Chapter 9

Could the Lyra Manifold be the Hidden Source of the Dark Energy ?

5

9.1   Introduction:

Many researchers and Scientists are putting huge effort to explain the dynamics of the universe and to understand the future evolution of the universe from the ancient times. However, till today we are not in a state to provide an exactly clear statement about the origin and evolution of our universe. From different literatures and philosophical point of views we found that different mines provide different opinions about our universe. The universe is full of mysterious elements and numerous effects of interactions which cannot be detected and difficult to explain even with advanced technology. Due to these reasons the most challenging problems in Astrophysics and modem cosmology is to understand the late time acceleration of the universe. In recent years most of researchers have drawn considerable attention in the context of dark energy and modified theories of gravity to study the various aspects of the universe. On the other hand many prominent results of the cosmological observations like Type SNeIa supernovae [Riess et al. (1988, 2004); Perlmutter et al. (1999); Amanullah et al. (2010); Astier et al. (2006); Suzuki et al. (2012)], CMB (Cosmic Microwave Background) anisotropies [Bennett et al. (2003); Spergel et al. (2003)], the large scale galaxies structures of universe [Daniel et al. (2008); Allen et al. (2004)], Baryon Acoustic Oscillations [Anderson et al. (2013); Eisenstein et al. (2005)], WMAP (Wilkinson Microwave Anisotropy Probe) [Komatsu et al. (2009); Hinshaw et al. (2013)] Sachs-Wolfe effects [Nishizawa (2014)] and

SDSS [Seljak et al. (2005); Tegmark et al. (2004); Abazajian et al. (2004); McCarthy et al. (2008)] are noticed to us for the cosmic acceleration with direct and indirect evidence. Not only the above mentioned observations and surveys, but some other new cosmological results and data sets like Planck [Ade et al. (2014, 2016)], Atacama Cosmology Telescope (ACT) [Sievers et al. (2013)], South Pole Telescope Sunyaev-Zel'dovice (SPT-SZ) survey [Story et al. (2013)] have measured the temperature and polarization of CMB to exquisite precision, are also supporting this fact. These results are more acceptable to the community and beneficial to the researchers to understand about the universe in this modern era.

From the recent literatures and findings we know that dark energy dominates the universe with positive energy density and negative pressure, responsible to produce sufficient acceleration in late time evolution of the Universe. Some of the important claimants of dark energy are tachyons [Padmanabhan and Choudhury (2002)], chaplygin gas [Elmardi et al. (2016)], phantom [Dabrowski (2008)], k-essence and quintessence [Putter and Linder (2007)] along with other four elements i.e. dark matter, baryons, radiation and neutrinos. But so far there is no direct detection of such exotic fluids. Although the literature is now flooded with hundreds of model for dark energy what we lack is precise cosmological data coming from variety of observations involving both background and inhomogeneous universe that can discriminate among these models. In this connection if we accept that Einstein was correct with his general relativity theory to explain accelerated expansion of the universe could also be explained by negative pressure working against gravity. The belief of Einstein to the static universe made him to think about negative pressure which will stop the attraction of the gravity. However, we know that we have non static universe, moreover we know that we have accelerated expansion. According to the above observational data analysis, it can be estimated the amount of the negative pressure in our universe, which we call it as dark energy. The simple question about the nature of the dark energy is still one of the intriguing questions and left free space for new speculations.

After Einstein many Physicists have been investigating about gravitation in different contexts. Hermann Weyl [Weyl (1918a)] attempted to generalize the idea of geometrizing the gravitation and electromagnetism by applying different techniques and methods. He described both gravitation and electromagnetism geometrically by formulating a new kind of gauge theory involving metric tensor with an intrinsic geometrical significance. With the concept of Einstein's general theory of relativity Lyra [Lyra (1951)] suggested a modification by introducing a gauge function into the structure-less manifold which removes the non-integrability condition of the length of a vector under parallel transport. Such theories are commonly

known as the modified theories of the gravitation or alternate theory of gravitation. Some important modified theories of gravitation are Brans-Dicke theory [Brans and Dicke (1961)], Scalar-tensor theories [Barker (1978)], Vector-tensor theory [Bekenstein (2004)], Weyl's theory [Weyl (1918b)], F(R) Gravity [Nojiri et al. (2003); Sotiriou (2010); Nojiri et al. (2011); Buchdahl (1970)], Mimetic Gravity [Chamseddine et al. (2013)], Mimetic F(R) gravity [Nojiri et al. (2014b); Myrzakulov et al. (2015)], Lyra geometry [Scheibe (1952)] and many more. We can explain about the accelerating expansion of the universe in the context of these modified theories of gravitation. Out of these modified theories of gravitation, here we will discuss about the Lyra geometry.

As we know that Lyra geometry is a modification of Riemannian geometry by introducing a gauge function into the structure less manifold which removes the non-integrability condition of the length of a vector under parallel transport. Lyra geometry along with constant gauge vector ϕ_i will either play the role of cosmological constant or creation field (equal to Hoyle's creation field [Hoyle (1948); Hoyle and Narlikar (1963); Hoyle and Narlikar (1964)]) which is discussed by Soleng (1987). Many researchers proposed different cosmological models in different context of Lyra geometry. Sen (1957) formulated a new scalar-tensor theory of gravitation based on Lyra geometry in which he found that static model with finite density in Lyra geometry is similar to the Einstein's static model, but it exhibited red shift which is a significant difference of the model. Later, Sen and Dunn (1971) and Rosen (1983), suggested that this theory was based on non-integrability of length transfer so that it had some unsatisfactory features and hence this theory did not gain general acceptance. Halford (1970, 1972) pointed out that in the normal general relativistic treatment the constant displacement vector field ϕ_i in Lyra's geometry plays the role of cosmological constant. Also, as in the Einstein's theory of relativity the scalar-tensor treatment based on Lyra's geometry predicts the same effect, within observational limits, as far as the classical solar system test are concerned . Many authors [Rahaman et al. (2003, 2005); Casana et al. (2005, 2006) ; Mohanty et al. (2007, 2009b); Mahanta et al. (2012); Asgar et al. (2014a); Mollah et al. (2015); Mollah and Singh (2016) attempted to solve Einstein's field equations in the framework of Lyra's geometry and successfully find their solutions under different circumstances.

From the critical study we know that space-time symmetry plays a vital role in the features of space-time that can be described as exhibiting some form of symmetry. The importance of symmetries in relativity and cosmology is to simplify Einstein's field equations and to provide a classification of the space time according to the structure of the corresponding Lie algebra. Some authors [Henriksen and Wesson (1978); Mohanty and Samanta (2009a);

Rao and Neelima (2013a) ; Cahill and Taub (1971)] solved Einstein's field equations for an axially symmetric space-time by imposing certain conditions upon the scale factor of the space time together with some conditions on matter which represents such space-time and some restrictions on its physical properties. To explain the accelerated expansion of the universe many prominent researchers like [Shchigolev (2013); Hova (2013); Ali and Rahaman (2013); Megied et al. (2014); Khurshudyan et al. (2014a, b); Saadat (2015); Darabi et al. (2015); Ziaie et al. (2013); Pucheu et al. (2016)] have investigated and proposed different cosmological models and ideas of the universe within the framework of Lyra's geometry and other theories of relativity in different context.

It is also seems that, majority of researchers in the field of Astrophysics, Cosmology and Particle Physics were doing their research in this area by considering only 4-dimensional case. But in this modern era peoples are more interested to study higher dimensional case, since the solutions of Einstein field equations in higher dimensional space times are believed to have physical relevance possibly at extremely early times before the universe underwent the compactification transitions. We also found that by using a suitable scalar field we can show that the phase transitions on the early universe can give rise to such objects which are nothing but the topological knots in the vacuum expectation value of the scalar field and most of their energy is concentrated in a small region. So it is necessary for us to study the cosmological problems by considering higher-dimensional space-time to unify gravity with other interactions. In cosmology, particularly for study of the early stage of universe the present four-dimensional stage of the universe might have been preceded by a multi-dimensional stage. Recently, Singh and Mollah (2016) studied about higher-dimensional LRS Bianchi type-I cosmological model universe interacting with perfect fluid in Lyra geometry with different cases. Many authors [Bali and Kumawat (2010); Pradhan et al. (2013); Accioly (1985); Kandalkar et al. (2009); Adhav et al. (2013); Rao and Neelima (2013b); Adhav (2011); Pradhan and Singh (2011b); Asgar and Ansary (2014b)] have studied Bianchi type models in order to examine the role of certain anisotropic sources during the formation of the large-scale structures that we see in the present universe and for better understanding the small amount of observed anisotropy in the universe. Banerjee et al., (1990) have investigated Bianchi type-I cosmological models with viscous fluid in higher dimensional space time. Also, Krori et al. (1994); Gron (1988); Collins et al. (1980) studied Bianchi type-I string cosmological model in higher dimensions.

Motivated from the above, in this chapter we discussed about five dimensional spatially homogeneous and anisotropic Locally Rotationally Symmetric (LRS) Bianchi Type I universe

with time dependent deceleration parameters in Lyra manifold considering different cases and find out the realistic solutions which are supporting to the present observational facts. We also found that the geometry itself of Lyra manifold behaves as a new source of dark energy which will be beneficial for further research work.

This chapter is organized as follows: In the section 9.2 we are presenting the formulation of the problem and the physical properties for the model are defined. In the next section i.e. in section 9.3, considering deceleration parameter in four different forms, we have obtained six different cosmological models (in four cases together with three sub cases of case-II), by solving the field equations obtained in the previous section. Some important physical parameters representing the models are also obtained in this section. In Section 9.4, Physical interpretations of all the model universes are discussed. In the final section, conclusion and summary of our research paper is presented.

9.2 Formulation of Problem:

The Einstein's field equations based on Lyra geometry is proposed by Sen (1957) and Sen and Dunn (1971) in normal gauge may be written as

$$R_{ij} - \frac{1}{2}g_{ij}R + \frac{3}{2}\phi_i\phi_j - \frac{3}{4}g_{ij}\phi^k\phi_k = -T_{ij} \tag{9.1}$$

where ϕ_i is the displacement vector and other symbols have their usual meanings as in the Riemannian geometry. The displacement vector ϕ_i is taken in the form

$$\phi_i = (0,0,0,0,\beta(t)) \tag{9.2}$$

In this Paper we consider the five dimensional LRS Bianchi type-I axially symmetric space time in the form

$$ds^2 = A^2(dx^2 + dy^2 + dz^2) + B^2 d\psi^2 - dt^2 \tag{9.3}$$

where A and B are functions of cosmic time t only, where the fifth coordinate is taken to be space-like. Here the spatial curvature has been taken as zero Gron (1988).

The energy momentum tensor T_{ij} for the perfect fluid is given by

$$T_{ij} = (\rho + p)u_i u_j + p g_{ij} \tag{9.4}$$

where, ρ is the energy density , p is the fluid pressure and u^i is the five velocity vector given by $u^i = (0,0,0,0,1)$ satisfying

$$g_{ij}u^i u^j = u^i u_i = -1 \tag{9.5}$$

The physical quantities like Spatial Volume V , Hubble's Parameter H , Expansion scalar θ , Shear scalar σ , mean anisotropy parameter Δ etc. plays a vital role for a cosmological model. These parameters for the axially symmetric LRS Bianchi type-I metric (9.3) are defined as

$$V = R^4(t) = A^3 B \tag{9.6}$$

$$H = \frac{\dot{R}}{R} = \frac{1}{4}\left(3\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) \tag{9.7}$$

$$\theta = 4H = 3\frac{\dot{A}}{A} + \frac{\dot{B}}{B} \tag{9.8}$$

$$\sigma^2 = \frac{1}{2}\left(\sum_{i=1}^{4} H_i^2 - 4H^2\right) \tag{9.9}$$

$$\Delta = \frac{1}{4}\sum_{i=1}^{4}\left(\frac{H_i - H}{H}\right)^2 \tag{9.10}$$

where H_i ; $i = 1,2,3,4$ represent the directional Hubble's parameters in x, y, z, ψ directions respectively and $\Delta = 0$ corresponds to isotropic expansion.

