CHAPTER 5

Fuzzy Boundary on the Basis of Reference Function

5.1 INTRODUCTION

After introduction of fuzzy sets by Zadeh's [86], Chang [20] defined and studied the notion of a fuzzy topological space in 1968. Since then, much attention has been paid to generalize the basic concepts of classical topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed. Though Pu and Liu [74] defined the notion of fuzzy boundary in fuzzy topological spaces in 1980, yet there is very little work available on this notion in present literature. Subsequently, Pu and Liu [74] have defined fuzzy boundary as a generalization of the crisp boundary in general topology. One reason, inter alia, of Tong [81] use of a limited version of Chang's fuzzy topological space was the non availability of sufficient material about properties of fuzzy boundary.

The results discussed in this chapter have published in International Journals

1. B. Basumatary, Towards Forming the Field of Fuzzy Closure with Reference to Fuzzy Boundary, *JPMNT*, Vol. 4, No.1, pp30-34.
2. B. Basumatary, S. Borgoyary, K. P. Singh, H. K. Baruah, "Towards Forming the Field of Fuzzy Boundary on the Basis of Reference Function", *GJPAM*, Vol. 13, No.6, pp2703-2716.

Fuzzy boundary in the context of fuzzy topological space was defined by Warren [84], which was later modified by Cuchillo-Ibanez and Tarres in [39].

Warren verified the following properties of the fuzzy boundary:

(1) The boundary is closed.

(2) The closure is the supremum of the interior and boundary.

(3) The boundary reduces to the usual topological boundary when all fuzzy sets are crisp.

(4) The boundary operator is an equivalent way of defining a fuzzy topology.

However, Warren's definition lacks the following properties:

(5) The boundary of a fuzzy set is identical to the boundary of the complement of the set.

(6) If a fuzzy set is closed (or open), then the interior of the boundary is empty.

(7) If a fuzzy set both open and closed then the boundary is empty.

Now let us discuss on some definition and some properties of classical boundary of a crisp set which are generally not same as fuzzy boundary.

5.2 DEFINITION A point x of a topological spaces (X, τ) is said to be boundary point of a set A of X if and only if it is neither an interior point nor an exterior point of A. The set of all boundary points of A is called the boundary of A and is denoted by Bd(A).

Note: From the above definition we are cleared that $Bd(A)=Cl(A)\cap Cl(A^C)$.

Example: Let $X=\{a, b, c, d, e\}$ and let $\tau=\{\phi, \{b\}, \{c, d\}, \{b, c, d\}, \{a, b, c, d\}, X\}$, then (X, τ) be topological space. Let $A=\{c\}$, $B=\{a, b\}$, $D=\{a, b, d\}$, $E=\{b, c, d\}$.

Now we try to find Bd(A), Bd(B), Bd(D) and Bd(E).

Now it is cleared from the definition of boundary that $Bd(A)=\{a, c, d, e\}$.

Here $B=\{a, b\}$ and $B^C=\{c, d, e\}$.

So, $Cl(B)=\cap\{P \text{ is closed set in } X: B\subseteq P\}$

$=\{a, b, e\}\cap\{a, b, c, d, e\}$

$=\{a, b, e\}$.

And $Cl(B^C)= \cap\{P \text{ is closed set in } X: B^C\subseteq P\}$

$=\{a, b, c, d\}$

Hence $Bd\{B\}=\{a, b, e\}\cap\{a, b, c, d, e\}$

$=\{a, b, e\}$.

Now for finding Bd(D), we repeat same process. Here $D=\{a, b, d\}$ and $D^C=\{c, e\}$

$Cl(D)=\cap\{P \text{ is closed set in } X: D\subseteq P\}$

$=\{a, b, c, d, e\}$

$Cl(D^C)=\cap\{P \text{ is closed set in } X: D^C\subseteq P\}$

Bd(D)={a, c, d, e}.

Now for E, here E={b, c, d} is open set and E^C={a, e} closed set.

