CHAPTER 6

FUZZY (τ_i, τ_j)-r-BOUNDARY OF FUZZY BITOPOLOGICAL SPACE ON THE BASIS OF REFERENCE FUNCTION

6.1 INTRODUCTION

Kely [101] defined the study of Bitopological spaces and then in 1983 Kubiak [86] introduced the bitopological aspects in the theory of fuzzy topological spaces, particularly concerned with fuzzy bitopological spaces related to fuzzy quasi-proximity spaces. He also established that such fuzzy bitopological spaces admits a fuzzy quasi proximity if it is pathwise fuzzy completely regular. On the other hand, in 1989 Kandil [87] introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological spaces and an extension of bitopological spaces [88] in fuzzy setting. Thereafter many fuzzy topologists have introduced and studied several concepts like fuzzy quasi-proximities [89], connectedness and semiconnectedness [90], pair wise fuzzy continuity [92], pairwise fuzzy connectedness between fuzzy sets [90] in fuzzy bitopological spaces using quasi-coincidence for fuzzy sets.

The results discussed in this chapter have published in UGC listed Journal and presented in national seminar

1. Basumatary B.,"A note on Fuzzy Boundary of Fuzzy Bitopological Spaces on the Basis of Reference Function", *Advances in Fuzzy Mathematics*,Vol-12, No-3, pp639-644.
2. Basumatary B, "A Study of Fuzzy Boundary on the Basis of Fuzzy Complement", national seminar presented held at Cotton College State University on 21-22 Oct. 2016.

Also notions like, fuzzy pairwise a-continuity (precontinuity) using fuzzy α-open sets (preopen sets) [91], pairwise fuzzy irresolute mappings [93] and semiopen sets, semicontinuity and semiopen mappings in fuzzy bitopological spaces [95] were defined and their properties were also investigated. Based on the concepts like fuzzy semiopen sets [96], fuzzy β-open sets [97], fuzzy regular open sets [98], generalized fuzzy open [99] (closed) sets, fuzzy g-locally closed sets[100] in fuzzy bitopological spaces and the results related to them, we generalize and extend some of them in both fuzzy topological spaces and fuzzy bitopological spaces.

6.2 PRELIMINARIES

6.2.1 Fuzzy Bitopological Space

A system (X, τ_i , τ_j) consisting of a set X with two fuzzy topological spaces τ_i and τ_j on X is called fuzzy bitopological space. Throughout this thesis indices i, j takes values in {1,2}

with $i \neq j$.

Let (X, τ_i , τ_j) be fuzzy bitopological spaces where $i, j \in \{1, 2\}$. We denote the closure of a fuzzy set A with respect to the fuzzy topologies τ_i , τ_j in a fuzzy bitopological spaces (X, τ_i , τ_j) by $\tau_i - cl(A)$.

Let (X, τ_i , τ_j) be a fuzzy bitopological space. A fuzzy set A in a X is called fuzzy (τ_i , τ_j)-generalized closed set if $\tau_j - cl(A) \leq U$ whenever $A \leq U$ and U is τ_i -fuzzy open set.

6.2.2 Fuzzy (τ_i, τ_j)-boundary

Let A be a fuzzy subset of a fuzzy bitopological space (X, τ_i, τ_j). Then the fuzzy

(τ_i, τ_j)-boundary of A is defined by $(\tau_i, \tau_j)\text{–}Bd(A)= \tau_i\text{-}cl(\tau_j\text{-}cl(A))\cap\tau_i\text{-}cl(\tau_j\text{-}cl(A^C))$.

6.3 NEW APPROACH TO FUZZY CLOSED SET OF FUZZY BITOPOLOGY ON THE BASIS OF REFERENCE FUNCTION

6.3.1 Fuzzy (τ_i, τ_j)-r-closed set

Let (X, τ_i, τ_j) be a fuzzy bitopological space and A be a fuzzy subset of X. Then A is said to be fuzzy (τ_i, τ_j)-r-closed set with reference function (in brief fuzzy (τ_i, τ_j)-r-closed set) if τ_j-r-cl(A)⊆U, when A⊆U, U∈τ_i.

6.3.2 Proposition If A be a fuzzy τ_j-r-closed set of (X, τ_i, τ_j) then it is fuzzy (τ_i, τ_j)-r-closed set.

Proof:

Let G∈τ_i such that A⊆G.

Now since A is τ_j-r-closed set so τ_j-r-cl(A)=A

Therefore τ_j-r-cl(A)⊆G.

Hence by the definition of fuzzy (τ_i, τ_j)-r-closed set we have A is fuzzy (τ_i, τ_j)-r-closed set.

Remarks: The converse is not true.