Using the comoving coordinate system, the field equations (9.1) for the metric (9.3) with the help of equation (9.4) yield

$$3\frac{\dot{A}^2}{A^2} + 3\frac{\dot{A}\dot{B}}{AB} - \frac{3}{4}\beta^2 = -\rho \tag{9.11}$$

$$2\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}^2}{A^2} + 2\frac{\dot{A}\dot{B}}{AB} + \frac{3}{4}\beta^2 = -p \tag{9.12}$$

$$3\frac{\ddot{A}}{A} + 3\frac{\dot{A}^2}{A^2} + \frac{3}{4}\beta^2 = -p \tag{9.13}$$

where the overhead dots $(.)$ denote the derivatives with respect to time t.

Again the energy conservation equation $T^{ij}_{;j} = 0$ gives us

$$\dot{\rho} + (\rho + p)\left(3\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = 0 \tag{9.14}$$

The field equations (9.11)-(9.13) are a system of three highly nonlinear differential equations with five unknown parameters A, B, ρ, p and β. Therefore in order to obtain deterministic solution of the above system of equations, it is required to assume two more additional physical conditions/mathematical relations involving these unknowns. Hence we assume

(a) The expansion scalar θ is proportional to the shear scalar (σ) [Collin et al. (1980)] which leads to

$$A = B^n \tag{9.15}$$

where $n \neq 1$ is a constant and

(b) Another additional condition is taken as the any one of the following time dependent deceleration parameter

(i) Average scale factor as an integrating function of time [Saha et al.(2012)] as

$$R = \left(t^a e^t\right)^{\frac{1}{b}}, \tag{9.16}$$

where a and b are positive constants in which scale factor has two factors: one factor behaving like exponential expansion and the other factor behaving like power law expansion. While the power law behavior dominate the cosmic dynamics in early phase of cosmic evolution, the exponential factor dominates at late phase. When $a = 0$, the exponential law is recovered and for $1/b = 0$, the scale factor reduces to the power law as suggested by Mishra and Tripathy (2015b) and Akarsu et al. (2014). Here we are very much interested in a transient universe with early deceleration and late acceleration so that at early time q can be positive whereas at late time q assumes a negative value in conformity with the recent observational data. Therefore, the value of deceleration parameter (DP) q becomes a function of time t as follows

$$q = -\frac{R\ddot{R}}{\dot{R}^2} = \frac{ab}{(a+b)^2} - 1$$

(ii) Variable deceleration parameter $q = -\frac{R\ddot{R}}{\dot{R}^2} = D$,

where R is the average scale factor and D is a variable.

(iii) Linearly varying deceleration parameter $q = -\frac{R\ddot{R}}{\dot{R}^2} = -kt + m - 1$,

where R is the average scale factor, $k \geq 0, m \geq 0$ are constants.

(iv) Special form of deceleration parameter $q = -\frac{R\ddot{R}}{\dot{R}^2} = -1 + \frac{c}{1+R^c}$,

where R is the average scale factor and $c > 0$ is a constant.

The above four conditions will respectively give us the four different different model universes as follows-

9.3 Solution of Field Equations in Four Different Cases (of Deceleration Parameter):

9.3.1 Case-I: Models with Time Dependent Deceleration Parameter:

In this case we consider the average scale factor as an integrating function of time as given in the above equation (9.16) as

$$R = \left(t^a e^t\right)^{\frac{1}{b}},$$

where a and b are positive constants.

From the above value of the average scale factor R, the value of deceleration parameter q can be obtained as

$$q = -\frac{R\ddot{R}}{\dot{R}^2} = \frac{ab}{(a+t)^2} - 1. \tag{9.17}$$

This law (9.17) gives us a time-dependent deceleration parameter which describes that initially our universe was decelerating but as the the time progresses our universe becomes an accelerating. That is our universe is in transition from early decelerating phase to the present accelerating phase.

From equation (9.6) we have

$$R(t) = \left(A^3 B\right)^{\frac{1}{4}},$$
(9.18)

The metric (9.1) can be determined completely by this average scale factor $R(t)$ given by equation (9.18).

From equations (9.15), (9.16) and (9.18) we have

$$A = \left(t^a e^t\right)^{\frac{4n}{b(3n+1)}}$$
(9.19)

and

$$B = \left(t^a e^t\right)^{\frac{4}{b(3n+1)}}$$
(9.20)

By using Equations (9.19) and (9.20) the metric (9.3) can be written as

$$ds^2 = \left(t^a e^t\right)^{\frac{8n}{b(3n+1)}} \left(dx^2 + dy^2 + dz^2\right) + \left(t^a e^t\right)^{\frac{8}{b(3n+1)}} d\psi^2 - dt^2$$
(9.21)

Equation (9.21) represents a Bianchi type-I cosmological model universe in Lyra geometry with time dependent deceleration parameter.

Some Physical Properties of the Model (9.21) with Time Dependent Deceleration Parameter

The energy conservation equation (9.14) leads to

$$\frac{3}{2}\beta\dot{\beta} + \frac{3}{2}\beta^2 \left(3\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = 0 .$$

Now since the displacement vector $\beta \neq 0$ so we have

$$\dot{\beta} + \beta \left(3\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) = 0 .$$
(9.22)

From equation (9.22) the displacement vector β is obtained as

$$\beta = \left(k_0 t^a e^t\right)^{-\frac{4}{b}} . \tag{9.23}$$

Energy density ρ and pressure p can be obtained from equations (9.11) and (9.13) as

$$\rho = -\frac{48n(n+1)}{b^2(3n+1)^2}\left(\frac{a}{t}+1\right)^2 + \frac{3}{4}\left(k_0 t^a e^t\right)^{-\frac{8}{b}}, \tag{9.24}$$

$$p = -\frac{96n^2}{b^2(3n+1)^2}\left(\frac{a}{t}+1\right)^2 + \frac{12an}{b(3n+1)^2} - \frac{3}{4}\left(k_0 t^a e^t\right)^{-\frac{8}{b}}. \tag{9.25}$$

The equations (9.6)-(9.10) gives us the Spatial Volume V , Hubble's Parameter H , Expansion scalar θ , Shear scalar σ , mean anisotropy parameter Δ as

$$V = \left(t^a e^t\right)^{\frac{4}{b}} \tag{9.26}$$

$$H = \frac{\dot{R}}{R} = \frac{1}{b}\left(\frac{a}{t}+1\right) \tag{9.27}$$

$$\theta = \frac{4}{b}\left(\frac{a}{t}+1\right) \tag{9.28}$$

$$\sigma^2 = \frac{6(n-1)^2}{b^2(3n+1)^2}\left(\frac{a}{t}+1\right)^2 \tag{9.29}$$

$$\Delta = \frac{3(n-1)^2}{(3n+1)^2} = constant \ (\neq 0 \ for \ n \neq 1) \tag{9.30}$$

From equations (9.28) and (9.29) we have

$$\frac{\sigma^2}{\theta^2} = \frac{3(n-1)^2}{(3n+1)^2} \ , \ which \ is \ a \ constant \ and \ is \ independent \ of \ time \ t.$$

Hence

$$\lim_{t \to \infty} \frac{\sigma^2}{\theta^2} = \frac{3(n-1)^2}{(3n+1)^2} \neq 0 \ for \ n \neq 1 . \tag{9.31}$$

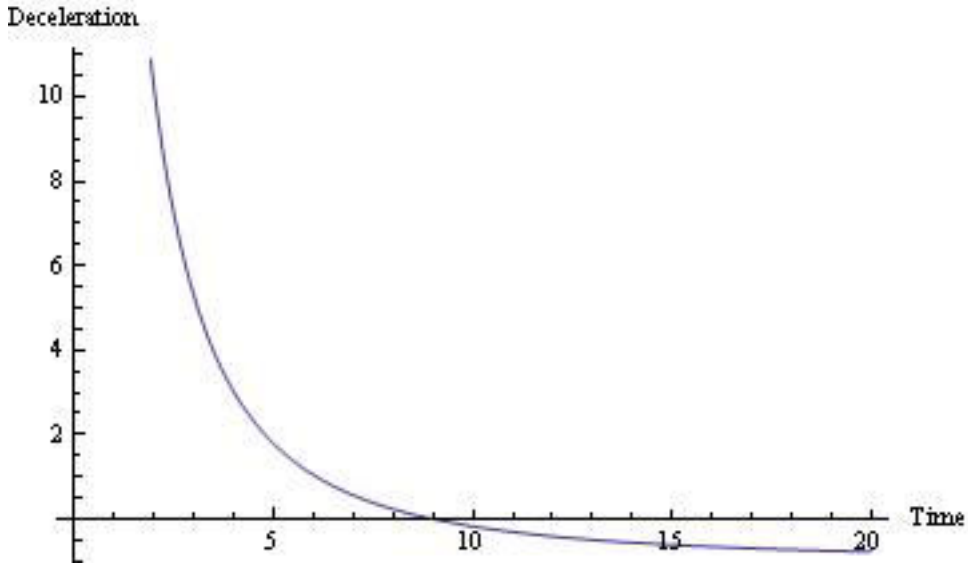The variations of some parameters with respect to time for this case are shown in **Figs. 1-3**.

Figure-9.1 : The plot of Deceleration Parameter q vs. Time t. Here, $a = k_0 = 1, b = 100$ and $n = 2$.
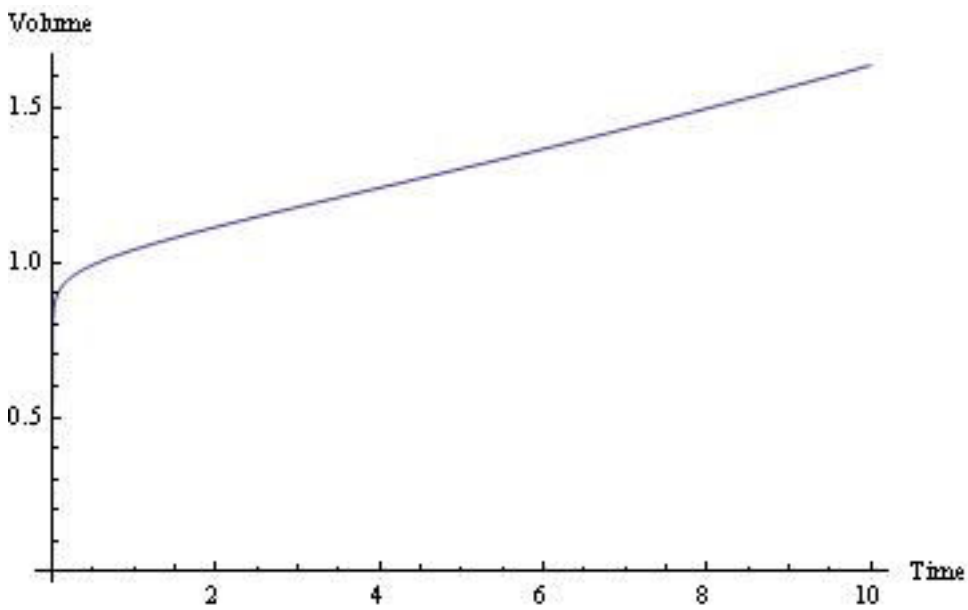

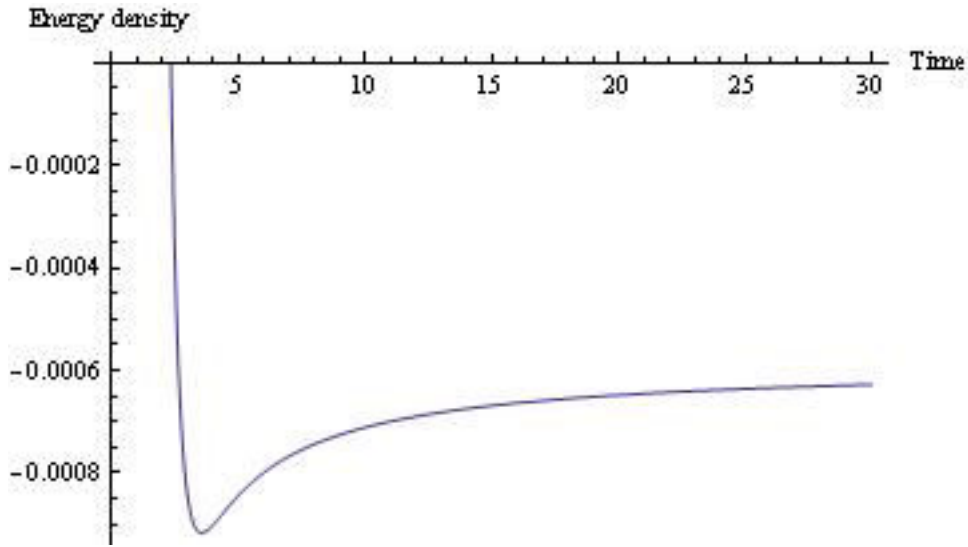**Figure-9.2** : The plot of Volume V vs. Time t. Here, $a = k_0 = 1, b = 100$ and $n = 2$.

