CHAPTER 2

CHAPTER 2

NEUTROSOPHIC SEMI CONTINUOUS AND NEUTROSOPHIC ALMOST CONTINUOUS MAPPING

To study topological groups and almost topological groups, continuous mapping, semi-continuous mapping, and almost continuous mapping are important. For that, in this chapter, the properties of the NSOS, NSCoS, NROS, NRCoS, NSCM, and NACM are studied.

Definition 2.0.1

Let $\mathcal{A}$ be a NS of NTS $(X, \daleth_{X_{\mathcal{N}}})$, then $\mathcal{A}$ is called a NSOS of X if $\exists$ a $\mathcal{B} \in \daleth_{X_{\mathcal{N}}}$ such that $\mathcal{A} \subseteq \mathcal{N} \sim Cl\Big(\mathcal{N} \sim Int(\mathcal{B})\Big)$.

Definition 2.0.2

Let $\mathcal{A}$ be a NS of NTS $(X, \daleth_{X_{\mathcal{N}}})$, then $\mathcal{A}$ is called a NSCoS of X if $\exists$ a $\mathcal{B}^c \in \daleth_{X_{\mathcal{N}}}$ such that $\mathcal{N} \sim Int\Big(\mathcal{N} \sim Cl(\mathcal{B})\Big) \subseteq \mathcal{A}$.

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Lemma 2.0.1

Let $\phi : X \to Y$ be a mapping and $\mathcal{A}_\alpha$ be a family of NSs of Y, then

(i) $\phi^{-1}(\cup\, \mathcal{A}_\alpha) = \cup\, \phi^{-1}(\mathcal{A}_\alpha)$ *and*

(ii) $\phi^{-1}(\cap\, \mathcal{A}_\alpha) = \cap\, \phi^{-1}(\mathcal{A}_\alpha)$.

The proof is straightforward.

Lemma 2.0.2

Let $\mathcal{A}$ and $\mathcal{B}$ be NSs of X and Y respectively, then

$$1_{X_\mathcal{N}} - \mathcal{A} \times \mathcal{B} = (\mathcal{A}^c \times 1_{X_\mathcal{N}}) \,\cup\, \big(1_{X_\mathcal{N}} \times \mathcal{B}^c\big).$$

Proof

Let (p, q) be any element of $X \times Y$, then

$$\begin{aligned}\big(1_{X_\mathcal{N}} - \mathcal{A} \times \mathcal{B}\big)(p, q) &= max\Big(1_{X_\mathcal{N}} - \mathcal{A}(p), 1_{X_\mathcal{N}} - \mathcal{B}(q)\Big)\\ &= max\Big\{\big(\mathcal{A}^c \times 1_{X_\mathcal{N}}\big)(p, q), \big(\mathcal{B}^c \times 1_{X_\mathcal{N}}\big)(p, q)\Big\}\\ &= \Big\{\big(\mathcal{A}^c \times 1_{X_\mathcal{N}}\big) \,\cup\, \big(1_{X_\mathcal{N}} \times \mathcal{B}^c\big)\Big\}(p, q),\end{aligned}$$

for each $(p, q) \in X \times Y$.

Lemma 2.0.3

Let $\phi_i : X_i \to Y_i$ and $\mathcal{A}_i$ be NSs of $Y_i,\ i = 1, 2$; then

$$\big(\phi_1 \times \phi_2\big)^{-1}\,(\mathcal{A}_1 \times \mathcal{A}_2) = \phi_1^{-1}(\mathcal{A}_1) \times \phi_2^{-1}(\mathcal{A}_2).$$

Proof

For each $(p_1, p_2) \in X_1 \times X_2$, we have

$$\begin{aligned}(\phi_1 \times \phi_2)^{-1}(\mathcal{A}_1 \times \mathcal{A}_2)(p_1, p_2) &= (\mathcal{A}_1 \times \mathcal{A}_2)\Big(\phi_1(p_1), \phi_2(p_1)\Big)\\ &= min\big\{\mathcal{A}_1\phi_1(p_1), \mathcal{A}_2\phi_2(p_2)\big\}\\ &= min\big\{\phi_1^{-1}(\mathcal{A}_1)(p_1), \phi_2^{-1}(\mathcal{A}_2)(p_2)\big\}\\ &= \Big(\phi_1^{-1}(\mathcal{A}_1) \times \phi_2^{-1}(\mathcal{A}_2)\Big)(p_1, p_2).\end{aligned}$$

Lemma 2.0.4

Let $\psi : X \to X \times Y$ be the graph of a mapping $\phi : X \to Y$. Then, if $\mathcal{A}, \mathcal{B}$ be NSs of X and Y, $\psi^{-1}(\mathcal{A} \times \mathcal{B}) = \mathcal{A} \cap \phi^{-1}(\mathcal{B})$.