Now for E if we follow the same process.

Cl(E) =∩{P is closed set in X: E⊆P}

={a, b, c, d}.

Now Cl(E^C) =∩{P is closed set in X: E^C⊆P}

={a, c, d, e}∩{a, b, e}

={a, e}

Hence Bd(E)={a, e}.

5.2.1 THEOREM Let (X, τ) be topological space and A be subset of X, then Cl(A)=Int(A)∪Bd(A).

5.2.1.1 Corollary: Bd(A)⊆ A.

5.2.1.2 Corollary: Cl(A)=A∪Bd(A).

5.2.2THEOREM Let (X, τ) be topological space and A, B be subsets of X. Then

(i) Bd(A) = Cl(A) – Int(A)

(ii) $[Bd(A)]^C$= Int(A)∪ Int(A^C)

(iii) $Bd(Int(A)) \subseteq Bd(A)$

(iv) $Bd(Cl(A)) \subseteq Bd(A)$

(v) $Bd(A \cup B) \subseteq Bd(A) \cup Bd(B)$

(vi) $Bd(A \cap B) \subseteq Bd(A) \cup Bd(B)$

5.2.3THEOREM Let (X, τ) be topological space and A be subsets of X. Then

(i) If A is open then $Bd(A) = Cl(A) – A$.

(ii) $Bd(A)=\phi$ if and only if A is both open as well as closed.

(iii) A is open if and only if $A \cap Bd(A)=\phi$, that is, if and only if $Bd(A) \subseteq A^C$.

(iv) A is closed if and only if $Bd(A) \subseteq A$.

5.2.4 THEOREM Let (X, τ) be topological space and A be subsets of X. Then

(i) $Bd(\phi)=\phi$.

(ii) $Bd(A) = Bd(A^C)$.

(iii) $Bd(Bd(A)) \subseteq Bd(A)$.

Now let us observe on some properties of fuzzy boundary which are not same as classical boundary of a crisp set.

Many authors have given various idea on the definition of fuzzy boundary. Here we discuss fuzzy boundary with the help of Chang [20] definition of fuzzy boundary.

5.3 DEFINITION Let A be a fuzzy set in fuzzy topological space X. Then the fuzzy boundary of A is defined as $Bd(A)=cl(A) \cap cl(A^C)$.

Remark1: Bd(A) is fuzzy closed set.

Remark2: For a fuzzy set A, A∪Bd(A)⊆Cl(A).

In classical topology for an arbitrary set A of a topological space X we have Cl(A)=A∪Bd(A), which is shown in **corollary5.2.1.2**.

Some Propositions of fuzzy boundary which are not satisfy the condition as classical boundary of a crisp set do.

5.4 PROPOSITION Let A and B be two fuzzy sets in fuzzy topological spaces X. then the following conditions are hold

(i) If A be fuzzy closed set then Bd(A)⊆A.

(ii) If A be fuzzy open set then Bd(A)⊆A^C.

In classical boundary converse parts of propositions (i) & (ii) are also true. But in fuzzy topology the converse part is not true.

In this chapter, our main concern is with defining fuzzy boundary of a fuzzy set in a manner which is consistent with the new definition of complement of fuzzy sets. It is necessary to define it accordingly because there are some drawbacks in defining existing definition of complement of fuzzy sets. As s consequence of which we tried to forward a new definition of fuzzy boundary for future work. It is very important to mention here that this new proposal does not affect the fuzzy boundary of usual fuzzy set because in the definition of complement of a fuzzy set, fuzzy membership value and the fuzzy membership function is taken to be different. In order to avoid any such confusion regarding this, we prepare to write in short about new definition of complementation of fuzzy sets on the basis of reference function as introduced by Baruah([8, 9, 10, 11, 12, 13, 14]). Here we do not like to examine their properties and procedures by which they are manipulated. Their detailed coverage can be found in Bauah([8, 9, 10, 11, 12, 13, 14]).

