

Case 2:

When reference function is not zero. Then B={y, μ_B(y), γ_B(y): yϵY} and

B^C ={y, γ_B(y), 0; yϵY}$\bigcup$ {y, 1, μ_B(y); yϵY}.

Then $f^{-1}(B^C)$= {x, γ_B(f(x)), 0; xϵX}$\bigcup$ {x, 1, μ_B((x)); xϵX}

={x, μ_B((x)), γ_B(f(x)); xϵX}^{C}

=[f^{-1}{y, μ_B(y), γ_B(y): yϵY}]^C

=[f^{-1}(B)]^C.

4 Proof:

Let, A={x, μ_A(x), γ_A(x): xϵX}.

Case 1:

When reference function is zero then A={x, μ_A(x), 0: xϵX} and

A^C={x, 1, μ_A(x): xϵX}.

f(A^C)=$\bigcup$ {x, 1, μ_A(x): xϵX}

$\supseteq \bigcap$ {x, 1, μ_A(x): xϵX}

={ $\bigcup$ {x, μ_A(x), 0: xϵX}}^{C}

={f(A)}^{C}

Case 2:

When reference function is not zero.

$A=\{x, \mu_A(x), \gamma_A(x): x\epsilon X\}$ and

$A^C=\{x, \gamma_A(x), 0: x\epsilon X\} \cup \{x, 1, \mu_A(x); x\epsilon X\ \}$

$f(A^C)= \cup [\{x, \gamma_A(x), 0: x\epsilon X\} \cup \{x, 1, \mu_A(x); x\epsilon X\}]$

$\supseteq \cap\ [\{x, \gamma_A(x), 0: x\epsilon X\} \cup \{x, 1, \mu_A(x); x\epsilon X\ \}]$

$=[\cup \{x, \mu_A(x),\ \gamma_A(x): x\epsilon X\}]^C$

$=[f(A)]^C$

Hence $\{f[A]\}^C \subseteq f[A^C]$.

3.4.3 PROPOSITIONS

Let f be a function from X to Y then

1. $f(1_U)=1_U$, if f is surjective

2. $f(0_U)=0_{U.}$

Proof: Prove of the propositions are true if complement is defined on the basis of reference function.

3.5 A NEW APPROACH TO FUZZY TOPOLOGY

After Zadeh created fuzzy sets in his classical paper [118], Chang [36] used them to introduce the concept of a fuzzy topology.

A fuzzy topology on a nonempty set X is a family τ of fuzzy set in X satisfying the following axioms

(T1) 0_X, $1_X \in \tau$

(T2) $G_1 \cap G_2 \in \tau$, for any $G_1, G_2 \in \tau$

(T3) $\cup G_i \in \tau$, for any arbitrary family $\{G_i : G_i \in \tau, i \in I\}$.

In this case the pair (X,τ) is called a fuzzy topological space and any fuzzy set in τ is known as fuzzy open set in X and clearly every element of τ^C is said to be fuzzy closed set.

Here

$1_X=\{x, 1, 0; x\epsilon X\}$ and $0_X=\{x, \mu(x), \mu(x); x\epsilon X\}$.

Example:

Let X={a, b} and

A={(a, 0.4, 0), (b, 0.5, 0)}.

Then the family $\Gamma=\{0_X, 1_X, A\}$ is fuzzy topology, where $1_X=\{x, 1, 0; x\epsilon X\}$ and $0_X=\{x, 0, 0; x\epsilon X\}$.

3.5.1 OBSERVATION

1. Every member of *T* is called a *T* – open fuzzy set.
2. A fuzzy set is *T*- closed if and only if its complement is *T* -open.

As in general topology the indiscrete fuzzy topology contains only Ø and X; while the discrete fuzzy topology contains all fuzzy sets.

3.5.2 DEFINITION Let (X, Γ_1) and (X, Γ_2) be *two* fuzzy topological spaces. If for any $G \epsilon\ \Gamma_1$ implies

$G \epsilon\ \Gamma_2$ then Γ_1 is said to be coarser than Γ_2 or Γ_2 is finer than Γ_1.

3.5.2 PROPOSITION Let $\{ \Gamma_i : i \epsilon I \}$ be family of fuzzy topological space on X. Then $\bigcap \Gamma_i$ is also an fuzzy topology on X.