Let X={a, b, c}, fuzzy sets A ={(a, 1, 0), (b, 1, 0), (c, 0, 0)}, B={(a, 1, 0), (b, 0, 0), (c, 0, 0)} and C = {(a, 0, 0), (b, 1, 0), (c, 0, 0)}.

Consider τ_1= {0, 1, A} and τ_2 = {0, 1, B}. Then (X, τ_1, τ_2) is fuzzy bitopological space. Then the set C is fuzzy (τ_2, τ_1)-r-closed set. But C is not τ_1-r-closed fuzzy set.

6.3.3 Proposition: If A and B are fuzzy (τ_i, τ_j)-r-closed sets then A∪B is also fuzzy (τ_i, τ_j)-r-closed sets.

6.3.4 Proposition: Finite union of fuzzy (τ_i, τ_j)-r-closed sets is fuzzy (τ_i, τ_j)-r-closed set.

The propositions **6.3.3** and **6.3.4** are also true when we use our definition of fuzzy set on the basis of reference function.

6.3.5 Proposition: Intersection of two fuzzy (τ_i, τ_j)-r-closed sets need not be a fuzzy (τ_i, τ_j)-r-closed set.

Let us cite an example

Let X={a, b, c}, fuzzy sets A, B and C be defined as follows: A={(a, 1, 0), (b, 1, 0), (c, 0, 0)}, B={(a, 1, 0), (b, 0, 0), (c, 0, 0)} and C={(a, 1, 0), (b, 0, 0), (c, 1, 0)}. Consider

τ_1 = {0, 1, A} and τ_2 = {0, 1, C}.Then (X, τ_1, τ_2) is a fuzzy bitopological space. Then the sets A and C are fuzzy (τ_2, τ_1)-r-closed sets, but $A \cap C = B$ is not a fuzzy (τ_2, τ_1)-r-closed set.

6.3.6 Proposition: If A be a fuzzy (τ_i, τ_j)-r-closed set and B is fuzzy τ_j-r-closed set then A∪B is fuzzy (τ_i, τ_j)-r-closed set.

6.4 Fuzzy (τ_i, τ_j)-r-boundary

Let A be a fuzzy subset of a fuzzy bitopological space (X, τ_i, τ_j). Then the fuzzy (τ_i, τ_j)-r-boundary of A is defined by (τ_i, τ_j)–r-Bd(A)= τ_i-r-cl(τ_j-r-cl(A))∩τ_i-r-cl(τ_j-r-cl(A^C)).

6.4.1 Proposition: Let A be any fuzzy set in fuzzy bitopological space (X, τ_i, τ_j).
Then (τ_i, τ_j)-r-Bd(A) = (τ_i, τ_j)-r-Bd(A^C).

Proof:

Let A be a fuzzy set of a fuzzy bitopological space (X, τ_i, τ_j).

Let A={x, μ_i(x), 0; xϵX }

(τ_i, τ_j) –r- Bd(A)= [τ_i–r-Cl(τ_j –r- Cl(A))]∩[τ_i–r- Cl(τ_j –r- Cl(A^C))]

= τ_i–r- Cl [{x, $\min_{\tau_j}$ {μ_i(x)}, 0; xϵX }]∩ τ_i– r-Cl [{x, 1, $\min_{\tau_j}${μ_i(x)}, 0; xϵX }]

=[{x, $\min_{\tau_i}$ $\min_{\tau_j}$ {μ_i(x)}, 0; xϵX }]∩ [{x, 1, $\max_{\tau_i}$ { $\max_{\tau_j}${μ_i(x)}, 0; xϵX }]

=[{x, 1, $\max_{\tau_i}$ $\max_{\tau_j}${μ_i(x)}, 0; xϵX }]∩[{x, $\min_{\tau_i}$ $\min_{\tau_j}$ {μ_i(x)}, 0; xϵX }]

=[{x, 1, $\max_{\tau_i}$ $\max_{\tau_j}${μ_i(x)}, 0; xϵX }]∩[{x, $\min_{\tau_i}$ $\min_{\tau_j}$ {μ_i(x)}, 0; xϵX $\}^C]^C$

=[τ_i–r-Cl(τ_j –r-Cl(A^C))]∩[τ_i– r-Cl(τ_j –r- Cl(A^C))$]^C$

=(τ_i, τ_j) –r- Bd(A^C).

6.4.2 Proposition: Let A be any fuzzy set in fuzzy bitopological space (X, τ_i, τ_j). A is fuzzy closed set if and only if (τ_i, τ_j) –r- Bd (A) $\subseteq$A.

The prove is straightforward.

Generally the converse part of the proposition is not true when we use the existing definition of fuzzy set. For this we cite an example.