Figure-9.3 : The plot of Energy density ρ vs. Time t. Here, $a = k_0 = 1, b = 100$ and $n = 2$.

9.3.2 Case-II: Models with Variable Deceleration Parameter:

Let us consider that the deceleration parameter q to be a variable parameter [Pradhan et al (2006)] as

$$q = -\frac{R\ddot{R}}{\dot{R}^2} = D \ (variable) , \tag{9.32}$$

where R is the average scale factor.

Equation (9.32) can be written as

$$\frac{\ddot{R}}{R} + D\frac{\dot{R}^2}{R^2} = 0 . \tag{9.33}$$

In order to solve equation (9.33), we may consider $D = D(R)$. Now since R is a function of time so we can assume $D = D(t) = D(R(t))$ and it is possible only when there is one to one correspondence between R and t . Again since both R and t are increasing function so this is possible only we can avoid singularity like big bang or big rip.

Therefore if we assume $D = D(R)$ then the general solution of equation (9.33) is obtained as

$$\int e^{\int \frac{B}{R} dR} dR = t + c_0 , \tag{9.34}$$

where c_0 is an integrating constant. Without loss of generality we may choose $c_0 = 0$

When $c_0 = 0$ then in order to solve (9.34), we have to choose $\int \frac{B}{R} dR$ in such a way that (9.34) becomes integrable. Hence we consider

$$\int \frac{B}{R} dR = \log L(R) \,, \tag{9.35}$$

where $L(R)$ is a function of R only and it does not affect the nature of generality of solution.

From equations (9.34) and (9.35) we have

$$\int L(R) \, dR = t \,, \tag{9.36}$$

Since the choice of function $L(R)$ in equation (9.36) is quite arbitrary and we are interested in obtaining the models of the universe which are not only physically viable but also consistent with observations, so we have chosen the function $L(R)$ in three different ways

$$L(R) = \frac{1}{k_1 R} \,,$$

where k_1 is an arbitrary constant.

$$L(R) = \frac{1}{2k_3 \sqrt{R + k_4}} \,,$$

where k_3 and k_4 are arbitrary constants.

$$L(R) = \frac{nR^{n-1}}{k_5 \sqrt{1 + R^{2n}}} \,,$$

where k_5 is an arbitrary constant.

Subcase-1 of Case II : In the First case the function $L(R)$ is chosen as

$$L(R) = \frac{1}{k_1 R} \,, \tag{9.37}$$

where k_1 is an arbitrary constant.

For this case, integrating equation (9.36) we may obtain the exact solution as

$$R = k_2 e^{k_1 t} \, , \tag{9.38}$$

Where k_2 is an integrating constant.

From equations (9.6), (9.15) and (9.38) we have

$$A = k_2^{\frac{4n}{3n+1}} e^{\frac{4nk_1 t}{3n+1}} \, , \tag{9.39}$$

$$B = k_2^{\frac{4}{3n+1}} e^{\frac{4k_1 t}{3n+1}} \, , \tag{9.40}$$

Using the values of A and B from equations (9.39) and (9.40), the metric (9.3) can be written as

$$ds^2 = k_2^{\frac{8n}{3n+1}} e^{\frac{8nk_1 t}{3n+1}} (dx^2 + dy^2 + dz^2) + k_2^{\frac{8}{3n+1}} e^{\frac{8k_1 t}{3n+1}} d\psi^2 - dt^2 \, . \tag{9.41}$$

Equation (9.41) represents a Bianchi type-I cosmological model universe in Lyra geometry with variable deceleration parameter.

Some Physical Properties of the Model (9.41) with variable Deceleration Parameter

The equations (9.6)-(9.10) gives us the Spatial Volume V , Hubble's Parameter H , Expansion scalar θ , Shear scalar σ , mean anisotropy parameter Δ as

$$V = k_2^4 e^{4k_1 t} \, , \tag{9.42}$$

$$H = k_1 = constant \, , \tag{9.43}$$

$$\theta = 4k_1 = constant \, , \tag{9.44}$$

$$\sigma^2 = \frac{6(n-1)^2 k_1^2}{(3n+1)^2} \, , \tag{9.45}$$

$$\Delta = \frac{3(n-1)^2}{(3n+1)^2} \, . \tag{9.46}$$

From equations (9.44) and (9.45) we have

$$\frac{\sigma^2}{\theta^2} = \frac{3(n-1)^2}{8(3n+1)^2} \ , \ \textit{which is a constant and is independent of time } t \ .$$

Hence

$$\lim_{t\to\infty} \frac{\sigma^2}{\theta^2} = \frac{3(n-1)^2}{8(3n+1)^2} \neq 0 \ \textit{for } n \neq 1 \ . \tag{9.47}$$

Again from (9.22), (9.11) and (9.13), the displacement vector β, the energy density ρ and the fluid pressure p are obtained as

$$\beta = l_2 e^{-4k_1 t} \ , \tag{9.48}$$

$$\rho = -\frac{48k_1^2 n(n+1)}{(3n+1)^2} + \frac{3}{4} l_2 e^{-4k_1 t} \ , \tag{9.49}$$

$$p = -\frac{96n^2 k_1^2}{(3n+1)^2} - \frac{3}{4} l_2 e^{-4k_1 t} \ , \tag{9.50}$$

where l_2 is a constant.

Also the deceleration parameter q is obtained as

$$q = -1 \ . \tag{9.51}$$

The variations of some parameters with respect to time for this sub case 1 are shown in **Figs. 4-8**.
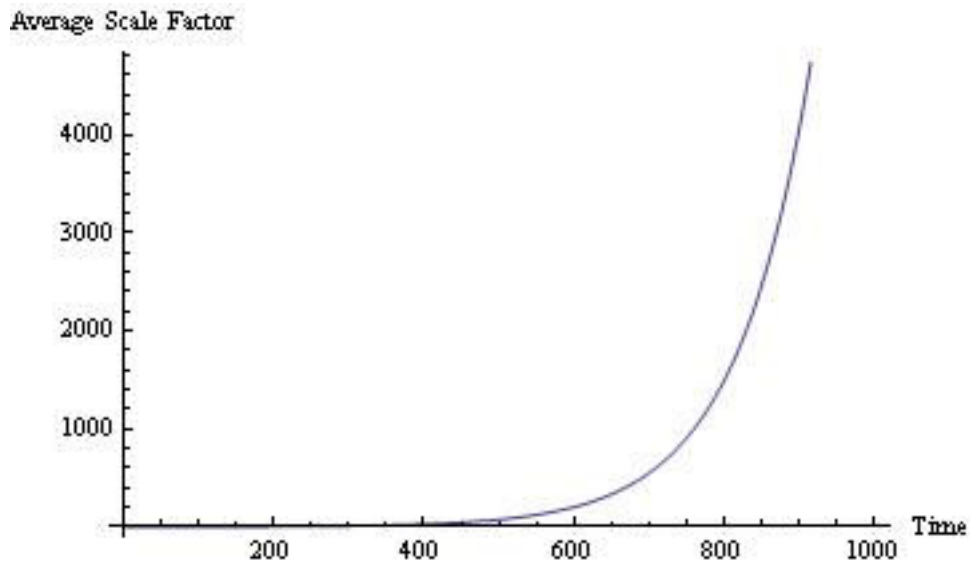
Figure-9.4 : The plot of Average Scale factor $R(t)$ vs. Time t. Here, $k_1 = 0.01$, $k_2 = 0.5$ and $n = 2$.
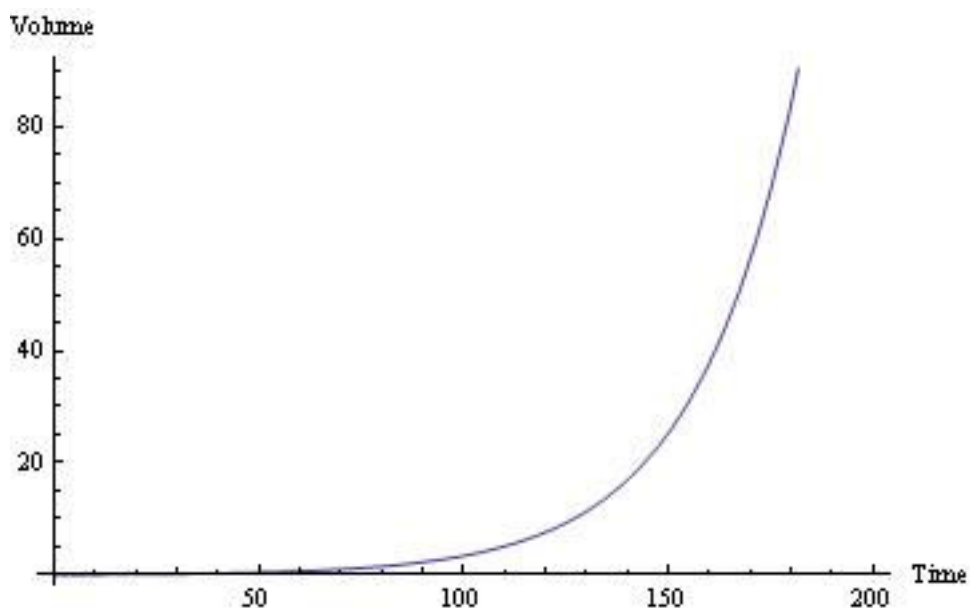

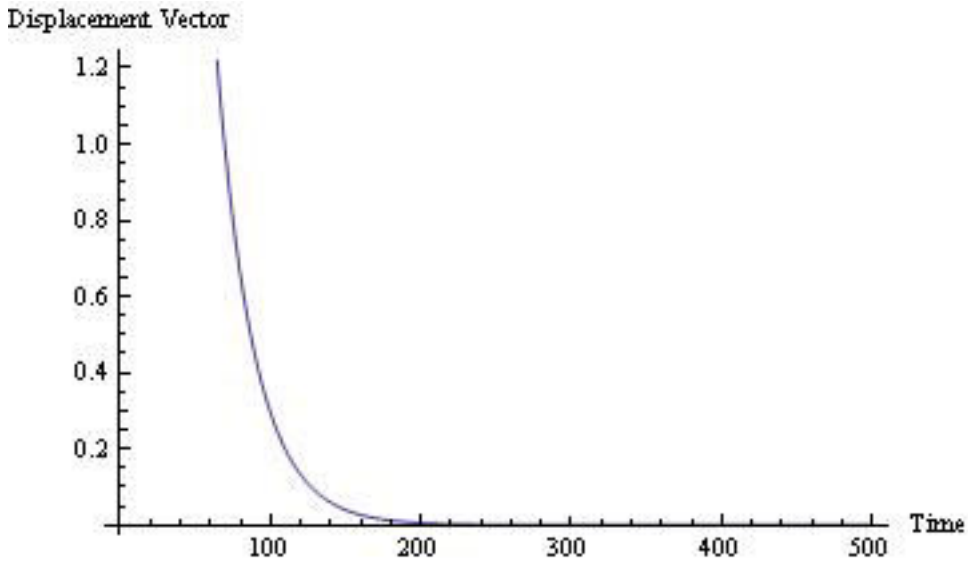**Figure-9.5** : The plot of Volume V vs. Time t. Here, $k_1 = 0.01$, $k_2 = 0.5$ and $n = 2$.