Proof

For each $p \in X$, we have

$$\begin{aligned}\psi^{-1}(\mathcal{A} \times \mathcal{B})(p) &= (\mathcal{A} \times \mathcal{B})\psi(p) \\ &= (\mathcal{A} \times \mathcal{B})\big(p, \phi(p)\big).\end{aligned}$$

Lemma 2.0.5

For a family $\{\mathcal{A}\}_\alpha$ of NSs of NTS $(X, \daleth_{X_\mathcal{N}})$, $\cup \mathcal{N} \sim Cl(\mathcal{A}_\alpha) \subseteq \mathcal{N} \sim Cl\big(\cup (\mathcal{A}_\alpha)\big)$. In case $\mathcal{B}$ is a finite set, $\cup \mathcal{N} \sim Cl(\mathcal{A}_\alpha) \subseteq \mathcal{N} \sim Cl\big(\cup (\mathcal{A}_\alpha)\big)$. Also, $\cup \mathcal{N} \sim Int(\mathcal{A}_\alpha) \subseteq \mathcal{N} \sim Int\big(\cup (\mathcal{A}_\alpha)\big)$, where a subfamily $\mathcal{B}$ of $(X, \daleth_{X_\mathcal{N}})$ is said to be subbase for $(X, \daleth_{X_\mathcal{N}})$ if the collection of all intersections of members of $\mathcal{B}$ forms a base for $(X, \daleth_{X_\mathcal{N}})$.

Lemma 2.0.6

For a NS $\mathcal{A}$ of NTS $(X, \daleth_{X_\mathcal{N}})$, then

(a) $1_{X_\mathcal{N}} - \mathcal{N} \sim Int(\mathcal{A}) = \mathcal{N} \sim Cl(1_{X_\mathcal{N}} - \mathcal{A})$, *and*

(b) $1_{X_\mathcal{N}} - \mathcal{N} \sim Cl(\mathcal{A}) = \mathcal{N} \sim Int(1_{X_\mathcal{N}} - \mathcal{A})$.

The proof is straightforward.

Theorem 2.0.1

The following statements are equivalent:

(i) $\mathcal{A}$ *is a NSCoS,*

(ii) $\mathcal{A}^c$ *is a NSOS,*

(iii) $\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A})\big) \subseteq \mathcal{A}$, *and*

(iv) $\mathcal{N} \sim Cl\big(\mathcal{N} \sim Int(\mathcal{A}^c)\big) \supseteq \mathcal{A}^c$.

Proof

(i) and (ii) are equivalent follows from Lemma 2.0.6, since for a NS $\mathcal{A}$ of NTS $(X, \urcorner_{X_\mathcal{N}})$ such that $1_{X_\mathcal{N}} - \mathcal{N} \sim Int(\mathcal{A}) = \mathcal{N} \sim Cl(1_{X_\mathcal{N}} - \mathcal{A})$, and $1_{X_\mathcal{N}} - \mathcal{N} \sim Cl(\mathcal{A}) = \mathcal{N} \sim Int(1_{X_\mathcal{N}} - \mathcal{A})$.

$(i) \Rightarrow (iii)$. By definition $\exists$ NCoS $\mathcal{B}$ such that $\mathcal{N} \sim Int(\mathcal{B}) \subseteq \mathcal{A} \subseteq \mathcal{B}$ and hence $\mathcal{N} \sim Int(\mathcal{B}) \subseteq \mathcal{A} \subseteq \mathcal{N} \sim Cl(\mathcal{A}) \subseteq \mathcal{B}$. Since $\mathcal{N} \sim Int(\mathcal{B})$ is the greatest NOS contained in $\mathcal{B}$, we have $\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{B})\big) \subseteq \mathcal{N} \sim Int(\mathcal{B}) \subseteq \mathcal{A}$.

$(iii) \Rightarrow (i)$ follows by taking $\mathcal{B} = \mathcal{N} \sim Cl(\mathcal{A})$.

$(ii) \Leftrightarrow (iv)$ can similarly be proved.

Theorem 2.0.2

(i) Arbitrary union of NSOSs is a NSOS, and

(ii) Arbitrary intersection of NSCoSs is a NSCoS.

Proof

(i) Let $\{\mathcal{A}_\alpha\}$ be a collection of NSOSs of NTS $(X, \urcorner_{X_\mathcal{N}})$. Then $\exists$ a $\mathcal{B}_\alpha \in \urcorner_{X_\mathcal{N}}$ such that $\mathcal{B}_\alpha \subseteq \mathcal{A}_\alpha \subseteq \mathcal{N} \sim Cl(\mathcal{B}_\alpha)$, for each α. Thus, $\cap\, \mathcal{B}_\alpha \subseteq \cup\, \mathcal{A}_\alpha \subseteq \cup\, \mathcal{N} \sim Cl(\mathcal{B}_\alpha) \subseteq \mathcal{N} \sim Cl\big(\cup\ (\mathcal{B}_\alpha)\big)$ [Lemma 2.0.5], and $\cup\, \mathcal{B}_\alpha \in \urcorner_{X_\mathcal{N}}$, this shows that $\cup\, \mathcal{B}_\alpha$ is a NSOS.