5.5 NEW APPROACH TO FUZZY BOUNDARY OF FUZZY SETS

5.5.1 DEFINITION Let A be a fuzzy set in fuzzy topological space X. Then the fuzzy boundary of A is defined as $Bd(A)=cl(A)\cap cl(A^C)$.

Remark 4.4 Bd(A) is closed set.

It is believed that in fuzzy topology for a fuzzy set A, $A\cup bd(A)\subseteq cl(A)$ but equality does not hold. Whereas in classical topology for an arbitrary set A of a topological space X, $A\cup bd(A)= cl(A)$. Many papers has been published in fuzzy boundary with remarks that in fuzzy topology $A\cup bd(A)\subseteq$ cl(A but converse does not hold. So in this work we would like to focus in this matter with example.

Now suppose X={a, b} and let A={(a, 0.3), (b, 0.6)} be a fuzzy set on X, then by existing definition of fuzzy set the collection $\tau=\{0_X, 1_{X,}\ A\}$ is fuzzy topology on X.

Let B={(a, 0.6), (b, 0.7)} be fuzzy set on X.

Then $cl(B)=1_X$ and $cl(B^C)=A^C$.

Now $bd(B)= cl(B)\cap cl(B^C)=A^C$.

Therefore $B\cup bd(B)= \{(a, 0.6), (b, 0.7)\} \cup A^C=\{(a, 0.7), (b, 0.7)\}$.

Hence $cl(B)\neq B\cup bd(B)$.

Now we use our definition of complement of fuzzy set on same problem.

Let X={a, b} and let A={(a, 0.3, 0), (b, 0.6, 0)} be a fuzzy set on X, then by existing definition of fuzzy set the collection τ={0_X, $1_{X,}$ A} is fuzzy topology on X, here 0_X={x, 0, 0} and 1_X={x, 1, 0}.

Let B={(a, 0.6, 0), (b, 0.7, 0)} be fuzzy set on X.

Then cl(B)=1_X and B^C={(a, 1, 0.6), (b, 1, 0.7)}. Also cl(B^C)=A^C.

Therefore bd(B)= cl(B)$\cap$cl(B^C)=A^C.

Now B$\cup$bd(B)={(a, 0.6, 0), (b, 0.7, 0)} $\cup A^C$

={(a, 0.6, 0), (b, 0.7, 0)} $\cup$ {(a, 1, 0.3), (b, 1, 0.6)}

={(a, 1, 0), (b, 1, 0)}

=1_X

Thus cl(B)=B$\cup$bd(B).

This would enable us to establish that cl(B)=B$\cup$bd(B) if we define complementation in our way.

Thus from this example we would enable to proposed the remark.

Remark: cl(B)=B$\cup$bd(B).

5.5.2 PROPOSITION Let A be any fuzzy set in fuzzy topological spaces X.

Then Bd(A) = Bd(A^C).

Proof:

$$Bd(A) = Cl(A) \cap Cl(A^C)$$

$$= Cl(A^C) \cap Cl(A)$$

$$= Cl(A^C) \cap (Cl(A^C))^C$$

$$=Bd(A^C).$$

Now let us try to prove this proposition with our extended definition of fuzzy set.

Suppose $A=\{x, \mu_i(x), 0; x\epsilon X \}$ then $A^C=\{x, 1, \mu_i(x); x\epsilon X \}$.

