A NOTE ON FUZZY CLOSURE OF A FUZZY SET

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Abstract: In this article fuzzy closure has been discussed with the help of extended definition of fuzzy set on the assumption that the union of a fuzzy set and its complement is universal set and intersection of a fuzzy set and its complement is empty set. Also we have discussed some proposition of fuzzy closure with the help of numerical example on the basis of extended definition of fuzzy set.

Keywords: *Fuzzy membership function, Fuzzy reference function, Fuzzy membership value, Fuzzy closed set, Fuzzy closure.*

1. Introduction

Fuzzy set theory was discovered by Zadeh [1] in 1965. Chang [2] introduced fuzzy topology. After the introduction of fuzzy sets and fuzzy topology, several researches were conducted on the generalizations of the notions of fuzzy sets and fuzzy topology. The theory of fuzzy sets actually has been a generalization of the classical theory of sets in the sense that the theory of sets should have been a special case of the theory of fuzzy sets. But unfortunately it has been accepted that for fuzzy set A and its complement A^C, neither $A \cap A^C$ is empty nor $A \cup A^C$ is the universal set. Whereas the operations of union and intersection of crisp sets are indeed special cases of the corresponding operation of two fuzzy sets, they end up giving peculiar results while defining $A\cap A^C$ and $A\cup A^C$. In this regard Baruah [3, 4, 5,] has forwarded an extended definition of fuzzy sets which enable us to define complement of fuzzy sets in a way that give us $A\cap A^C$ is empty and $A\cup A^C$ is universal set

2. Extended definition of fuzzy set

Baruah [3, 4, 5,] gave an extended definition of complementation of fuzzy sets. According to Baruah [3, 4, 5] to define a fuzzy set, two functions namely fuzzy membership function and fuzzy reference function are necessary. Fuzzy membership value is the difference between fuzzy membership function and fuzzy reference function.

Let $\mu_1(x)$ and $\mu_2(x)$ be two functions such that $0\leq \mu_2(x) \leq \mu_1(x) \leq 1$. For fuzzy number denoted by $\{x, \mu_1(x), \mu_2(x); x\in U\}$, we call $\mu_1(x)$ as fuzzy membership function and $\mu_2(x)$ a reference function such that $(\mu_1(x) - \mu_2(x))$ is the fuzzy membership value.

Let $A = \{x, \mu_1(x), \mu_2(x); x\in U\}$ and $B =\{x, \mu_3(x), \mu_4(x); x\in U\}$ be two fuzzy sets.

Then we would have according to our way:

$A\cap B= \{x, \min(\mu_1(x), \mu_3(x)), \max(\mu_2(x), \mu_4(x)); x\in U\}$ and

$A\cup B = \{x, \max(\mu_1(x), \mu_3(x)), \min(\mu_2(x), \mu_4(x)); x\in U\}$.

Two fuzzy sets $C= \{x, \mu_C(x); x\in U\}$ and $D= \{x, \mu_D(x); x\in U\}$ in the usual definition would be expressed as

C= {x, $\mu_C(x)$, 0; xϵ U } and D= {x, $\mu_D(x)$, 0; xϵ U }.

Now for two fuzzy sets A={x, μ(x), 0; xϵX } and

B={x, 1, μ(x); xϵX} are defined over the same universe X. Then we would have

A∩B= {x, min (μ(x), 1) , max (μ(x), μ(x) ; xϵ U}

= {x, μ(x), μ(x) ; xϵ U},

which is nothing but the null set. In other words, B defined above is nothing but A^C complement in the classical sense of the set theory. That is if we define fuzzy sets A^C ={x, 1, μ(x); xϵX}, it can be seen that it is nothing but the complement of the fuzzy sets A={x, μ(x), 0 ; xϵX } .

Also we have

A ⋃ B ={x, max (μ(x), 1) , min (0, μ(x)); xϵU}

={x, 1, 0; xϵU}, which is nothing but the universal set.

We therefore conclude that if we express the complement of a fuzzy set A={x, μ(x), 0; xϵX} as

A^C ={x, 1, μ(x); xϵX}, then we get

1. $A \cap A^C$ = the null set ϕ , and
2. $A \cup A^C$ = the universal set X.

This would enable us to establish that the fuzzy sets do form a field if we define complementation in our way.

3. Basic operations

Let A={x, $\mu_1(x)$, $\mu_2(x)$; xϵU} and B={x, $\mu_3(x)$, $\mu_4(x)$; xϵU} be two fuzzy sets defined over the same universe U.

i. $A \subseteq B$ iff $\mu_1(x) \le \mu_3(x)$ and $\mu_4(x) \le \mu_2(x)$ for all xϵU.

ii. A ⋃ B ={x, max($\mu_1(x)$, $\mu_3(x)$), min($\mu_2(x)$, $\mu_4(x)$)} for all xϵU.

iii. A ⋂ B ={x, min($\mu_1(x)$, $\mu_3(x)$), max($\mu_2(x)$, $\mu_4(x)$)} for all xϵU.