(ii) Let $\{\mathcal{A}_\alpha\}$ be a collection of NSCoSs of NTS $(X, \urcorner_{X_\mathcal{N}})$. Then $\exists$ a $\mathcal{B}_\alpha \in \urcorner_{X_\mathcal{N}}$ such that $\mathcal{N} \sim Int(\mathcal{B}_\alpha) \subseteq \mathcal{A}_\alpha \subseteq \mathcal{B}_\alpha$, for each α. Thus, $\mathcal{N} \sim Int\big(\cap\, (\mathcal{B}_\alpha)\big) \subseteq \cap\, \mathcal{N} \sim Int(\mathcal{B}_\alpha) \subseteq \cap\, \mathcal{A}_\alpha \subseteq \cap\, \mathcal{B}_\alpha$ [Lemma 2.0.5], and $\cup\, \mathcal{B}_\alpha \in \urcorner_{X_\mathcal{N}}$, this shows that $\cap\, \mathcal{B}_\alpha$ is a NSCoS.

Remark 2.0.1

It is clear that every NOS (NCoS) is a NSOS (NSCoS). The converse *is false, it is seen in Example 2.0.1. It also shows that the intersection (union) of any two NSOSs (NSCoSs) need not be a NSOS (NSCoS). Even the intersection (union) of a NSOS (NSCoS) with a NOS (NCoS) may fail to be a NSOS (NSCoS). It should be noted that the ordinary topological setting the intersection of a NSOS with an NOS is a NSOS.*

Further, the closure of NOS is a NSOS and the interior of NCoS is a NSCoS.

Example 2.0.1

Let $X = \{a, b\}$ and $\mathcal{A}, \mathcal{B}$ be neutrosophic subsets of X such that

$$\mathcal{A} = \left\{\langle\frac{a}{(0.6,\ 0.3,\ 0.2)}\rangle, \langle\frac{b}{(0.5,\ 0.2,\ 0.3)}\rangle\right\},$$
$$\mathcal{B} = \left\{\langle\frac{a}{(0.5,\ 0.4,\ 0.3)}\rangle, \langle\frac{b}{(0.4,\ 0.2,\ 0.3)}\rangle\right\}.$$

Then $\urcorner_{X_{\mathcal{N}}} = \left\{1_{X_{\mathcal{N}}}, 0_{X_{\mathcal{N}}}, \mathcal{A}, \mathcal{B}, \mathcal{A} \cup \mathcal{B}, \mathcal{A} \cap \mathcal{B}\right\}$ is a NTS on X.
Let $\mathcal{P} = \left\{\langle\frac{a}{(0.8,\ 0.2,\ 0.1)}\rangle, \langle\frac{b}{(0.7,\ 0.2,\ 0.3)}\rangle\right\}$ be any neutrosophic set $\urcorner_{X_{\mathcal{N}}}$, then $\mathcal{N} \sim Int(\mathcal{P}) = \cup \left\{G : G\ is\ open\ set,\ G \subseteq \mathcal{P}\right\} = \mathcal{A} \cup \mathcal{B} = \mathcal{A}$ and $\mathcal{N} \sim Cl(\mathcal{P}) = \cap \left\{K \supseteq \mathcal{P} : K\ is\ closed\ set\ in\ \urcorner_{X_{\mathcal{N}}}\right\} = 1_{X_{\mathcal{N}}}$. Therefore, $\mathcal{P}$ is a NSOS, which is not a NOS and also by Theorem 2.0.1, $\mathcal{P}^c$ is a NSCoS, which is not an NCS.

Theorem 2.0.3

If $(X, \urcorner_{X_{\mathcal{N}}})$ and $(Y, \urcorner_{Y_{\mathcal{N}}})$ are NTSs. Then the product $\mathcal{A} \times \mathcal{B}$ of a NSOS $\mathcal{A}$ of X and a NSOS $\mathcal{B}$ of Y is NSOS of the neutrosophic product space $X \times Y$.

Proof

Let $\mathcal{P} \subseteq \mathcal{A} \subseteq \mathcal{N} \sim Cl(\mathcal{P})$ and $\mathcal{Q} \subseteq \mathcal{B} \subseteq \mathcal{N} \sim Cl(\mathcal{Q})$ where $\mathcal{P} \in \urcorner_{X_{\mathcal{N}}}$ and $\mathcal{Q} \in \urcorner_{Y_{\mathcal{N}}}$. Then $\mathcal{P} \times \mathcal{Q} \subseteq \mathcal{A} \times \mathcal{B} \subseteq \mathcal{N} \sim Cl(\mathcal{P}) \times \mathcal{N} \sim Cl(\mathcal{Q})$. For NSs $\mathcal{P}$'s of X and $\mathcal{Q}$'s of Y, we have

(a) $\inf\{\mathcal{P}, \mathcal{Q}\} = \min\{\inf \mathcal{P}, \inf \mathcal{Q}\}$,

(b) $\inf\{\mathcal{P} \times 1_{X_{\mathcal{N}}}\} = (\inf \mathcal{P}) \times 1_{X_{\mathcal{N}}}$, and

(c) $\inf\{1_{X_{\mathcal{N}}} \times \mathcal{Q}\} = 1_{X_{\mathcal{N}}} \times (\inf \mathcal{Q})$.