Displacement Vector

Figure-9.6 : The plot of Displacement vector β vs. Time t. Here, $k_1 = 0.01$, $k_2 = 0.5$, $n = 2$ and $l_2 = 16$.
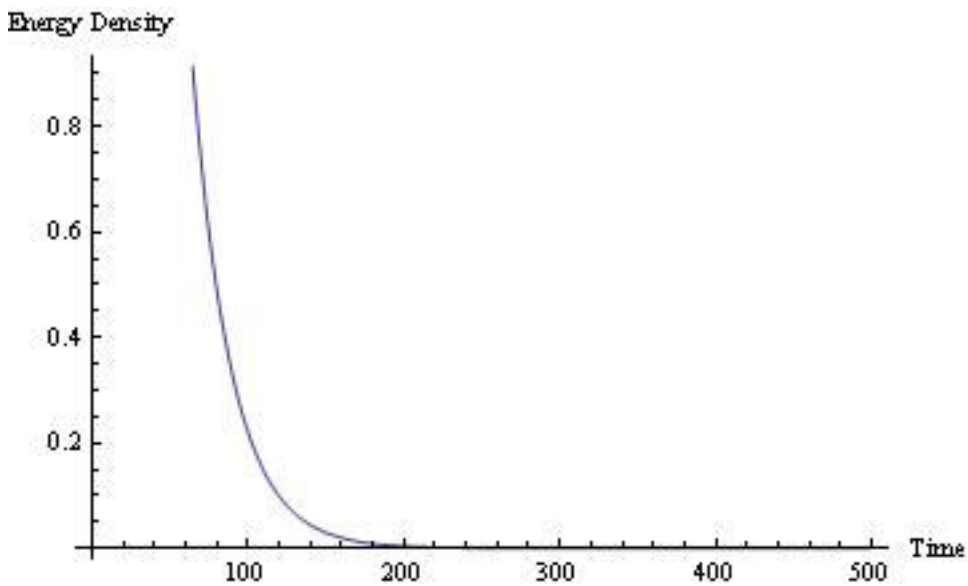
Energy Density

Figure-9.7 : The plot of Energy density ρ vs. Time t. Here, $k_1 = 0.01$, $k_2 = 0.5$, $n = 2$ and $l_2 = 16$.
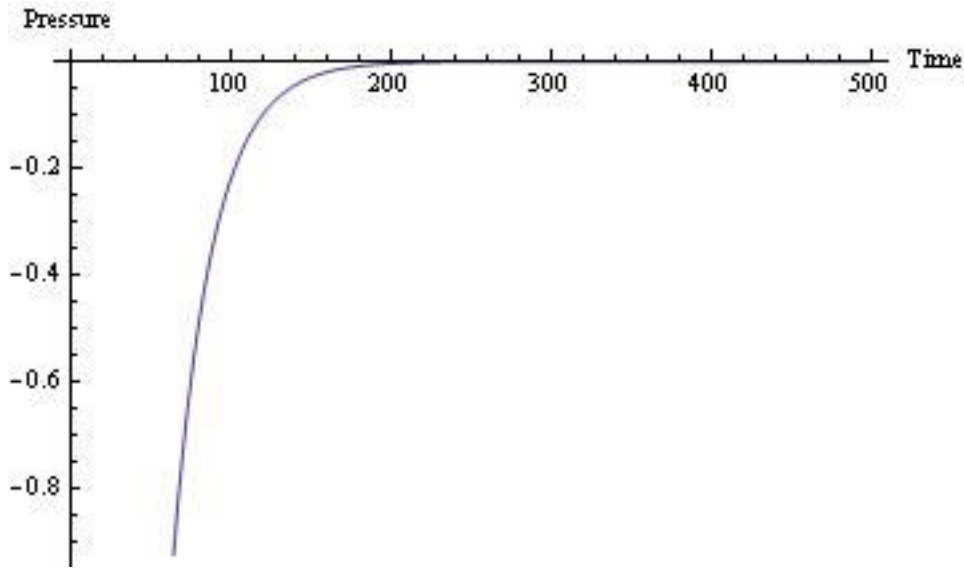
Figure-9.8 : The plot of Pressure p vs. Time t. Here, $k_1 = 0.01$, $k_2 = 0.5$, $n = 2$ and $l_2 = 16$
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Subcase-2 of Case II : In the second case we choose $L(R)$ as

$$L(R) = \frac{1}{2k_3 \sqrt{R + k_4}} \, , \tag{9.52}$$

where k_3 and k_4 are arbitrary constants.

Therefore from equation (9.36) we get

$$R = \alpha_1 t^2 + \alpha_2 t + \alpha_3 \, , \tag{9.53}$$

where α_1, α_2 and α_3 are arbitrary constants.

This value of R gives us the values of A and B, from equations (9.6), (9.15), as

$$A = \left(\alpha_1 t^2 + \alpha_2 t + \alpha_3 \right)^{\frac{4n}{3n+1}} \, , \tag{9.54}$$

$$B = \left(\alpha_1 t^2 + \alpha_2 t + \alpha_3 \right)^{\frac{4}{3n+1}} \, . \tag{9.55}$$

Using the values of A and B from equations (9.54) and (9.55) in the metric (9.3) we have

$$ds^2 = (\alpha_1 t^2 + \alpha_2 t + \alpha_3)^{\frac{8n}{3n+1}}(dx^2 + dy^2 + dz^2) + (\alpha_1 t^2 + \alpha_2 t + \alpha_3)^{\frac{8}{3n+1}}d\psi^2 - dt^2 . \quad (9.56)$$

Equation (9.56) give us also another form of Bianchi type-I cosmological model universe in Lyra geometry with variable deceleration parameter.

Some Physical Properties of the Model (9.56) with variable Deceleration Parameter

The equations (9.6)-(9.10) gives us the Spatial Volume V , Hubble's Parameter H , Expansion scalar θ , Shear scalar σ , mean anisotropy parameter Δ as

$$V = (\alpha_1 t^2 + \alpha_2 t + \alpha_3)^4 , \quad (9.57)$$

$$H = \frac{2\alpha_1 t + \alpha_2}{\alpha_1 t^2 + \alpha_2 t + \alpha_3} , \quad (9.58)$$

$$\theta = \frac{4(2\alpha_1 t + \alpha_2)}{\alpha_1 t^2 + \alpha_2 t + \alpha_3} , \quad (9.59)$$

$$\sigma^2 = \frac{6(n-1)^2}{(3n+1)^2}\frac{(2\alpha_1 t + \alpha_2)^2}{(\alpha_1 t^2 + \alpha_2 t + \alpha_3)^2} , \quad (9.60)$$

$$\Delta = \frac{3(n-1)^2}{(3n+1)^2} = constant \ (\neq 0 \ for \ n \neq 1) . \quad (9.61)$$

From equations (9.59) and (9.60) we have

$$\frac{\sigma^2}{\theta^2} = \frac{3(n-1)^2}{8(3n+1)^2} , \ which \ is \ a \ constant \ and \ is \ independent \ of \ time \ t .$$

Hence

$$\lim_{t \to \infty}\frac{\sigma^2}{\theta^2} = \frac{3(n-1)^2}{8(3n+1)^2} \neq 0 \ for \ n \neq 1 . \quad (9.62)$$

Again from (9.22), (9.11) and (9.13), the displacement vector β, the energy density ρ and the fluid pressure p are obtained as

$$\beta = \frac{\alpha_4}{(\alpha_1 t^2 + \alpha_2 t + \alpha_3)^4} \, , \tag{9.63}$$

$$\rho = -\frac{48(n+1)}{(3n+1)^2} \frac{(2\alpha_1 t + \alpha_2)^2}{(\alpha_1 t^2 + \alpha_2 t + \alpha_3)^2} + \frac{3}{4} \frac{\alpha_4^2}{(\alpha_1 t^2 + \alpha_2 t + \alpha_3)^8} \, , \tag{9.64}$$

$$p = -\frac{12n(5n-1)}{(3n+1)^2} \frac{(2\alpha_1 t + \alpha_2)^2}{(\alpha_1 t^2 + \alpha_2 t + \alpha_3)^2} - \frac{24n}{3n+1} \frac{\alpha_1}{(\alpha_1 t^2 + \alpha_2 t + \alpha_3)^2}$$
$$-\frac{3}{4} \frac{\alpha_4^2}{(\alpha_1 t^2 + \alpha_2 t + \alpha_3)^8} \, . \tag{9.65}$$

Also the deceleration parameter q is obtained as

$$q = -\frac{2\alpha_1(\alpha_1 t^2 + \alpha_2 t + \alpha_3)}{(2\alpha_1 t + \alpha_2)^2} \, . \tag{9.66}$$

The variations of some parameters with respect to time for this sub case 2 are shown in **Figs. 9-11**.
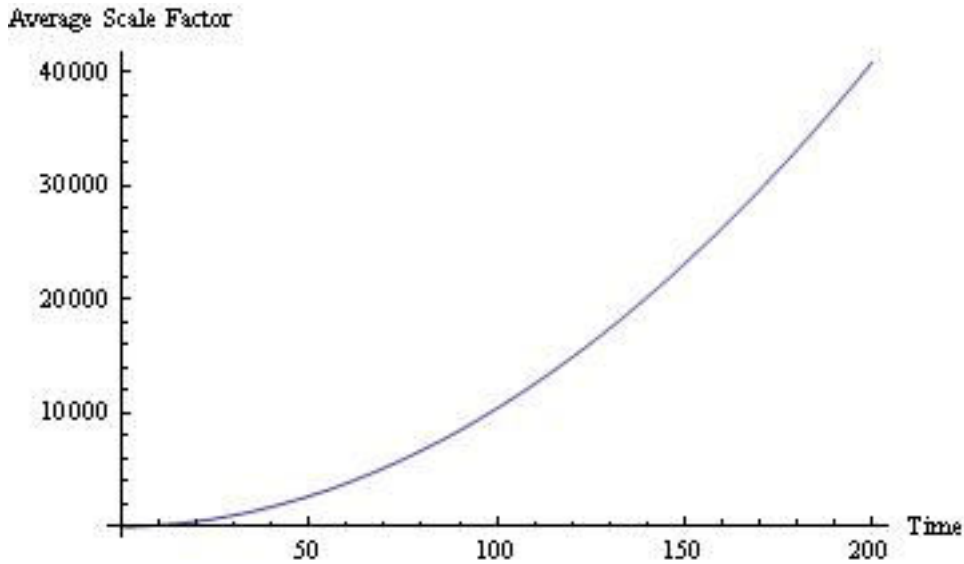

Figure-9.9 : The plot of Average Scale Factor $R(t)$ vs. Time t. Here, $\alpha_1 = 1$, $\alpha_2 = 4$, $\alpha_3 = 3$, $\alpha_4 = 1$ and $n = 2$.
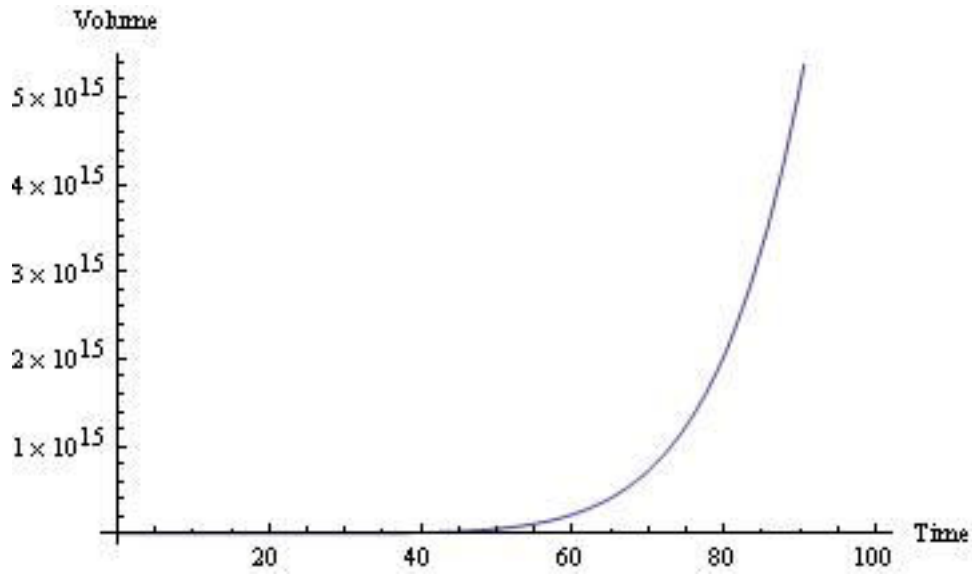
Figure-9.10 : The plot of Volume V vs. Time t. Here, $\alpha_1 = 1$, $\alpha_2 = 4$, $\alpha_3 = 3$, $\alpha_4 = 1$ and $n = 2$.
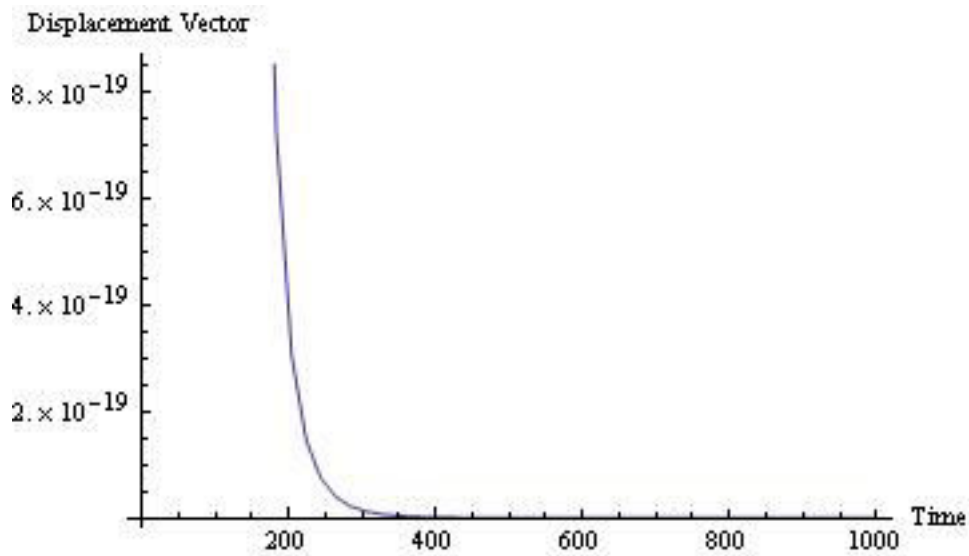