Let X={a, b, c}and A={x, (a, 0.4), (b, 0.7), (c, 0.2)}, B={x, (a, 0.6), (b, 0.9), (c, 0.1)}, C={x, (a, 0.4), (b, 0.7), (c, 0.3)} and D={x, (a, 0.6), (b, 0.9), (c, 0.2)} be fuzzy sets on X. Then τ_1={0_X,1_X A, B, A⋃B, A∩B} and τ_2={0_X,1_X ,C, D, C⋃D, C∩D } be fuzzy topologies in X. Let P={ x, (a, 0.6), (b, 0.7), (c, 0.9)} be any fuzzy set in X and P^C={ x, (a, 0.4), (b, 0.3), (c, 0.1)}.

Now τ_j –r-Cl(P))=1_X and τ_i –r- Cl(1_X))= 1_X.

τ_j –r-Cl(P^C))= { x, (a, 0.4), (b, 0.3), (c, 0.8)} and

τ_i –r-Cl({ x, (a, 0.4), (b, 0.3), (c, 0.8)})= { x, (a, 0.6), (b, 0.3), (c, 0.9)}

Now (τ_i, τ_j) –r- bd (A)= $1_X \cap$ { x, (a, 0.6), (b, 0.3), (c, 0.9)}= { x, (a, 0.6), (b, 0.3), (c, 0.9)}.

Therefore (τ_i, τ_j)–r-Bd (P) $\subseteq$ P. But P is not fuzzy closed set.

Now if we express the same problem in our definition with respect to reference function

Let X={a, b, c}and A={x, (a, 0.4, 0), (b, 0.7, 0), (c, 0.2, 0)}, B={x, (a, 0.6, 0), (b, 0.9, 0), (c, 0.1, 0)}, C={x, (a, 0.4, 0), (b, 0.7, 0), (c, 0.3, 0)} and D={x, (a, 0.6, 0), (b, 0.9, 0), (c, 0.2, 0)} be fuzzy sets on X. Then τ_1={0_X,1_X A, B, A$\cup$B, A$\cap$B} and τ_2={0_X,1_X C, D, C$\cup$D, C$\cap$D } be fuzzy topologies in X. Let P={ x, (a, 0.6, 0), (b, 0.7, 0), (c, 0.9, 0)} be any fuzzy sets in X and P^C={ x, (a, 1, 0.4), (b, 1, 0.3), (c, 1, 0.1)}.

Then τ={0_X,1_X A, B, A$\cup$B, A$\cap$B} be fuzzy topology in X.

Let P={ x, (a, 0.6, 0), (b, 0.7, 0), (c, 0.9, 0)} be any fuzzy set in X and P^C ={ x, (a, 1, 0.6), (b, 1, 0.7), (c, 1, 0.9)}.

Now τ_j –r- Cl(P))=1_X and τ_i –r- Cl(1_X))= 1_X.

τ_j –r- Cl(P^C))={ x, (a, 1, 0.4), (b, 1, 0.7), (c, 1, 0.2)}.

τ_i –r- Cl{x, (a, 1, 0.4), (b, 1, 0.6), (c, 1, 0.2)}= { x, (a, 1, 0.4), (b, 1, 0.7), (c, 1, 0.1)}.

Now (τ_i, τ_j) –r-Bd (A)= $1_X \cap$ { x, (a, 1, 0.4), (b, 1, 0.7), (c, 1, 0.1)}

= { x, (a, 1, 0.4), (b, 1, 0.7), (c, 1, 0.1)}.

Now we have (τ_i, τ_j) –r- Bd (A)={ x, (a, 1, 0.4), (b, 1, 0.7), (c, 1, 0.1)} and P={ x, (a, 0.6, 0), (b, 0.7, 0), (c, 0.9, 0)} which is not comparable. This is happened because P is not closed set.

Thus it is clearly seen that the converse part of the proposition is also true when we use our definition of fuzzy set with respect to reference function.

6.4.3 Proposition: Let A be any fuzzy set in fuzzy bitopological space (X, τ_i, τ_j). A is fuzzy open set if and only if (τ_i, τ_j)-r-Bd(A) $\subseteq A^C$.

Prove is straightforward.

6.5 Conclusion

The main purpose of this chapter was to represent fuzzy closed set and fuzzy boundary of fuzzy bitopological space on the basis of reference function. We defined the definition of fuzzy(τ_i, τ_j)-r-closed set on the basis of reference and some propositions on fuzzy (τ_i, τ_j)-r-closed set are also discussed. Also, in this chapter we discussed on definition of fuzzy (τ_i, τ_j)-r-Boundary and some of their propositions on the basis of reference function. It is seen that all the propositions on fuzzy closed sets and fuzzy Boundary of fuzzy bitopological space can be explained very interestingly with the help of our new definition of fuzzy set on the basis of reference function.