It is sufficient to prove that $\mathcal{N} \sim Cl(\mathcal{A} \times \mathcal{B}) \supseteq \mathcal{N} \sim Cl(\mathcal{A}) \times \mathcal{N} \sim Cl(\mathcal{B})$. Let $\mathcal{P} \in \daleth_{X_{\mathcal{N}}}$ and $\mathcal{Q} \in \daleth_{Y_{\mathcal{N}}}$. Then

$$\begin{aligned}
\mathcal{N} \sim Cl(\mathcal{A} \times \mathcal{B}) &= \inf\{(\mathcal{P} \times \mathcal{Q})^c | (\mathcal{P} \times \mathcal{Q})^c \supseteq \mathcal{A} \times \mathcal{B}\} \\
&= \inf\{(\mathcal{P}^c \times 1_{X_{\mathcal{N}}}) \cup (1_{X_{\mathcal{N}}} \times \mathcal{Q}^c) | (\mathcal{P}^c \times 1_{X_{\mathcal{N}}}) \cup (1_{X_{\mathcal{N}}} \\
&\quad \times \mathcal{Q}^c) \supseteq \mathcal{A} \times \mathcal{B}\} \\
&= \inf\{(\mathcal{P}^c \times 1_{X_{\mathcal{N}}}) \cup (1_{X_{\mathcal{N}}} \times \mathcal{Q}^c) | \mathcal{P}^c \supseteq \mathcal{A} \text{ or } \mathcal{Q}^c \supseteq \mathcal{B}\} \\
&= \min \begin{bmatrix} \inf\left\{(\mathcal{P}^c \times 1_{X_{\mathcal{N}}}) \cup (1_{X_{\mathcal{N}}} \times \mathcal{Q}^c) | \mathcal{P}^c \supseteq \mathcal{A}\right\}, \\ \inf\left\{(\mathcal{P}^c \times 1_{X_{\mathcal{N}}}) \cup (1_{X_{\mathcal{N}}} \times \mathcal{Q}^c) | \mathcal{Q}^c \supseteq \mathcal{B}\right\} \end{bmatrix}
\end{aligned}$$

Since, $\inf\left\{(\mathcal{P}^c \times 1_{X_{\mathcal{N}}}) \cup (1_{X_{\mathcal{N}}} \times \mathcal{Q}^c) | P^c \supseteq \mathcal{A}\right\}$

$$\begin{aligned}
&\supseteq \inf\left\{(\mathcal{P}^c \times 1_{X_{\mathcal{N}}}) | \mathcal{P}^c \supseteq \mathcal{A}\right\} \\
&= \inf\left\{\mathcal{P}^c | \mathcal{P}^c \supseteq \mathcal{A}\right\} \times 1_{X_{\mathcal{N}}} \\
&= \mathcal{N} \sim Cl(\mathcal{A}) \times 1_{X_{\mathcal{N}}}
\end{aligned}$$

and $\inf\left\{(\mathcal{P}^c \times 1_{X_{\mathcal{N}}}) \cup (1_{X_{\mathcal{N}}} \times \mathcal{Q}^c) | \mathcal{Q}^c \supseteq \mathcal{B}\right\}$

$$\begin{aligned}
&\supseteq \inf\left\{(1_{X_{\mathcal{N}}} \times \mathcal{Q}^c) | \mathcal{Q}^c \supseteq \mathcal{B}\right\} \\
&= 1_{X_{\mathcal{N}}} \times \inf\left\{\mathcal{Q}^c | \mathcal{Q}^c \supseteq \mathcal{B}\right\} \\
&= 1_{X_{\mathcal{N}}} \times \mathcal{N} \sim Cl(\mathcal{B})
\end{aligned}$$

we have,

$$\begin{aligned}
\mathcal{N} \sim Cl(\mathcal{A} \times \mathcal{B}) &\supseteq min\left\{\mathcal{N} \sim Cl(\mathcal{A}) \times 1_{X_{\mathcal{N}}}, 1_{X_{\mathcal{N}}} \times \mathcal{N} \sim Cl(\mathcal{B})\right\} \\
&= \mathcal{N} \sim Cl(\mathcal{A}) \times \mathcal{N} \sim Cl(\mathcal{B})\}. \text{ Hence the result.}
\end{aligned}$$

Definition 2.0.3

A NS $\mathcal{A}$ of NTS $(X, \urcorner_{X_{\mathcal{N}}})$ is called a NROS of X if $\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A})\big) = \mathcal{A}$.