Figure-9.11(a): The plot of Displacement Vector $\beta(t)$ vs. Time t. Here, $\alpha_1 = 1$, $\alpha_2 = 4$, $\alpha_3 = 3$, $\alpha_4 = 1$ and $n = 2$.
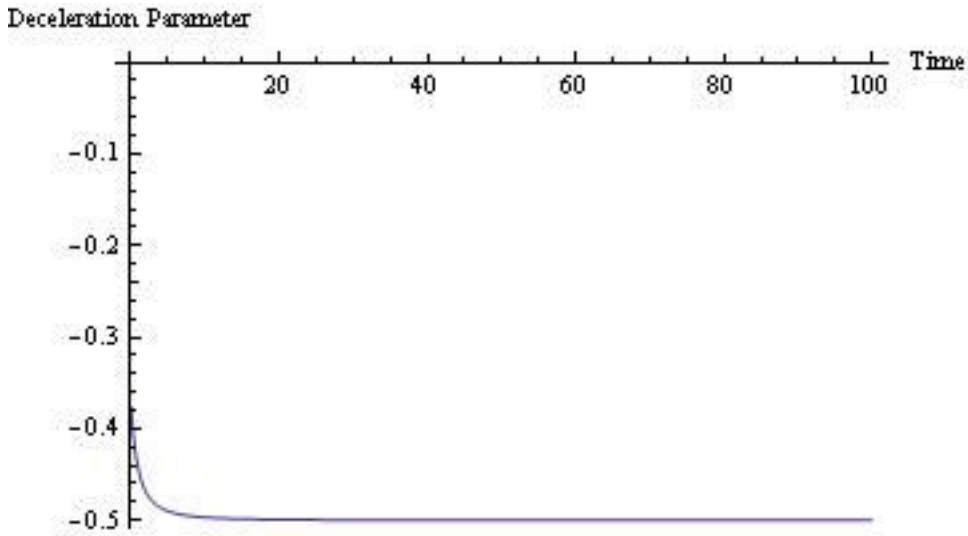
Figure-9.11(b): The plot of Deceleration Parameter q vs. Time t. Here, $\alpha_1 = 1$, $\alpha_2 = 4$, $\alpha_3 = 3$, $\alpha_4 = 1$ and $n = 2$.

Subcase-3 of Case II : Also in the third case we choose $L(R)$ as

$$L(R) = \frac{nR^{n-1}}{k_5\sqrt{1+R^{2n}}} \, ,$$
(9.67)

where k_5 is an arbitrary constant.

Using (9.67) in (9.36) we have

$$R = [\sinh(k_5 t)]^{\frac{1}{n}} \, .$$
(9.68)

From equations (9.6), (9.15) and (9.68) we have

$$A = [\sinh(k_5 t)]^{\frac{4}{3n+1}} \, ,$$
(9.69)

$$B = [\sinh(k_5 t)]^{\frac{4}{n(3n+1)}} \, .$$
(9.70)

By the use of equations (9.69) and (9.70) the metric (9.3) takes the form

$$ds^2 = [\sinh(k_5 t)]^{\frac{8}{3n+1}}(dx^2 + dy^2 + dz^2) + [\sinh(k_5 t)]^{\frac{4}{n(3n+1)}} d\psi^2 - dt^2 \, .$$
(9.71)

Equation (9.71) represents a Bianchi type-I cosmological model universe in Lyra geometry with variable deceleration parameter.

Some Physical Properties of the Model (9.71) with variable Deceleration Parameter:

The equations (9.6)-(9.10) gives us the Spatial Volume V , Hubble's Parameter H , Expansion scalar θ , Shear scalar σ , mean anisotropy parameter Δ as

$$V = [\sinh(k_5 t)]^{\frac{4}{n}} , \tag{9.72}$$

$$H = \frac{\dot{R}}{R} = \frac{k_5}{n} \coth(k_5 t) , \tag{9.73}$$

$$\theta = \frac{4k_5}{n} \coth(k_5 t) , \tag{9.74}$$

$$\sigma^2 = \frac{6(n-1)^2}{n^2(3n+1)^2} k_5^2 \coth^2(k_5 t) , \tag{9.75}$$

$$\Delta = \frac{3(n-1)^2}{(3n+1)^2} = \ constant \ (\neq 0 \ \ for \ \ n \neq 1) . \tag{9.76}$$

From equations (9.74) and (9.75) we have

$$\frac{\sigma^2}{\theta^2} = \frac{3(n-1)^2}{8(3n+1)^2} , \ which \ is \ a \ constant \ and \ is \ independent \ of \ time \ t .$$

Hence

$$\lim_{t \to \infty} \frac{\sigma^2}{\theta^2} = \frac{3(n-1)^2}{8(3n+1)^2} \neq 0 \ for \ n \neq 1 . \tag{9.77}$$

Again from (9.22), (9.11) and (9.13), the displacement vector β, the energy density ρ and the fluid pressure p are obtained as

$$\beta = [\sinh(k_5 t)]^{-\frac{4}{n}} , \tag{9.78}$$

$$\rho = -\frac{48(n+1)}{n(3n+1)}k_5^2\coth^2(k_5t) + \frac{3}{4}[\sinh(k_5t)]^{-\frac{8}{n}},$$
(9.79)

$$p = -\frac{96k_5^2}{(3n+1)^2}\coth^2(k_5t) + \frac{12k_5^2}{(3n+1)}[\coth^2(k_5t)-1] - \frac{3}{4}[\sinh(k_5t)]^{-\frac{8}{n}}.$$
(9.80)

Also the deceleration parameter q is obtained as

$$q = n[1-\tanh^2(k_5t)]-1.$$
(9.81)

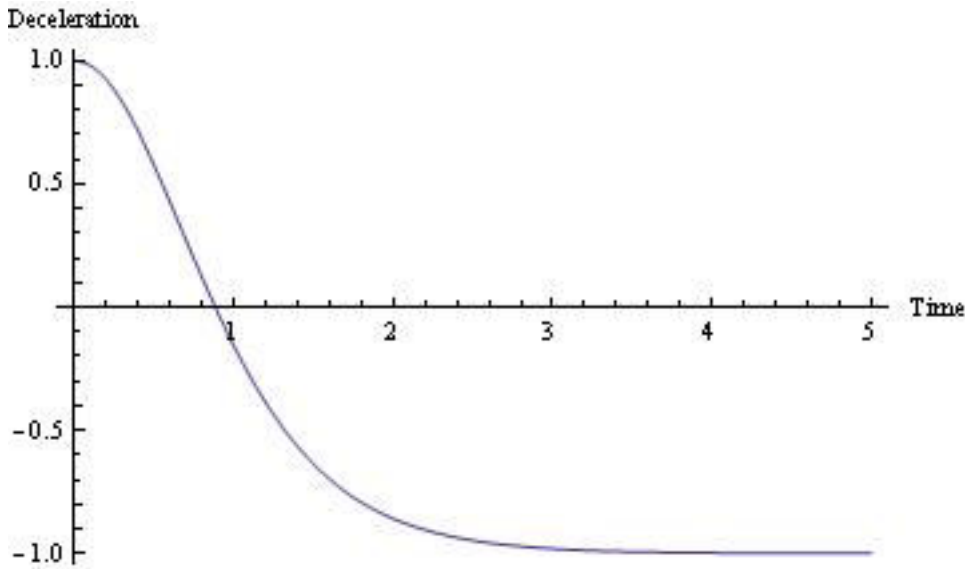The variations of some parameters with respect to time for this sub case 3 are shown in **Figs. 12-14**.

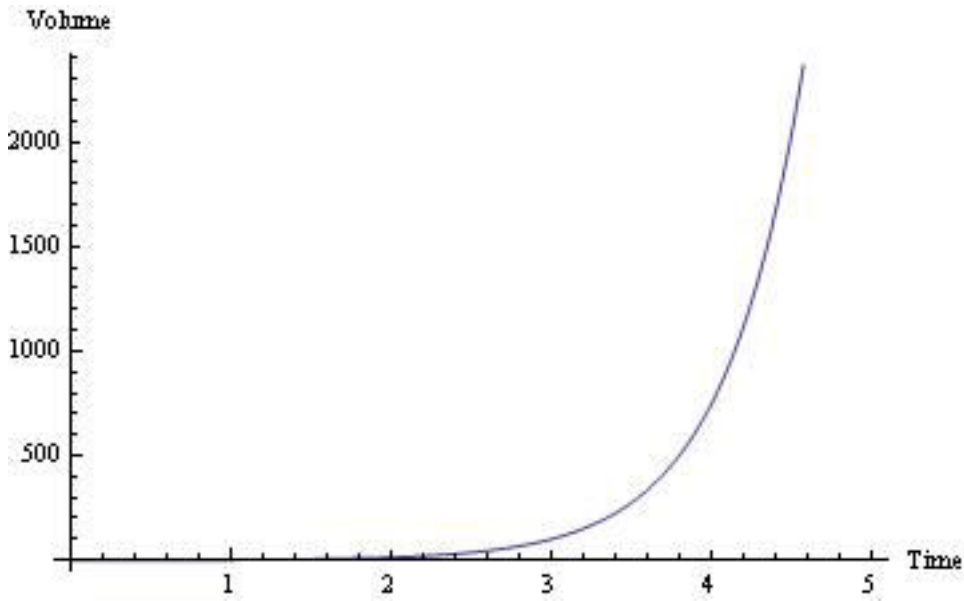**Figure-9.12** : The plot of Deceleration Parameter q vs. Time t. Here $k_5 = 1$ and $n = 2$.

Figure-9.13 : The plot of Volume V vs. Time t. Here $k_5 = 1$ and $n = 2$.
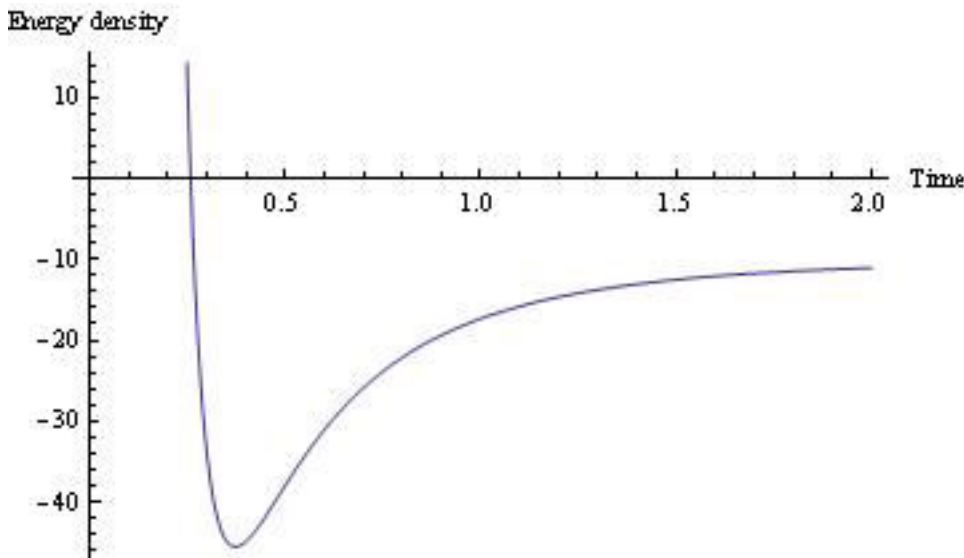

Figure-9.14 : The plot of Energy density ρ vs. Time t. Here $k_5 = 1$ and $n = 2$.