If for some xϵU, min($\mu_1(x)$, $\mu_3(x)$) ≤ max($\mu_2(x)$, $\mu_4(x)$)}, then our conclusion will be $A \cap B = \phi$

iv. A^C={x, $\mu_1(x)$, $\mu_2(x)$; xϵU}^{C}

={x, $\mu_2(x)$, 0; xϵU} ⋃ {x, 1, $\mu_1(x)$; xϵU}

v. If D = {x, μ(x), 0; xϵU} then D^C={x, 1, μ(x); xϵU} for all xϵU.

Proposition 1: For fuzzy sets A, B, C over the same universe X, we have the following proposition

i. $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$
ii. $A \cap B \subseteq A, A \cap B \subseteq B$
iii. $A \subseteq A \cup B, B \subseteq A \cup B$
iv. $A \subseteq B \Rightarrow A \cap B = A$
v. $A \subseteq B \Rightarrow A \cup B = B$

Proposition 2:

Let τ= {A_r : rϵI} be a collection of fuzzy sets over the same universe U. Then

i. $\bigcup_i A_i$={x, max(μ_{1i}), min(μ_{2i}); xϵU}

ii. $\bigcap_i A_i$={x, min(μ_{1i}), max(μ_{2i}); xϵU}

iii. $\{\bigcup_i A_i\}^C = \bigcap_i A_i^C$

iv. $\{\bigcap_i A_i\}^C = \bigcup_i \{A_i\}^C$

4. Fuzzy Topology

Definition: A fuzzy topology on a nonempty set X is a family τ of fuzzy set in X satisfying the following axioms

(T1) $0_X, 1_X \in \tau$
(T2) $G_1 \cap G_2 \in \tau$, for any $G_1, G_2 \in \tau$
(T3) $\bigcup G_i \in \tau$, for any arbitrary family $\{G_i : G_i \in \tau, i \in I\}$.

In this case the pair (X,τ) is called a fuzzy topological space and any fuzzy set in τ is known as fuzzy open set in X and clearly every element of τ^C is said to be closed set.
Here $1_x=\{x, 1, 0; x \epsilon X\}$ and $0_X=\{x, \mu(x), \mu(x)\}$, where μ(x) is taken from reference function compared to relative fuzzy set.

5. **Closure of fuzzy set**: Let (X, τ) be fuzzy topology and $A=\{x, \mu(x), \gamma(x); x \epsilon X\}$ be fuzzy set in X. Then fuzzy closure of A are defined by
$Cl(A)=\bigcap\{G$: G is fuzzy closed set in X and $A \subseteq G\}$

Example:
Let X= {a, b} and
A= { {a, 0.2, 0}, {b, 0.1, 0}}
B= {{a, 0.4, 0}, {b, 0.3, 0}}.
Then the family $\tau=\{1_X, 0_X, A, B\}$ of fuzzy set in X is fuzzy topology on X.
Let C={{a, 0.5, 0.4}, {b, 0.5, 0.3}}, where in fuzzy set C the membership value of a is 0.1 and the membership value of b is 0.2.
Then G_1={{a, 1, 0.2}, {b, 1, 0.1}} and G_2= {{a, 1, 0.4}, {b, 1, 0.3}} are closed sets.
Therefore $cl(C)=G_1 \cap G_2$={{a, 1, 0.4}, {b, 1, 0.3}}.

Proposition 3: Let (X, τ) be fuzzy topology and C be fuzzy set in X, then $cl(0_X)=0_X$.

Example :
Let X= {a, b} and
A= { {a, 0.2, 0}, {b, 0.1, 0}}
B= {{a, 0.4, 0}, {b, 0.3, 0}}.
Then the family $\tau=\{1_X, 0_X, A, B\}$ of fuzzy set in X is fuzzy topology on X.
Clearly we can show $cl(0_X)=0_X$.

Proposition 4: Let (X, τ) be fuzzy topology and C be fuzzy set in X, then $C \subseteq cl(C)$.

Example : Let X= {a, b} and
A= { {a, 0.2, 0}, {b, 0.1, 0}}
B= {{a, 0.4, 0}, {b, 0.3, 0}}.
Then the family $\tau=\{1_X, 0_X, A, B\}$ of fuzzy set in X is fuzzy topology on X.
Let C={{a, 0.5, 0.4}, {b, 0.5, 0.3}}, where in fuzzy set C the membership value of a is 0.1 and the membership value of b is 0.2.
Then G_1={{a, 1, 0.2}, {b, 1, 0.1}} and G_2= {{a, 1, 0.4}, {b, 1, 0.3}} are closed sets.
Therefore $cl(C)=G_1 \cap G_2$={{a, 1, 0.4}, {b, 1, 0.3}}.
From this example it is clear $C \subseteq cl(C)$.