Definition 2.0.4

A NS $\mathcal{A}$ of NTS $(X, \urcorner_{X_{\mathcal{N}}})$ is called a NRCoS of X if $\mathcal{N} \sim Cl\big(\mathcal{N} \sim Int(\mathcal{A})\big) = \mathcal{A}$.

Theorem 2.0.4

A NS $\mathcal{A}$ of NTS $(X, \urcorner_{X_{\mathcal{N}}})$ is a NROS iff $\mathcal{A}^c$ is NRCoS.

The proof follows from Lemma 2.0.6.

Remark 2.0.2

It is obvious that every NROS (NRCoS) is NOS (NCoS). The converse need not be true. For this we cite an example-

Example 2.0.2

From Example 2.0.1, it is clear that $\mathcal{A}$ is NOS. Now $\mathcal{N} \sim Cl(\mathcal{A}) = 1_{X_{\mathcal{N}}}$ and $\mathcal{N} \sim Int(\mathcal{N} \sim Cl(\mathcal{A})) = 1_{X_{\mathcal{N}}}$. Therefore, $\mathcal{N} \sim Int(\mathcal{N} \sim Cl(\mathcal{A})) \neq \mathcal{A}$, hence$\mathcal{A}$ is not NROS.

Remark 2.0.3

The union (intersection) of any two NROSs (NRCoS) need not be a NROS (NRCoS).

Example 2.0.3

Let $X = \{a, b, c\}$ and $\urcorner_{X_{\mathcal{N}}} = \{0_{X_{\mathcal{N}}}, 1_{X_{\mathcal{N}}}, \mathcal{A}, \mathcal{B}, \mathcal{C}\}$ be NTS on X, where

$$\mathcal{A} = \left\{ \left\langle \frac{a}{(0.4,\ 0.5,\ 0.6)} \right\rangle, \left\langle \frac{b}{(0.7,\ 0.5,\ 0.3)} \right\rangle, \left\langle \frac{c}{(0.5,\ 0.5,\ 0.5)} \right\rangle \right\},$$
$$\mathcal{B} = \left\{ \left\langle \frac{a}{(0.6,\ 0.5,\ 0.4)} \right\rangle, \left\langle \frac{b}{(0.3,\ 0.5,\ 0.7)} \right\rangle, \left\langle \frac{c}{(0.5,\ 0.5,\ 0.5)} \right\rangle \right\},$$
$$\mathcal{C} = \left\{ \left\langle \frac{a}{(0.6,\ 0.5,\ 0.4)} \right\rangle, \left\langle \frac{b}{(0.7,\ 0.5,\ 0.3)} \right\rangle, \left\langle \frac{c}{(0.5,\ 0.5,\ 0.5)} \right\rangle \right\}.$$

Then $\mathcal{N} \sim Cl(\mathcal{A}) = \mathcal{B}^c, \mathcal{N} \sim Int(\mathcal{B}^c) = \mathcal{A}$.
Clearly, $\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A})\big) = \mathcal{A}$.
Similarly, $\mathcal{N} \sim Int\big(Cl(\mathcal{B})\big) = \mathcal{B}$.
Now, $\mathcal{A} \cup \mathcal{B} = \mathcal{C}$.
But $\mathcal{N} \sim Cl(\mathcal{A} \cup \mathcal{B}) = 1_{X_\mathcal{N}}$ *and* $\mathcal{N} \sim Int\big(Cl(\mathcal{A} \cup \mathcal{B})\big) = 1_{X_\mathcal{N}}$.
Hence, $\mathcal{A}$ *and* $\mathcal{B}$ *are two NROSs but* $\mathcal{A} \cup \mathcal{B}$ *is not NROS.*

Theorem 2.0.5

(i) The intersection of any two NROSs is a NROS, and

(ii) The union of any two NRCoSs is a NRCoS.

Proof

(i) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be any two NROSs of NTS $(X, \daleth_{X_\mathcal{N}})$. Since $\mathcal{A}_1 \cap \mathcal{A}_2$ is NOS [from Remark 2.0.2], we have $\mathcal{A}_1 \cap \mathcal{A}_2 \subseteq \mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A}_1 \cap \mathcal{A}_2)\big)$.
Now, $\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A}_1 \cap \mathcal{A}_2)\big) \subseteq \mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A}_1)\big) = \mathcal{A}_1$ and $\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A}_1 \cap \mathcal{A}_2)\big) \subseteq \mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A}_2)\big) = \mathcal{A}_2$ implies that $\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A}_1 \cap \mathcal{A}_2)\big) \subseteq \mathcal{A}_1 \cap \mathcal{A}_2$. Hence the theorem.