9.3.3   Case-III: Models with Linearly Varying Deceleration Parameter:

In this case we consider the linearly varying deceleration parameter [Akarsu and Dereli (2011)] as

$$q = -\frac{R\ddot{R}}{\dot{R}^2} = -kt + m - 1 \,, \tag{9.82}$$

where R is the average scale factor, $k \geq 0$, $m \geq 0$ are constants.

For $k = 0$, the equation (9.52) reduces to

$$q = m - 1 , \tag{9.83}$$

which is a constant and it corresponds to cosmological model with constant deceleration parameter.

Therefore, the cosmological models that are obtained via constant deceleration parameter may be generalized by using this law.

By solving equation (9.82), We obtain three different forms of the average scale factors as

$$R = R_0 e^{\frac{2}{\sqrt{m^2 - kc_0}} \tanh^{-1}\left(\frac{kt - m}{m^2 - kc_0}\right)} \; ; \; for \; k > 0 \; and \; m \geq 0 , \tag{9.84}$$

$$R = R_1 \left(mt + c_1\right) \qquad \; ; \; for \; k = 0 \; and \; m > 0 , \tag{9.85}$$

$$R = R_2 e^{c_2 t} \qquad \; ; \; for \; k = 0 \; and \; m = 0 , \tag{9.86}$$

where R_0 , R_1 , R_2 , c_0 , c_1 and c_2 are constants of integration. The last two equations (9.85) and (9.86) are the solutions for constant deceleration parameters. We are not interested in these two solutions but we are interested only on the first solution, which is new. For convenience, in the following we consider the solution for $k > 0$ and $m > o$ and omit the integrating constant c_0 by setting $c_0 = 0$ in (9.84). By doing this, we also set the initial time of the universe to $t = 0$. The reason for considering the solution only for $k > 0$ and $m > o$ is not only for simplicity but also for compatibility with the observed universe. $k > 0$ means we are dealing with increasing acceleration $q = -k < 0$. Because $t = 0$ and $k > 0$, is the only way to shift the deceleration parameter to values higher than -1 is to set $m > 0$. Under the above considerations, the equation (9.84) is further reduces to

$$R = R_0 e^{\frac{2}{m} \tanh^{-1}\left(\frac{kt - m}{m}\right)} \; ; \; for \; k > 0 \; and \; m \geq 0 . \tag{9.87}$$

From equations (9.6) , (9.15) and (9.87) we got

$$A = R_0^{\frac{4n}{3n+1}} e^{\frac{8n}{m(3n+1)} \tanh^{-1}\left(\frac{kt-m}{m}\right)} , \qquad (9.88)$$

$$B = R_0^{\frac{4}{3n+1}} e^{\frac{8}{m(3n+1)} \tanh^{-1}\left(\frac{kt-m}{m}\right)} . \qquad (9.89)$$

With the help of equations (9.88) and (9.89) , the metric (9.3) becomes

$$
\begin{aligned}
ds^2 =& R_0^{\frac{8n}{3n+1}} e^{\frac{16n}{m(3n+1)} \tanh^{-1}\left(\frac{kt-m}{m}\right)} (dx^2 + dy^2 + dz^2) \\
&+ R_0^{\frac{8}{3n+1}} e^{\frac{16}{m(3n+1)} \tanh^{-1}\left(\frac{kt-m}{m}\right)} d\psi^2 - dt^2 .
\end{aligned}
\qquad (9.90)
$$

Equation (9.90) represents a Bianchi type-I cosmological model universe in Lyra geometry with linearly varying deceleration parameter.

Some Physical Properties of the Model (9.90) with linearly varying Deceleration Parameter

The equations (9.6)-(9.10) gives us the Spatial Volume V , Hubble's Parameter H , Expansion scalar θ , Shear scalar σ , mean anisotropy parameter Δ as

$$V = R_0^4 e^{\frac{8}{m} \tanh^{-1}\left(\frac{kt-m}{m}\right)} , \qquad (9.91)$$

$$H = \frac{\dot{R}}{R} = \frac{2}{2mt - kt^2} , \qquad (9.92)$$

$$\theta = \frac{8}{2mt - kt^2} , \qquad (9.93)$$

$$\sigma^2 = \frac{24(n-1)^2}{(3n+1)^2(2mt - kt^2)^2} , \qquad (9.94)$$

$$\Delta = \frac{3(n-1)^2}{(3n+1)^2} = \ constant \ (\ \neq 0 \ for \ n \neq 1) . \qquad (9.95)$$

From equation (9.93) and (9.94) we have

$$\frac{\sigma^2}{\theta^2} = \frac{3(n-1)^2}{8(3n+1)^2} \cdot$$

Therefore

$$\lim_{t\to\infty} \frac{\sigma^2}{\theta^2} = \frac{3(n-1)^2}{8(3n+1)^2} \neq 0 \ \ for \ \ n \neq 1 \,. \tag{9.96}$$

Again from (9.22), (9.11) and (9.13), the displacement vector β, the energy density ρ and the fluid pressure p are obtained as

$$\beta = \beta_0 e^{-\frac{8}{m}\tanh^{-1}\left(\frac{kt-m}{m}\right)} \,, \tag{9.97}$$

$$\rho = -\frac{192n(n+1)}{(3n+1)^2(2mt-kt^2)^2} + \frac{3}{4}\beta_0^2 e^{-\frac{16}{m}\tanh^{-1}\left(\frac{kt-m}{m}\right)} \,, \tag{9.98}$$

$$p = -\frac{48n(kt-m)}{(3n+1)^2(2mt-kt^2)^2} - \frac{3}{4}\beta_0^2 e^{-\frac{16}{m}\tanh^{-1}\left(\frac{kt-m}{m}\right)} \,. \tag{9.99}$$

Variations of the some important parameters with respect to time for Case III are shown in **Figs. 15-17**.
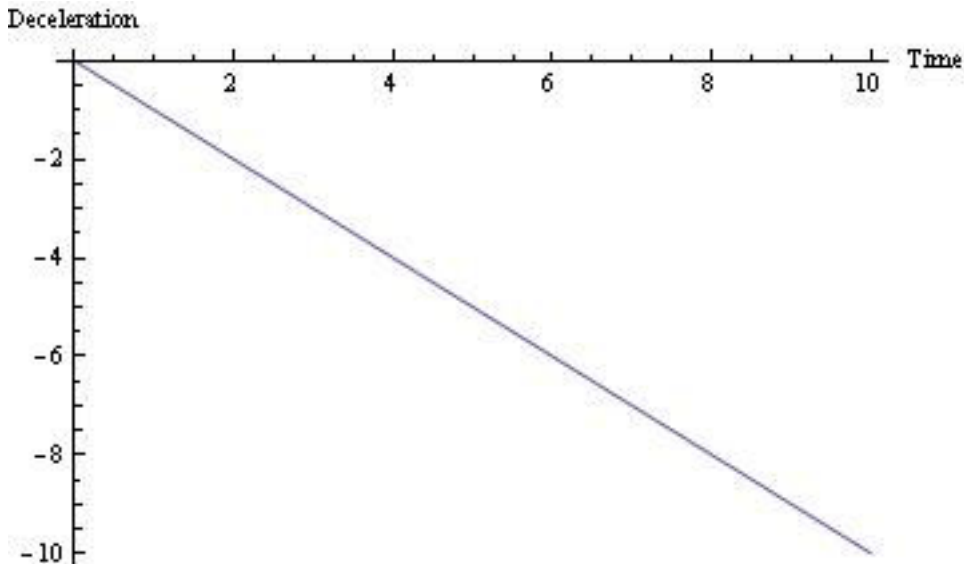

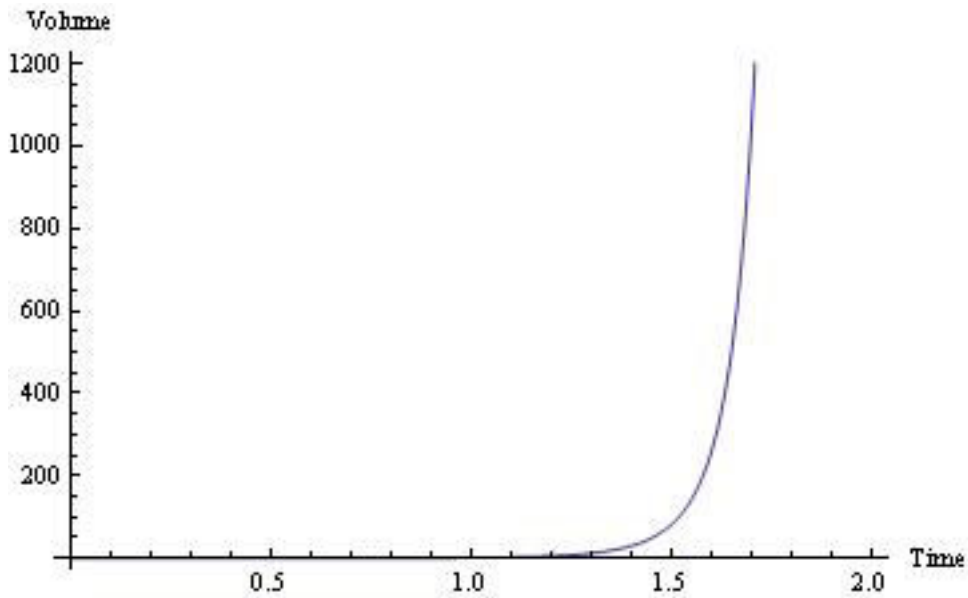**Figure-9.15** : The plot of Deceleration Parameter q vs. Time t. Here $k = 1(k > 0), m = 1(m \geq 0)$.

Figure-9.16 : The plot of Volume V vs. Time t. Here $k = 1(k > 0), m = 1(m \geq 0), n = 2$ and $R_0 = 1$.
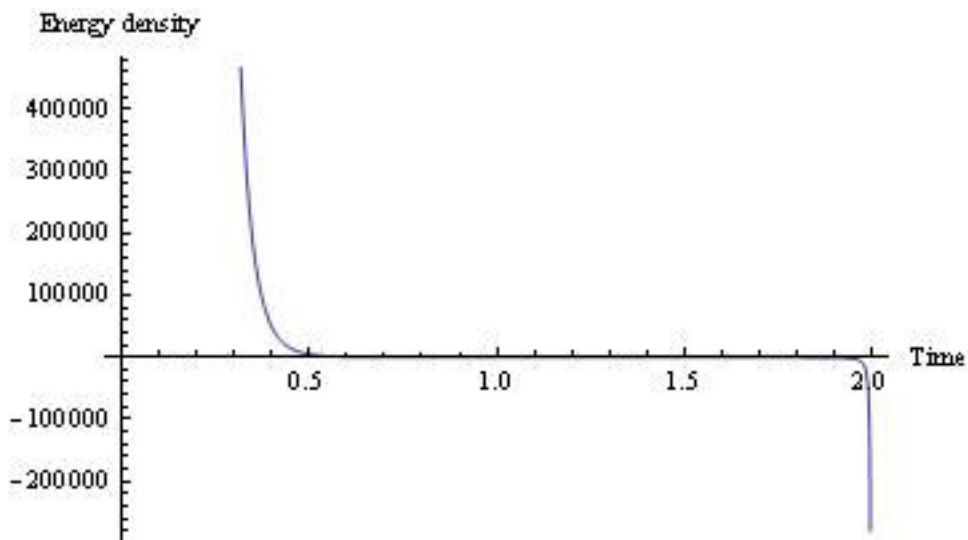

Figure-9.17 : The plot of Volume Energy density ρ vs. Time t. Here $k = 1(k > 0), m = 1(m \geq 0), n = 2$ and $R_0 = 1$.