Therefore $Cl(A) =\cap\{x, \mu_i(x), 0; x\epsilon X \}=\{x, \min\{\mu_i(x)\}, 0; x\epsilon X \}$ and

$$Cl(A^C) =\cap\{x, 1, \mu_i(x); x\epsilon X \}=\{x, 1, \max\{\mu_i(x)\}, 0; x\epsilon X \}$$

So,

$$Bd(A) = Cl(A) \cap Cl(A^C)$$

$$=\{x, \min\{\mu_i(x)\}, 0; x\epsilon X \}\cap \{x, 1, \max\{\mu_i(x)\}, 0; x\epsilon X \}$$

$$=\{ x, 1, \max\{\mu_i(x)\}, 0; x\epsilon X \}\cap\{x, \min\{\mu_i(x)\}, 0; x\epsilon X \}$$

$$=\{x, 1, \max\{\mu_i(x)\}, 0; x\epsilon X \}\cap\{x, 1, \min\{\mu_i(x)\}; x\epsilon X \}^C$$

$$=\{x, 1, \max\{\mu_i(x)\}, 0; x\epsilon X \}\cap\{\{x, \min\{\mu_i(x)\}, 0; x\epsilon X \}^C\}^C$$

$$= Cl(A^C) \cap (Cl(A^C))^C$$

$$=Bd(A^C).$$

5.5.3 PROPOSITION: Let A be any fuzzy set in fuzzy topological spaces X. A is fuzzy closed set if and only if bd (A)$\subseteq$A.

Proof:

Let A be fuzzy closed set. So, cl(A) = A.

Bd(A) = cl(A) $\cap$ cl(A^C) $\subseteq$ cl(A) = A.

Therefore Bd(A) $\subseteq$A.

Conversely let Bd(A) $\subseteq$A.

Also A$\subseteq$ cl(A).

Now cl(A)=A$\cup$ bd(A)

$\subseteq$ A$\cup$ A

=A

$\Rightarrow$Cl(A) $\subseteq$ A

Therefore cl(A)=A.

Hence A is fuzzy closed set.

Note: If we expressed the fuzzy set and fuzzy closed set as existence definition of fuzzy set then the converse of the proposition is not true. But if we expressed with our new definition of fuzzy set then the converse part is also true. Let us cite an example.

Let $X=\{a, b, c\}$and $A=\{x, (a, 0.4), (b, 0.7), (c, 0.2)\}$ and $B=\{x, (a, 0.6), (b, 0.9), (c, 0.1)\}$ be fuzzy sets on X. Then $\tau=\{0_X, 1_X\ A, B, A\cup B, A\cap B\}$ be fuzzy topology in X.

Let $P=\{ x, (a, 0.6), (b, 0.7), (c, 0.9)\}$ be any fuzzy set in X.

Now $Cl(P)=\cap\{M \mid M$ is fuzzy closed set in X and $P\subseteq M \}$

$=1_X$

And $Cl(P^C)= \cap\{M \mid M$ is fuzzy closed set in X and $P^C\subseteq M \}$

$=A^C\cap(A\cap B)^C$

$=A^C$

$Bd(P)=Cl(P)\cap Cl(P^C)=1_X\cap A^C= A^C$

Hence $Bd(P) \subseteq P$. But P is not fuzzy closed set.

Now let us express the same problem in our definition with respect to reference function.

Let $X=\{a, b, c\}$and $A=\{x, (a, 0.4, 0), (b, 0.7, 0), (c, 0.2, 0)\}$ and $B=\{x, (a, 0.6, 0), (b, 0.9, 0), (c, 0.1, 0)\}$ be fuzzy sets on X. Then $\tau=\{0_X, 1_X\ A, B, A\cup B, A\cap B\}$ be fuzzy topology in X.

Let $P=\{ x, (a, 0.6, 0), (b, 0.7, 0), (c, 0.9, 0)\}$ be any fuzzy set in X.

Now $Cl(P)=\cap\{M \mid M$ is fuzzy closed set in X and $P\subseteq M \}$

$=1_X$

And $Cl(P^C) = \cap\{M \mid M$ is fuzzy closed set in X and $P^C \subseteq M\}$

$=A^C \cap (A \cap B)^C$

$=A^C$

$Bd(P)=Cl(P)\cap Cl(P^C)=1_X \cap A^C=A^C=\{x, (a, 1, 0.4), (b, 1, 0.7), (c, 1, 0.2)\}$.