(ii) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be any two NROSs of NTS $(X, \daleth_{X_\mathcal{N}})$. Since $\mathcal{A}_1 \cup \mathcal{A}_2$ is NOS [from Remark 2.0.2], we have $\mathcal{A}_1 \cup \mathcal{A}_2 \supseteq \mathcal{N} \sim Cl\big(\mathcal{N} \sim Int(\mathcal{A}_1 \cup \mathcal{A}_2)\big)$.
Now, $\mathcal{N} \sim Cl\big(\mathcal{N} \sim Int(\mathcal{A}_1 \cup \mathcal{A}_2)\big) \supseteq \mathcal{N} \sim Cl\big(\mathcal{N} \sim Int(\mathcal{A}_1)\big) = \mathcal{A}_1$ and $\mathcal{N} \sim Cl\big(\mathcal{N} \sim Int(\mathcal{A}_1 \cup \mathcal{A}_2)\big) \supseteq \mathcal{N} \sim Cl\big(\mathcal{N} \sim Int(\mathcal{A}_2)\big) = \mathcal{A}_2$ implies that $\mathcal{A}_1 \cup \mathcal{A}_2 \subseteq \mathcal{N} \sim Cl\big(\mathcal{N} \sim Int(\mathcal{A}_1 \cup \mathcal{A}_2)\big)$. Hence the theorem.

Theorem 2.0.6

(i) The closure of a NOS is NRCoS, and

(ii) The interior of a NCoS is NROS.

Proof

(i) Let $\mathcal{A}$ be a NOS of NTS $(X, \urcorner_{X_{\mathcal{N}}})$, clearly, $\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A})\big) \subseteq \mathcal{N} \sim Cl(\mathcal{A}) \Rightarrow \mathcal{N} \sim Cl\big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A})\big)\big) \subseteq \mathcal{N} \sim Cl(\mathcal{A})$.
Now, $\mathcal{A}$ is NOS implies that $\mathcal{A} \subseteq \mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A})\big)$ and hence $\mathcal{N} \sim Cl(\mathcal{A}) \subseteq \mathcal{N} \sim Cl\big(N \sim Int\big(N \sim Cl(\mathcal{A})\big)\big)$. Thus, $\mathcal{N} \sim Cl(\mathcal{A})$ is NRCoS.

(ii) Let $\mathcal{A}$ be a NCoS of a NTS $(X, \urcorner_{X_{\mathcal{N}}})$, clearly, $\mathcal{N} \sim Cl\big(\mathcal{N} \sim Int(\mathcal{A})\big) \supseteq \mathcal{N} \sim Int(\mathcal{A}) \Rightarrow \mathcal{N} \sim Int\big(\mathcal{N} \sim Cl\big(\mathcal{N} \sim Int(\mathcal{A})\big)\big) \supseteq \mathcal{N} \sim Int(\mathcal{A})$.
Now, $\mathcal{A}$ is NCoS implies that $\mathcal{A} \supseteq \mathcal{N} \sim Cl\big(\mathcal{N} \sim Int(\mathcal{A})\big)$ and hence $\mathcal{N} \sim Int(\mathcal{A}) \supseteq \mathcal{N} \sim Int\big(\mathcal{N} \sim Cl\big(\mathcal{N} \sim Int(\mathcal{A})\big)\big)$. Thus, $\mathcal{N} \sim Int(\mathcal{A})$ is NROS.

Definition 2.0.5

Let $\phi : (X, \urcorner_{X_{\mathcal{N}}}) \rightarrow (Y, \urcorner_{Y_{\mathcal{N}}})$ be a mapping from NTS $(X, \urcorner_{X_{\mathcal{N}}})$ to another NTS $(Y, \urcorner_{X_{\mathcal{N}}})$, then ϕ is called a NCM, if $\phi^{-1}(\mathcal{A}) \in \urcorner_{X_{\mathcal{N}}}$ for each $\mathcal{A} \in \urcorner_{X_{\mathcal{N}}}$; or equivalently $\phi^{-1}(\mathcal{B})$ is a NCoS of X for each NCoS $\mathcal{B}$ of Y.

Definition 2.0.6

Let $\phi : (X, \urcorner_{X_{\mathcal{N}}}) \rightarrow (Y, \urcorner_{Y_{\mathcal{N}}})$ be a mapping from NTS $(X, \urcorner_{X_{\mathcal{N}}})$ to another NTS $(Y, \urcorner_{X_{\mathcal{N}}})$, then ϕ is said to be a NOM, if $\phi(\mathcal{A}) \in \urcorner_{Y_{\mathcal{N}}}$ for each $\mathcal{A} \in \urcorner_{X_{\mathcal{N}}}$.

Definition 2.0.7

Let $\phi : (X, \urcorner_{X_{\mathcal{N}}}) \rightarrow (Y, \urcorner_{Y_{\mathcal{N}}})$ be a mapping from NTS $(X, \urcorner_{X_{\mathcal{N}}})$ to another NTS $(Y, \urcorner_{X_{\mathcal{N}}})$, then ϕ is said to be a NCoM, if $\phi(\mathcal{B})$ is a NCoS of Y for each NCoS $\mathcal{B}$ of X.