9.3.4   Case-IV: Models with Special form of Deceleration Parameter:

In this case we have considered a special form of deceleration parameter [Singha and Debnath (2008)] as

$$q = -\frac{R\ddot{R}}{\dot{R}^2} = -1 + \frac{c}{1 + R^c} \ , \tag{9.100}$$

where the average scale factor R is a function of time t and $c > 0$ is a constant.

The equation (9.100) gives us

$$R = \left(d_2 e^{d_1 ct} - 1 \right)^{\frac{1}{c}} \ . \tag{9.101}$$

Therefore from equations (9.6) , (9.15) and (9.101) we have

$$A = \left(d_2 e^{d_1 ct} - 1 \right)^{\frac{4n}{c(3n+1)}} \ , \tag{9.102}$$

$$B = \left(d_2 e^{d_1 ct} - 1 \right)^{\frac{4}{c(3n+1)}} \ . \tag{9.103}$$

Therefore by using equations (9.102) and (9.103) , the metric (9.3) can be obtained as

$$ds^2 = \left(d_2 e^{d_1 ct} - 1 \right)^{\frac{8n}{c(3n+1)}} (dx^2 + dy^2 + dz^2) + \left(d_2 e^{d_1 ct} - 1 \right)^{\frac{8}{c(3n+1)}} d\psi^2 - dt^2 \ . \tag{9.104}$$

Equation (9.104) represents a Bianchi type-I cosmological model universe in Lyra geometry with special form of deceleration parameter.

Some Physical Properties of the Model (9.104) with special form of Deceleration Parameter

The equations (9.6)-(9.10) gives us the Spatial Volume V , Hubble's Parameter H , Expansion scalar θ , Shear scalar σ , mean anisotropy parameter Δ as

$$V = \left(d_2 e^{d_1 ct} - 1 \right)^{\frac{4}{c}} \ , \tag{9.105}$$

$$H = \frac{\dot{R}}{R} = d_1 d_2 \left(d_2 e^{d_1 ct} - 1 \right)^{-1} \ , \tag{9.106}$$

$$\theta = 4d_1 d_2 \left(d_2 e^{d_1 ct} - 1 \right)^{-1} , \tag{9.107}$$

$$\sigma^2 = \frac{6(n-1)^2 d_1^2 d_2^2}{(3n+1)^2} \left(d_2 e^{d_1 ct} - 1 \right)^{-2} , \tag{9.108}$$

$$\Delta = \frac{3(n-1)^2}{(3n+1)^2} = constant \ (\neq 0 \ for \ n \neq 1) . \tag{9.109}$$

From equation (9.107) and (9.108) we have

$$\frac{\sigma^2}{\theta^2} = \frac{3(n-1)^2}{8(3n+1)^2} .$$

Therefore

$$\lim_{t \to \infty} \frac{\sigma^2}{\theta^2} = \frac{3(n-1)^2}{8(3n+1)^2} \neq 0 \ for \ n \neq 1 . \tag{9.110}$$

Again from (9.22), (9.11) and (9.13), the displacement vector β, the energy density ρ and the fluid pressure p are obtained as

$$\beta = \beta_1 \left(d_2 e^{d_1 ct} - 1 \right)^{-\frac{4}{c}} , \tag{9.111}$$

$$\rho = -\frac{48n(n+1)d_1^2 d_2^2}{(3n+1)^2} \left(d_2 - e^{-d_1 ct} \right)^{-2} + \frac{3}{4}\beta_1^2 \left(d_2 e^{d_1 ct - 1} \right)^{-\frac{8}{c}} , \tag{9.112}$$

$$p = -\frac{96n^2 d_1^2 d_2^2}{(3n+1)^2} \left(d_2 - e^{-d_1 ct} \right)^{-2} + \frac{12nd_1^2 d_2 ce^{-d_1 ct}}{3n+1} \left(d_2 - e^{-d_1 ct} \right)^{-2}$$
$$- \frac{3}{4}\beta_1^2 \left(d_2 e^{d_1 ct - 1} \right)^{-\frac{8}{c}} . \tag{9.113}$$

Also from equations (9.100) and (9.101), the deceleration parameter q is found as

$$q = -1 + \frac{1}{d_2} e^{-d_1 ct} . \tag{9.114}$$

Variations of the some important parameters with respect to time for **case IV** are shown in **Figs. 18-20**.
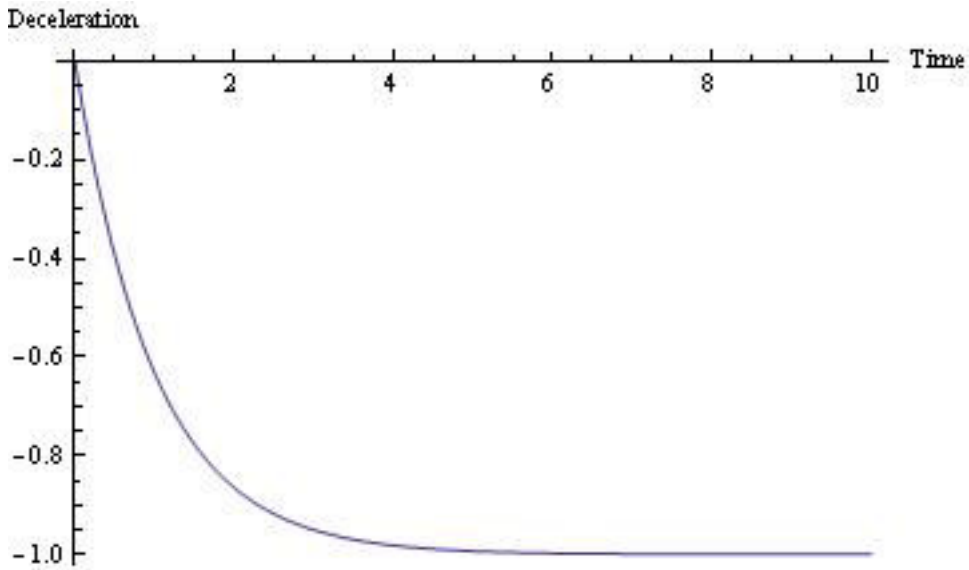
Figure-9.18 : The plot of Deceleration Parameter q vs. Time t. Here $c = d_1 = d_2 = \beta_1 = 1$ and n=2 .
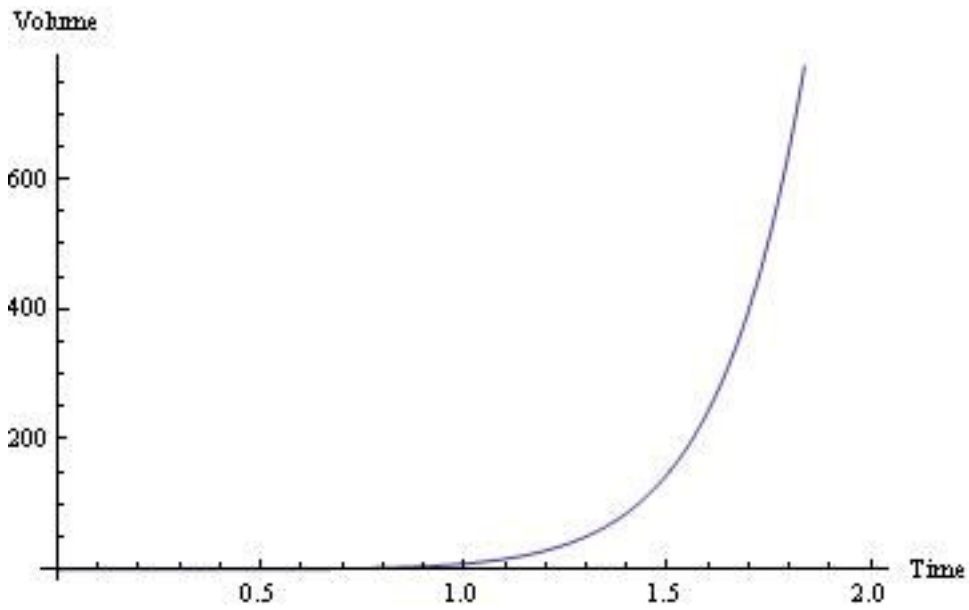

Figure-9.19 : The plot of Volume V vs. Time t. Here $c = d_1 = d_2 = \beta_1 = 1$ and n=2.
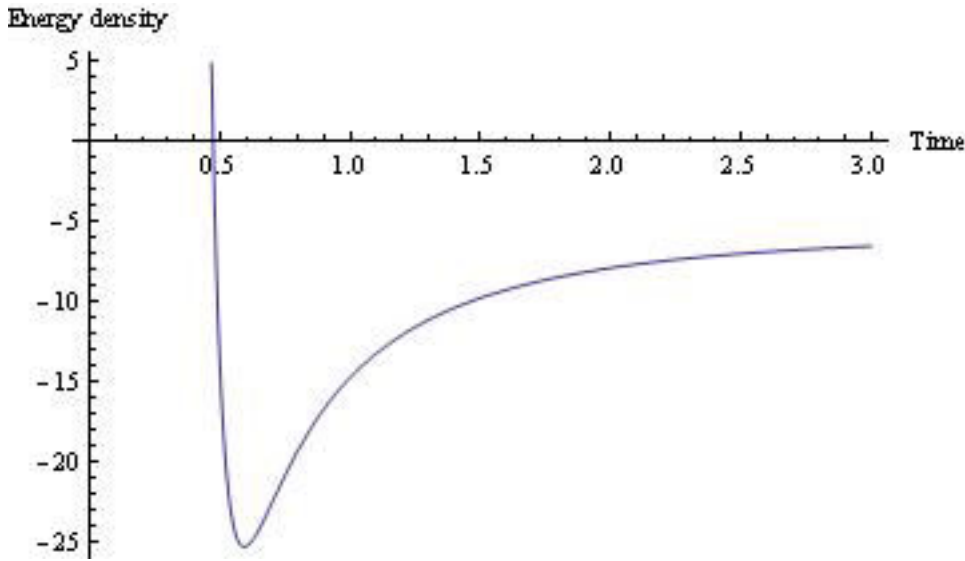
Figure-9.20 : The plot of Energy density ρ vs. Time t. Here $c = d_1 = d_2 = \beta_1 = 1$ and n=2 .

9.4   Physical Interpretations of the Solutions:

9.4.1   Case-I: Models with Time Dependent Deceleration Parameter:

(i) The equation (9.23) gives the displacement vector β as $\beta = (k_0 t^a e^t)^{-\frac{4}{b}}$, which is a decreasing function of cosmic time t and finally it becomes zero when $t \to \infty$.

(ii) From equations (9.24) and (9.25) we have seen that the energy density ρ and pressure p diverge at $t = 0$ but when $t \to \infty$ then the energy density ρ becomes zero whereas the pressure takes the finite value $\frac{12an}{b(3n+1)^2}$.

(iii) Again from equation (9.26) it is clear that initially when $t = 0$ the spatial volume $V = 0$ gradually increases with the passage of time t and it becomes infinite when $t \to \infty$. Also from equations (9.27) and (9.28) we have seen that both Hubble's parameter H and expansion scalar θ are infinite at $t = 0$ and expands with the increase of cosmic time. Thus our model universe is evolves at time $t = 0$ and is expanding that explains the Big-Bang scenario of our model universe.

(iv) The average scale factor given by equation (9.16) is zero at initial epoch of time (t=0), so our model has a point type singularity.

(v) Also from equations (9.17) and (9.27) we observed that at $t \to \infty$ the deceleration

parameter $q = -1$ and $\frac{dH}{dt} = 0$. Thus for $t \to \infty$ the Hubble's parameter H is maximum and the model will be increasing at fastest rate for $t \to \infty$.

(vi) The equations (9.30) and(9.31) give us that the mean anisotropy parameter Δ and $\lim_{t \to \infty} \frac{\sigma^2}{\theta^2}$ are non zero constants for all values of time t whenever $n \neq 1$, so from the mathematical point of view our model will be an anisotropic one. From this result we seem that during the process of evolution of our model universe there may have some possibilities to have an anisotropic universe for some duration. But it does not contradict to the present observational fact that our present universe is an isotropic one, because the initial anisotropy of the Bianchi Type-I universe quickly dies away and eventually it evolves into a FRW universe as suggested by Jacobs (1968). Also, a Bianchi Type-I universe has a different scale factor for each of the three spatial directions, and so anisotropy is introduced to the system automatically representing an anisotropic universe. Departures from isotropy that preserve homogeneity are described by Bianchi model with the overall geometry of space. One may show [Ellis and MacCallum (1969)] that only certain Bianchi types specially type I, V, IX etc allow for isotropic limit. But by conducting general test of isotropy using CMB temperature and polarization data from Plank Saadeeh et al. (2016) strongly disfavoured the anisotropic expansion of the universe with odds of 121,000:1 against. But still some share of anisotropic is there, so we required for further investigation in this area mathematically as well as observationally.