3.5.3 DEFINITION Let (X,τ) and (Y,δ) be two fuzzy topological spaces and $f: X \rightarrow Y$ be a mapping then f is called continuous mapping if and only if preimage of each fuzzy set in δ is fuzzy set in τ.

3.6 PROPOSITION The function $f: (X,\tau) \rightarrow (Y,\delta)$ is fuzzy continuous if and only if preimage of each fuzzy closed set in δ is fuzzy closed set in τ.

Proof:

Case1:

When reference function is zero.

Let f: $(X,\tau) \rightarrow (Y,\delta)$ be fuzzy continuous function. Let B={y, $\mu_B(y)$, 0: yϵY } be fuzzy set in δ. Then B^C={y, 1, $\mu_B(y)$, : yϵY} is fuzzy closed set in δ.

Since we have $f^{-1}[B^C]=\{f^{-1}[B]\}^C$.

Now as f is fuzzy continuous so by definition

$f^{-1}[B^C]=[f^{-1}(B)]^C \epsilon\ \tau$.

Conversely let f: $(X,\tau) \rightarrow (Y,\delta)$ be a fuzzy function and the preimage of each fuzzy closed set in δ is fuzzy closed set in τ. Let B={y, $\mu_B(y)$, 0: yϵY } be fuzzy set in δ.

So B^C={y, 1, $\mu_B(y)$: yϵY} is fuzzy closed set in δ and $f^{-1}[B^C]=\{f^{-1}[B]\}^C$. Since f is function from (X,τ) to (Y,δ) so f^{-1} is function from (Y,δ) to (X,τ). B^C is fuzzy closed set in δ so

$f^{-1}[B^C]=\{f^{-1}[B]\}^C$ is fuzzy closed set in τ. Therefore $f^{-1}[B^C]\ \epsilon\ \tau$ and hence f is fuzzy continuous function.

Case 2:

when reference function is not zero.

Let f: $(X,\tau) \rightarrow (Y,\delta)$ be fuzzy continuous function . Let B={y, $\mu_B(y)$, $\gamma_B(y)$: yϵY} be fuzzy set in δ. B^C={y, 1, $\mu_B(y)$}$\cup$ {y, $\gamma_B(y)$, 0: yϵY} is fuzzy closed set in δ.

$f^{-1}[B^C]= f^{-1}$ [{y, 1, $\mu_B(y)$}$\cup$ {y, $\gamma_B(y)$, 0: yϵY}]

$=[f^{-1}(B)]^C$.

Now as f is fuzzy continuous so by definition

$f^{-1}[B^C]=[f^{-1}(B)]^C \epsilon\ \tau$.

Conversely let f: $(X,\tau) \rightarrow (Y,\delta)$ be a fuzzy function and the preimage of each fuzzy closed set in δ is fuzzy closed set in τ.

Let $B=\{y, \mu_B(y), \gamma_B(y): y\epsilon Y\}$ be fuzzy set in δ.

$B^C=\{y, 1, \mu_B(y)\} \cup \{y, \gamma_B(y), 0: y\epsilon Y\}$ is fuzzy closed set in δ. $f^{-1}[B^C]=\{f^{-1}[B]\}^C$. Since f is function from (X,τ) to (Y,δ) so f^{-1} is function from (Y,δ) to (X,τ). B^C is fuzzy closed set in δ so $f^{-1}[B^C]=\{f^{-1}[B]\}^C$ is fuzzy closed set in τ. Therefore $f^{-1}[B^C]\epsilon\ \tau$ and hence f is fuzzy continuous function.

3.7 PROPOSITION The function f: $(X,\tau) \rightarrow (Y,\delta)$ is fuzzy continuous if and only if preimage of each fuzzy open set in δ is fuzzy open set in τ.

Proof: Prove of the proposition is obvious.

3.8 CONCLUSION

In this Chapter, our main contribution is to revisit and comment on the definition of fuzzy set and in the process it is found that, when we dealing with complement of fuzzy set then it is not reasonable. We tried to give example of fuzzy set geometrically on the basis of reference function. We have introduced definition of fuzzy function on the basis of reference function. Also, we have discussed on some proposition of fuzzy function. It is seen that when we expressed fuzzy function in our way then propositions on fuzzy functions are satisfied very nicely. In this Chapter we put new definition of fuzzy topology on the basis of reference function with example. Definition of fuzzy continuous function is discussed with new definition of fuzzy set. Some propositions on fuzzy continuous function is discussed on the basis of reference function.