Definition 2.0.8

Let $\phi : (X, \urcorner_{X_{\mathcal{N}}}) \to (Y, \urcorner_{Y_{\mathcal{N}}})$ *be a mapping from NTS* $(X, \urcorner_{X_{\mathcal{N}}})$ *to another NTS* $(Y, \urcorner_{X_{\mathcal{N}}})$*, then* ϕ *is said to be a NSCM, if* $\phi^{-1}(\mathcal{A})$ *is a NSOS of* X*, for each* $\mathcal{A} \in \urcorner_{Y_{\mathcal{N}}}$*.*

Definition 2.0.9

Let $\phi : (X, \urcorner_{X_{\mathcal{N}}}) \to (Y, \urcorner_{Y_{\mathcal{N}}})$ *be a mapping from NTS* $(X, \urcorner_{X_{\mathcal{N}}})$ *to another NTS* $(Y, \urcorner_{X_{\mathcal{N}}})$*, then* ϕ *is said to be a NSOM, if* $\phi(\mathcal{A})$ *is a NSOS for each* $\mathcal{A} \in \urcorner_{X_{\mathcal{N}}}$*.*

Definition 2.0.10

Let $\phi : (X, \urcorner_{X_{\mathcal{N}}}) \to (Y, \urcorner_{Y_{\mathcal{N}}})$ *be a mapping from NTS* $(X, \urcorner_{X_{\mathcal{N}}})$ *to another NTS* $(Y, \urcorner_{Y_{\mathcal{N}}})$*, then* ϕ *is said to be a NSCoM, if* $\phi(\mathcal{B})$ *is a NSCoS for each NCoS* $\mathcal{B}$ *of* X*.*

Remark 2.0.4

From Remark 2.0.1, a NCM (NOM, NCoM) is also a NSCM (NSOM, NSCoM). But the converse is not true.

Example 2.0.4

Let $X = \{a, b\}, Y = \{x, y\}$*, and*

$$\begin{aligned}
\mathcal{A} &= \left\{ \langle \frac{a}{(0.6,\ 0.3,\ 0.2)} \rangle, \langle \frac{b}{(0.5,\ 0.2,\ 0.3)} \rangle \right\}, \\
\mathcal{B} &= \left\{ \langle \frac{x}{(0.5,\ 0.4,\ 0.3)} \rangle, \langle \frac{y}{(0.4,\ 0.2,\ 0.3)} \rangle \right\}, \\
\mathcal{C} &= \left\{ \langle \frac{x}{(0.8,\ 0.2,\ 0.1)} \rangle, \langle \frac{y}{(0.7,\ 0.2,\ 0.3)} \rangle \right\}.
\end{aligned}$$

Then $\urcorner_{X_{\mathcal{N}}} = \{0_{X_{\mathcal{N}}}, 1_{X_{\mathcal{N}}}, \mathcal{A}\}$ *and* $\urcorner_{Y_{\mathcal{N}}} = \{0_{X_{\mathcal{N}}}, 1_{X_{\mathcal{N}}}, \mathcal{B}, \mathcal{C}\}$ *are NTSs on* X *and* Y*.*

Let $\phi : (X, \urcorner_{X_{\mathcal{N}}}) \to (Y, \urcorner_{Y_{\mathcal{N}}})$ *be a mapping defined as* $\phi(a) = y, \phi(b) = x$*. Then* $\phi : (X, \urcorner_{X_{\mathcal{N}}}) \to (Y, \urcorner_{Y_{\mathcal{N}}})$ *is NSCM but not NCM.*

Theorem 2.0.7

Let X_1, X_2, Y_1 *and* Y_2 *be NTSs such that* X_1 *is product related to* X_2*.*

Then, the product $\phi_1 \times \phi_2 : X_1 \times X_2 \to Y_1 \times Y_2$ of NSCMs $\phi_1 : X_1 \to Y_1$ and $\phi_2 : X_2 \to Y_2$ is NSCM.

Proof

Let $\mathcal{A} \equiv \cup(\mathcal{A}_\alpha \times \mathcal{B}_\beta)$, where $\mathcal{A}_\alpha$'s and $\mathcal{B}_\beta$'s are NOSs of Y_1 and Y_2 respectively, be a NOS of $Y_1 \times Y_2$.
By using Lemma 2.0.1 (i) and Lemma 2.0.3, we have

$$(\phi_1 \times \phi_2)^{-1}(\mathcal{A}) = \cup[\phi_1^{-1}(\mathcal{A}_\alpha) \times \phi_2^{-1}(\mathcal{A}_\beta)].$$

That $(\phi_1 \times \phi_2)^{-1}(\mathcal{A})$ is a NSOS follows from Theorem 2.0.3 and Theorem 2.0.2 (i).

Theorem 2.0.8

Let X, X_1 and X_2 be NTSs and $p_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection of $X_1 \times X_2$ onto X_i. Then, if $\phi : X \to X_1 \times X_2$ is a NSCM, $p_i\phi$ is also NSCM.