So, $Bd(P)= \{x, (a, 1, 0.4), (b, 1, 0.7), (c, 1, 0.2)\}$ and $P = \{x, (a, 0.6, 0), (b, 0.7, 0), (c, 0.9, 0)\}$, which is not comparable. This is happened because P is not closed set.

Thus it is clearly seen that the converse part of the proposition is also true when we use our definition of fuzzy set with respect to reference function.

5.5.4 PROPOSITION

Let A be any fuzzy set in fuzzy topological spaces X. A is fuzzy open set if and only if

$bd(A) \subseteq A^C$.

Proof:

Let A be fuzzy open set.

So, A^C is fuzzy closed set and hence from earlier proposition $Bd(A^C) \subseteq A^C$. Since we have $Bd(A)=Bd(A^C)$, hence $Bd(A) \subseteq A^C$.

Conversely, let $Bd(A) \subseteq A^C$.

Now since $Bd(A) \subseteq A^C \Rightarrow Bd(A)\cap A=\phi$

$\Rightarrow (Cl(A^C) \cap Cl(A)) \cap A = \phi$

$\Rightarrow Cl(A^C) \cap (Cl(A) \cap A) = \phi$

$\Rightarrow Cl(A^C) \cap A = \phi$, since $A \subseteq Cl(A)$

$\Rightarrow A \subseteq (Cl(A^C))^C$

$\Rightarrow A \subseteq Int(A)$, since $(Cl(A^C))^C = Int(A)$.

But we have $Int(A) \subseteq A$. Hence $A = Int(A)$.

This shows that A is fuzzy open.

Note: If we expressed the fuzzy set and fuzzy closed set that is complement of fuzzy open set as existence definition of fuzzy set then the converse of the proposition is not true. But if we expressed with our new definition of fuzzy set then the converse part is also true. Let us cite an example.

Let X={a, b, c}and A={x, (a, 0.5), (b, 0.8), (c, 0.4)} and B={x, (a, 0.6), (b, 0.9), (c, 0.4)} and D={ x, (a, 0.5), (b, 0.7), (c, 0.3)} be fuzzy sets on X. Then $\tau=\{0_X, 1_X$ A, B, C, $A \cup B$, $A \cap B$, $A \cap D$, $A \cup D$, $D \cup B$, $D \cap B$ } be fuzzy topology in X.

Let E={ x, (a, 0.5), (b, 0.3), (c, 0.3)} be any fuzzy set in X.

$Cl(E)=D^C$ and $CL(E^C)=1_X$.

$Bd(E)=D^C \subseteq E^C$.

But D is not fuzzy open set.

Hence converse is not true when we used existing definition of complement of fuzzy set.

Now let us express the same problem in our definition with respect to reference function.

Let X={a, b, c}and A={x, (a, 0.5, 0), (b, 0.8, 0), (c, 0.4, 0)} and B={x, (a, 0.6, 0), (b, 0.9, 0), (c, 0.4, 0)} and D={ x, (a, 0.5, 0), (b, 0.7, 0), (c, 0.3, 0)} be fuzzy sets on X. Then $\tau=\{0_X,1_X$ A, B, C, A∪B, A∩B, A∩D, A∪D, D∪B, D∩B } be fuzzy topology in X.

Let E={x, (a, 0.5, 0), (b, 0.3, 0), (c, 0.3, 0)} be any fuzzy set in X.

$Cl(E)=1_X$ and $CL(E^C)=1_X$.

$Bd(E)=1_X \nsubseteq E^C$.

This is happened because E is not fuzzy open set.

To clarify the proposition more clearly let us try to give example.

Let X={a, b, c}and A={x, (a, 0.5, 0), (b, 0.8, 0), (c, 0.4, 0)} and B={x, (a, 0.6, 0), (b, 0.9, 0), (c, 0.4, 0)} and D={ x, (a, 0.5, 0), (b, 0.7, 0), (c, 0.3, 0)} be fuzzy sets on X. Then $\tau=\{0_X,1_X$ A, B, C, A∪B, A∩B, A∩D, A∪D, D∪B, D∩B } be fuzzy topology in X.