Proposition 5. Let (X, τ) be fuzzy topology and C, D be any two fuzzy sets in X, then $Cl(C \cup D)= Cl(C) \cup Cl(D)$. Proposition3. Let (X, τ) be fuzzy topology and C, D be any two fuzzy sets in X, then
if $C \subseteq D \Rightarrow cl(C) \subseteq cl(D)$.

Example:
Let X= {a, b} and
A= { {a, 0.2, 0}, {b, 0.1, 0}}
B= {{a, 0.4, 0}, {b, 0.3, 0}}.
Then the family $\tau=\{1_X, 0_X, A, B\}$ of fuzzy set in X is fuzzy topology on X.
Let C={{a, 0.5, 0.4}, {b, 0.5, 0.3}}, where in fuzzy set C the membership value of "a" is 0.1 and the membership value of "b" is 0.2.
D={{a, 0.6, 0.3}, {b, 0.5, 0.2}}, where in fuzzy set C the membership value of "a" is 0.3 and the membership value of "b" is 0.3.
Clearly from our definition of fuzzy set $C \subseteq D$.

Now cl(C)= {{a, 1, 0.4}, {b, 1, 0.3}} and cl(D)= {{a, 1, 0.2}, {b, 1, 0.1}}, which shows that cl (C) $\subseteq$ cl (D).
Hence C $\subseteq$ D $\Rightarrow$ cl (C) $\subseteq$ cl (D).

Proposition 6. Let (X, τ) be fuzzy topology and C be any fuzzy sets in X, then cl(cl(C))=cl(C).

Example:
Let X= {a, b} and
A= { {a, 0.2, 0}, {b, 0.1, 0}}
B= {{a, 0.4, 0}, {b, 0.3, 0}}.
Then the family τ={1_X, 0_X, A, B} of fuzzy set in X is fuzzy topology on X.
Let C={{a, 0.5, 0.4}, {b, 0.5, 0.3}}, where in fuzzy set C the membership value of a is 0.1 and the membership value of b is 0.2.
Then G_1={{a, 1, 0.2}, {b, 1, 0.1}} and G_2= {{a, 1, 0.4}, {b, 1, 0.3}} are closed sets.
Therefore cl(C)=$G_1 \cap G_2$={{a, 1, 0.4}, {b, 1, 0.3}}.
Now if we consider cl(C)=D, then cl(D)={{a, 1, 0.4}, {b, 1, 0.3} }=cl(C).
Hence cl(cl(C))=cl(C).

Proposition 7 . Let (X,τ) be fuzzy topology and C, D be any two fuzzy sets in X, then Cl(C $\cup$ D)= Cl(C) $\cup$ Cl(D).

Example:
Let X= {a, b} and
A= { {a, 0.2, 0}, {b, 0.1, 0}}
B= {{a, 0.4, 0}, {b, 0.3, 0}}.
Then the family τ={1_X, 0_X, A, B} of fuzzy set in X is fuzzy topology on X.
Let C={{a, 0.5, 0.4}, {b, 0.5, 0.3}}, where in fuzzy set C the membership value of "a" is 0.1 and the membership value of "b" is 0.2.
D={{a, 0.6, 0.3}, {b, 0.5, 0.2}}, where in fuzzy set C the membership value of "a" is 0.3 and the membership value of "b" is 0.3.

Now cl(C)= {{a, 1, 0.4}, {b, 1, 0.3}} and cl(D)= {{a, 1, 0.2}, {b, 1, 0.1}}.
Also C $\cup$ D= {{a, 0.5, 0.4}, {b, 0.5, 0.3}} $\cup$ {{a, 0.6, 0.3}, {b, 0.5, 0.2}}
={{a, 0.6, 0.3}, {b, 0.5, 0.2}}
So Cl(C $\cup$ D)= {{a, 1, 0.2}, {b, 1, 0.1}}.

Cl(C) $\cup$ Cl(D)= {{a, 1, 0.4}, {b, 1, 0.3}} $\cup$ {{a, 1, 0.2}, {b, 1, 0.1}}.
={{a, 1, 0.2}, {b, 1, 0.1}}.
Hence Cl(C $\cup$ D)= Cl(C) $\cup$ Cl(D).

6. Conclusions

We have seen that, in article [5] if a fuzzy set is characterized with respect to a extended definition of fuzzy set, we can define the complement of a fuzzy set in its actual perspective. As for the definition using a reference function, we are clear that the original definition is sufficient for fuzzy arithmetic. However, from original definition $A \cap A^C = \phi$ does not follow. From Baruah's [3, 4, 5] definition of complement of a fuzzy set we can remove this difficulty. Similar problem we have faced in fuzzy topology too. In this article we have seen that we can define fuzzy closure and its propositions very nicely. Also we have given numerical examples on the basis of extended definition of fuzzy set. Hope our work will give some contribution in further study of fuzzy topology.

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