But interestingly when n = 1our model universe approach to an isotropic one which is a good indication favouring the recent observational findings.

(vii) Again from equation (9.17) we have as $t \to \infty$ the value of the deceleration parameter q becomes -1. Also when we plot the deceleration parameter versus time as shown in **Figure-9.1**, it seems that our model universe is decelerating at the initial phase and changing from decelerating to accelerating as time progresses. It means that our model universe might have undergone a transition from early deceleration to late time acceleration at certain point of time supporting the recent observational findings. Such type of situation can be stimulated due to the time varying deceleration parameter which may be positive at the early stage and evolves negative values at the late time. Here, in our problem the deceleration parameter is generated from the scale factor $R = (t^a e^t)^{\frac{1}{b}}$ which have a hybrid form containing the factors of exponential behavior and power law behavior which are widely used in the very common literatures for the investigation of background cosmology. Since the power law factor of this scale factor dominates the early part of cosmic dynamics and the exponential factor part

dominates at late time providing a realistic model universe.

(viii) From **Figure-9.2** of equation (9.26) we have seen that the that the volume V is an increasing function of time t and it becomes infinite as $t \to \infty$. Initially the volume V increases very rapidly but as the time progresses the rate of increase becomes slow.

(ix) The **Figure-9.3** explains that the energy density of our model universe has a finite value at initial epoch of time (t=0), that is at the beginning of evolution; and even when $t \to \infty$ it has some finite value and there is ni singularity; there by there is possibility that our model is that of oscillating type.

9.4.2 Case-II: Models with Variable Deceleration Parameter:

Sub Case-1 of Case-II :

(i) From equation (9.38), it is clear that the average scale factor R(t) can never be negative if $k_2 > 0$. From **Figure-9.4**, we have seen that in the early stages of the universe, i.e. when $t \to 0$ then the scale factor of the universe is almost constant and increasing very slowly as the time progresses. But, after a certain period of time our universe exploded suddenly and starts expansion at large scale, which is consistent with Big Bang scenario.

(ii) The **Figure-9.5** explains that initially the volume V of the universe is constant and it increases with the passage of time and finally the volume becomes infinite as $t \to \infty$.

(iii) From equation (9.48) and its graph i.e. **Figure-9.6**, it has been observed that β is always positive and is a decreasing function of time. In our model, displacement vector β plays the role of cosmological constant and preserves the same character as Λ-term in the Einstein's theory of relativity, in fact with respect to the recent observations.

(iv) In the early stage i.e. When $t \to 0$ then for very very small values of k_1, we have seen from equations (9.49)-(9.50) and their graphs i.e. **Figure-9.7, Figure-9.8** that the energy density is a decreasing function of time and tend to a small positive quantity whereas the pressure p is an increasing function of time and tend to a small negative quantity.

(v) Again, the equation (9.51) give us $q = -1$, so that our cosmological model represents an accelerating universe.

Sub Case-2 of Case-II :

(i) Equations (9.53) and (9.57) shows that both scale factor R(t) and volume V are positive for $0 \leq t < \infty$ if α_1, α_2 and α_3 are positive constants. Also **Figure-9.9** and **Figure-9.10**, explains that both scale factor R(t) and volume V are increasing functions of time, implying that our universe is expanding.

(ii) From **Figure-9.11-a** and equation (9.63), it has been observed that β is positive and is a decreasing function of time and it tends to zero in infinite time. Characteristically β is similar to that of Λ term in Einstein's theory of gravity. In this model β also plays the same role as cosmological constant and preserves the same character as Λ term in Einstein's theory.

(iii) Again, from equations (9.64) and (9.65), we have seen that both energy density and pressure are negative, but whenever $t \rightarrow \infty$ both of them converges to zero.

(iv) From equations (9.61) and (9.62) it is observed that Both mean anisotropic parameter Δ and $\lim_{t \rightarrow \infty} \frac{\sigma^2}{\theta^2}$ are constant and non zero for all values of time t whenever $n \neq 1$ and $n \neq \frac{1}{3}$, so our model remains anisotropic throughout the evolution of the universe. But whenever $n = 1$ then both Δ and $\lim_{t \rightarrow \infty} \frac{\sigma^2}{\theta^2}$ become zero for all values of time t so our model universe will be an isotropic universe for this particular case.

Sub Case-3 of Case-II :

(i) From equations (9.72) and (9.74) it is clear that initially when $t = 0$ the spatial volume $V = 0$ and gradually increases with the passage of time t and it becomes infinite when $t \rightarrow \infty$ whereas the expansion scalar θ is infinite at $t = 0$.

Again at $t \rightarrow \infty$ the equations (9.73) and (9.81) give us $\frac{dH}{dt} = 0$ and deceleration parameter $q = -1$ so the Hubble's parameter H is maximum. Thus at $t \rightarrow \infty$ our model is expanding at fastest rate. Hence our model universe is evolves at time $t = 0$ and is expanding that explains the Big-Bang scenario of our model universe.

(ii) The equation (9.78) shows that the displacement vector $\beta = [\sinh(k_5 t)]^{-\frac{4}{n}}$ is a decreasing function of cosmic time t and finally it becomes zero when $t \rightarrow \infty$.

(iii) From equations (9.79) and (9.80) we have seen that the energy density ρ and pres-

sure p diverge at $t = 0$ and tend to zero at $t \to \infty$.

(iv) The average scale factor given by equation (9.68) and the spatial scale factors given by equations (9.69) and (9.70) are all zero at initial epoch of time $t = 0$, so our model has a point type singularity.

(v) The equations (9.76) and (9.77) give us that the mean anisotropy parameter Δ and $\lim_{t \to \infty} \frac{\sigma^2}{\theta^2}$ are non zero constants for all values of time t whenever $n \neq 1$, so our model remains anisotropic throughout the evolution of the universe and our model universe does not approach to isotropy.

(vi) Again from equation (9.81) and from the graph of deceleration parameter vs. time given by **Figure-9.12**, it is clear that the deceleration parameter q in this case also changes sign from positive to negative, so we can conclude that initially our model universe is decelerating at initial phase and then changes from decelerating to accelerating.

(vii) Also from **Figure-9.14** of equation (9.79) it is seen that initially our model universe has infinite density and as the time progresses it becomes negative and finally energy density tend to a negative finite value. Thus this model universe approaches to a steady state.

Hence all the three model universes in Case-II are consistent with the results of recent cosmological observations.

9.4.3   Case-III: Models with Linearly Varying Deceleration Parameter:

(i) From equations (9.91) and (9.33) it is clear that initially when $t = 0$ the spatial volume V has a constant value $V = R_0^4$ and gradually increases with the passage of time t and it becomes infinite when $t \to \infty$ whereas the expansion scalar θ is a decreasing function of time t which is infinite at $t = 0$ and finally it becomes zero when $t \to \infty$.

The equations (9.92)-(9.94) implies that the Hubble's parameter H, expansion scalar θ and shear scalar σ diverge for $t = \frac{2m}{k}$.

(ii) The model (9.90) does not have any singularity, because the spatial scale factors A and B given by equations (9.88) and (9.89) respectively are non zero for all values of t.

(iii) From equation (9.82) it is seen that initially when $t = 0$ then $q = m - 1$ so that the

model universe is decelerating for $m > 1$ and accelerating for $0 \leq m \leq 1$. But for $t > \frac{m-1}{k}$, $q > 0$ the model universe starts accelerating expansion and whenever $t = \frac{m}{k}$, $q = -1$ implying that the universe experiences super exponential expansion and ends with $q = -m - 1$ at $t = \frac{2m}{k}$.

(iv) From equations (9.95) and (9.96) we have the mean anisotropy parameter $\Delta = \frac{3(n-1)^2}{(3n+1)^2} =$ *constant* $(\neq 0 \ for \ n \neq 1)$ and $\lim_{t \to \infty} \frac{\sigma^2}{\theta^2} = \frac{3(n-1)^2}{8(3n+1)^2} (\neq 0 \ for \ n \neq 1)$ so the model is anisotropic throughout the evolution of the universe except at $m = 1$, hence the model does not approach to isotropy.

(v) From equation (9.97) we have also got that the displacement vector β is a decreasing function of time t and it has a finite value at the initial epoch of time i.e. at $t = 0$.

(vi) The equations (9.98) and (9.99) give that both the energy density $\rho = -\infty$ and pressure $p = -\infty$ for $t = 0$ and they will take finite value $\frac{3}{4}\beta_0^2$ as $t \to \infty$.

(vii) Again from equation (9.82) and **Figure-9.15**, it is clear that the deceleration parameter q changes sign from positive to negative, so we can conclude that initially our model universe is decelerating (at initial phase) and then changes from decelerating to accelerating.

Hence this model universe is also consistent with the result of recent cosmological observations.

9.4.4 Case-IV: Models with Special form of Deceleration Parameter:

(i) From equations (9.105) and (9.107) it is clear that initially when $t = 0$ the spatial volume V has a finite value $V = (d_2 - 1)^{\frac{4}{c}}$ and gradually increases with the passage of time t and it becomes infinite when $t \to \infty$ whereas the expansion scalar θ is a decreasing function of time t which is finite at $t = 0$ and finally it becomes zero when $t \to \infty$.

(ii) The model (9.101) does not have any singularity, because the spatial scale factors A and B given by equations (9.102) and (9.103) respectively are non-zero for all values of t.

(iii) From equations (9.109) and (9.110) we have the mean anisotropy parameter $\Delta = \frac{3(n-1)^2}{(3n+1)^2} = constant$ $(\neq 0 \ for \ n \neq 1)$ and $\lim_{t \to \infty} \frac{\sigma^2}{\theta^2} = \frac{3(n-1)^2}{8(3n+1)^2} (\neq 0 \ for \ n \neq 1)$ so the model is anisotropic throughout the evolution of the universe except at $m = 1$, hence the

model does not approach to isotropy.

(iv) From equation (9.111) we have got that the displacement vector β is a finite quantity at $t = 0$ and become zero as $t \to \infty$.

(v) The equations (9.112) and (9.113) give that both the energy density $\rho = -\infty$ and pressure $p = -\infty$ for $t = 0$ and they will take finite values as $t \to \infty$.

(vi) Again from equation (9.100) it is clear that the deceleration parameter q changes sign from positive to negative, so we can conclude that initially our model universe is decelerating (at initial phase) and then changes from decelerating to accelerating one.

From all the above different cases we find that the energy density and gauge function are positive and decreasing function of time. We also observed the value of deceleration parameter obtained from all the cases of our model universe are in fair agreement with the result of cosmological observations like Type SNeIa supernova, CMB anisotropies, the large scale galaxies structures of universe, Baryon Acoustic Oscillations, WMAP, and new data sets like Planck results, ACT and SPT which have measured the CMB temperature and polarisation anisotropies. From the values of pressure p and critical density ρ we found that our model universe behaves as the dark energy model universe which is accelerated at the late phase of the cosmic dynamics before it might be decelerating at the early phase.

9.5   Conclusion:

It may be a good idea to try to search for the hidden source of the dark energy which dominates the universe with positive energy density and negative pressure, and responsible to produce sufficient acceleration in late time evolution of the Universe. So within the frame work of Lyra Geometry while investigating five dimensional LRS Bianchi type-I model universe with time dependent deceleration parameter interacting with vector field ϕ_i interestingly we found the above model universes behaves as a dark energy model universes which are consistent with the observational findings. Here the displacement field, which is considered as a component of total energy, plays the role of dark energy. In all the cases which we consider here with are expanding and anisotropic through the evolution supporting the present day observational findings. Thus it is seen that our models behaves as a dark energy field universe. From our findings, we seem that Lyra manifold itself contribute to dark energy consistent with the recent cosmological observations. Further study of such type

of universe will be helpful for explaining the present accelerated expansion behavior of the universe.