Let F={ x, (a, 0.5, 0), (b, 0.7, 0), (c, 0.3, 0)} so $Cl(F)=1_X$ and $Cl(F^C)=D^C$.

So, $Bd(F)=D^C \subseteq F^C$ and clearly it is seen that F is fuzzy open set because F is nothing but D.

Thus it is clearly seen that the converse part of the proposition is also true when we use our definition of fuzzy set with respect to reference function.

5.5.5 PROPOSITION: Let A be any fuzzy set in fuzzy topological spaces X.

Then $Bd(Cl(A)) \subseteq Bd(A)$.

Proof:

$Bd(Cl(A)) = Cl(Cl(A)) \cap Cl((Cl(A))^C)$

$= Cl(Cl(A)) - (Cl((Cl(A))^C))^C$

$= Cl(Cl(A)) - Int(Cl(A))$

$= Cl(A) - Int(Cl(A))$

$\subseteq Cl(A) - Int(A)$

$= Bd(A)$.

Example:

Let X={a, b, c}and A={x, (a, 0.4, 0), (b, 0.7, 0), (c, 0.2, 0)} and B={x, (a, 0.6, 0), (b, 0.9, 0), (c, 0.1, 0)} be fuzzy sets on X. Then $\tau=\{0_X, 1_X\ A, B, A\cup B, A\cap B\}$ be fuzzy topology in X.

Let Q={ x, (a, 0.6, 0), (b, 0.7, 0), (c, 0.8, 0)} be any fuzzy set in X.

$Cl(Q)=1_X$ and $Bd(Cl(Q))=Bd(1_X)=Cl(1_X) \cap Cl((1_X)^C)=0_X$.

Now $Cl(Q)=1_X$ and $Cl(Q^C)=A^C \cap (A\cap B)^C$

$Bd(Q)= 1_X \cap A^C \cap (A\cap B)^C = A^C \cap (A\cap B)^C$

Hence from this example $Bd(Cl(Q)) \subseteq Bd(Q)$.

5.5.6 PROPOSITION

Let A and B be fuzzy sets in fuzzy topological space X.

Then $Bd(A\cup B) \subseteq Bd(A) \cup Bd(B)$.

Proof:

$Bd(A \cup B) = Cl(A \cup B) \cap Cl((A \cup B)^C)$

$\subseteq Cl(A) \cup Cl(B) \cap (Cl(A^C) \cap Cl(B^C))$

$= (Cl(A) \cap (Cl(A^C) \cap Cl(B^C))) \cup (Cl(B) \cap (Cl(A^C) \cap Cl(B^C)))$

$= (Bd(A)) \cap Cl(B^C)) \cup (Bd(B) \cap Cl(A^C))$

$\subseteq Bd(A) \cup Bd(B)$.

Example:

Let X={a, b, c}and A={x, (a, 0.4, 0), (b, 0.7, 0), (c, 0.2, 0)} and B={x, (a, 0.6, 0), (b, 0.9, 0), (c, 0.1, 0)} be fuzzy sets on X. Then $\tau=\{0_X, 1_X$ A, B, A∪B, A∩B} be fuzzy topology in X.

Let U={x, (a, 0.3, 0), (b, 0.2, 0), (c, 0.1, 0)} and V={x, (a, 0.7, 0), (b, 0.7, 0), (c, 0.8, 0)}

U∪V={x, (a, 0.7, 0), (b, 0.7, 0), (c, 0.8, 0)}

$Cl(U \cup V)=1_X$ and $Cl((U \cup V)^C)=A^C$ so $Bd(U \cup V)=A^C$

Also $Cl(U)=1_X$, $Cl(U^C)=1_X$ and $Cl(V)=1_X$, $Cl(v^C)=A^C$

$Bd(U)= 1_X$ and $Bd(V)=A^C$.

Now $Bd(U) \cup Bd(V)=1_X$.

Hence $Bd(U \cup V) \subseteq Bd(U) \cup Bd(V)$.

5.5.7 PROPOSITION: Let A and B be fuzzy sets in fuzzy topological space X.

Then $Bd(A\cap B) \subseteq Bd(A) \cup Bd(B)$.

Proof:

$Bd(A\cap B) = Cl(A\cap B)\cap Cl((A\cap B)^C)$

$\subseteq [Cl(A)\cup Cl(B)]\cap(Cl(A^C)\cup Cl(B^C))$

$= [(Cl(A)\cup Cl(B))\cap Cl(A^C)]\cup[(Cl(A)\cup Cl(B))\cap(Cl(B^C)]$

$= [Cl(A)\cup Cl(B)\cap Cl(A^C)]\cup[Cl(A)\cup Cl(B)\cap(Cl(B^C)]$

$= [(Bd(A)\cup Cl(B))]\cup[(Cl(A)\cup Bd(B))]$

$\subseteq Bd(A) \cup Bd(B)$.

Example:

Let X={a, b, c}and A={x, (a, 0.4, 0), (b, 0.7, 0), (c, 0.2, 0)} and B={x, (a, 0.6, 0), (b, 0.9, 0), (c, 0.1, 0)} be fuzzy sets on X. Then $\tau=\{0_X,1_X$ A, B, $A\cup B$, $A\cap B\}$ be fuzzy topology in X.

Let U={x, (a, 0.3, 0), (b, 0.2, 0), (c, 0.1, 0)} and V={x, (a, 0.7, 0), (b, 0.7, 0), (c, 0.8, 0)}

$U\cup V$={x, (a, 0.7, 0), (b, 0.7, 0), (c, 0.8, 0)}

$Cl(U\cup V)=1_X$ and $Cl((U\cup V)^C)=A^C$ so $Bd(U\cup V)=A^C$.

5.5.8 PROPOSITION: Let A and B be fuzzy sets in fuzzy topological space X.

Then $Bd(Bd(A)) \subseteq Bd(A)$.

Proof:

$Bd(Bd(A)) = Cl(Bd(A)) \cap Cl((Bd(A))^C)$

$\subseteq Cl(Bd(A))$

$= Bd(A)$.

Example:

Let X={a, b, c}and A={x, (a, 0.4, 0), (b, 0.7, 0), (c, 0.2, 0)} and B={x, (a, 0.6, 0), (b, 0.9, 0), (c, 0.1, 0)} be fuzzy sets on X. Then $\tau=\{0_X, 1_X\ A, B, A\cup B, A\cap B\}$ be fuzzy topology in X.

Let D={x, (a, 0.6, 0), (b, 0.6, 0), (c, 0.7, 0)}.

Then $Cl(D)=1_X$ and $Cl(D^C)=1_X$ so, $Bd(D)=1_X$

Therefore $Bd(Bd(D))=Cl(1_X)\cap Cl(0_X)=0_X$.

Hence $Bd(Bd(A)) \subseteq Bd(A)$.

5.6 CONCLUSION

In this Chapter, we have commented on the fuzzy boundary of fuzzy set. Here, we have given definition of fuzzy set on the basis of reference function. Also, some propositions of fuzzy boundary are prove on the basis of reference function. This Chapter also discuss that existing method on expression of fuzzy boundary of fuzzy sets has some drawbacks that is why some propositions on fuzzy boundary are not same as the propositions of classical boundary of crisp set. To stimulate future impact of the introduce concept, we believe that more efforts are needed to find a proper representation of complement of fuzzy set as well as defining fuzzy boundary. How to find the fuzzy boundary of a fuzzy set properly so that we could get perfect boundary still remained to be worked out. Thus the recommendation for future work includes the development of a procedure to extend the concept of complementation of fuzzy sets on the basis of reference function in defining fuzzy boundary of a fuzzy set.