Proof

For a NOS $\mathcal{A}$ of X_i, we have $(p_i\phi)^{-1}(\mathcal{A}) = \phi^{-1}\big(p_i^{-1}(\mathcal{A})\big)$. That p_i is a NCM and ϕ is a NSCM imply that $(p_i\phi)^{-1}(\mathcal{A})$ is a NSOS of X.

Theorem 2.0.9

Let $\phi : X \to Y$ be a mapping from NTS X to another NTS Y. Then if the graph $\psi : X \to X \times Y$ of ϕ is NSCM, then ϕ is also NSCM.

Proof

From Lemma 2.0.4, we have $\phi^{-1}(\mathcal{A}) = 1_{X_\mathcal{N}} \cap \phi^{-1}(\mathcal{A}) = \psi^{-1}(1_{X_\mathcal{N}} \times \mathcal{A})$, for each NOS $\mathcal{A}$ of Y. Since ψ is a NSCM and $1_{X_\mathcal{N}} \times \mathcal{A}$ is a NOS $X \times Y$, $\phi^{-1}(\mathcal{A})$ is a NSOS of X and hence ϕ is a NSCM.

Remark 2.0.5

The converse of Theorem 2.0.9 is not true.

Definition 2.0.11

A mapping $\phi : (X, \daleth_{X_\mathcal{N}}) \to (Y, \daleth_{Y_\mathcal{N}})$ *from NTS* X *to another NTS* Y *is said to be a NACM, if* $\phi^{-1}(\mathcal{A}) \in \daleth_{X_N}$ *for each NROS* $\mathcal{A}$ *of* Y.

Theorem 2.0.10

Let $\phi : (X, \daleth_{X_\mathcal{N}}) \to (Y, \daleth_{Y_\mathcal{N}})$ *be a mapping. Then the following statements are equivalent:*

(a) ϕ *is a NACM,*

(b) $\phi^{-1}(\mathcal{F})$ *is a NCoS, for each NRCoS* F *of* Y*,*

(c) $\phi^{-1}(\mathcal{A}) \subseteq \mathcal{N} \sim Int\Big(\phi^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A})\big)\Big)\Big)$ *, for each NOS* $\mathcal{A}$ *of* Y*,*

(d) $\mathcal{N} \sim Cl\Big(\phi^{-1}\Big(\mathcal{N} \sim Cl\big(\mathcal{N} \sim Int(\mathcal{N})\big)\Big)\Big) \subseteq \phi^{-1}(\mathcal{N})$*, for each NCoS* F *of* Y.

Proof

Consider that $\phi^{-1}(\mathcal{A}^c) = (\phi^{-1}(\mathcal{A}))^c$, for any NS $\mathcal{A}$ of Y, $(a) \Leftrightarrow (b)$ follows from Theorem 2.0.4.

$(a) \Rightarrow (c)$. Since $\mathcal{A}$ is a NOS of Y, $\mathcal{A} \subseteq \mathcal{N} \sim Int\big(Cl(\mathcal{A})\big)$ and hence $\phi^{-1}(\mathcal{A}) \subseteq \phi^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A})\big)\Big)$. From Theorem 2.0.6 (ii), $\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A})\big)$ is a NROS of Y, hence $\phi^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A})\big)\Big)$ is a NOS of X. Thus, $\phi^{-1}(\mathcal{A}) \subseteq \phi^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A})\big)\Big) = \mathcal{N} \sim Int\Big(\phi^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A})\big)\Big)\Big)$.

$(c) \Rightarrow (a)$. Let $\mathcal{A}$ be a NROS of Y, then we have $\phi^{-1}(\mathcal{A}) \subseteq \mathcal{N} \sim Int\Big(\phi^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A})\big)\Big)\Big) = \mathcal{N} \sim Int\big(\phi^{-1}(\mathcal{A})\big)$. Thus, $\phi^{-1}(\mathcal{A}) = \mathcal{N} \sim Int\big(\phi^{-1}(\mathcal{A})\big)$. This shows that $\phi^{-1}(\mathcal{A})$ is a NOS of X.

$(b) \Leftrightarrow (d)$ similarly can be proved.

Remark 2.0.6

Clearly, a NCM is NACM. But the converse needs not be true.

Example 2.0.5

Let $X = \{a, b\}, Y = \{x, y\}$*, and*

$$\mathcal{A} = \left\{ \langle \frac{a}{(0.6,\ 0.5,\ 0.3)} \rangle, \langle \frac{b}{(0.4,\ 0.5,\ 0.5)} \rangle \right\},$$
$$\mathcal{B} = \left\{ \langle \frac{a}{(0.2,\ 0.5,\ 0.7)} \rangle, \langle \frac{b}{(0.4,\ 0.5,\ 0.5)} \rangle \right\},$$
$$\mathcal{C} = \left\{ \langle \frac{x}{(0.6,\ 0.5,\ 0.3)} \rangle, \langle \frac{y}{(0.4,\ 0.5,\ 0.5)} \rangle \right\},$$
$$\mathcal{D} = \left\{ \langle \frac{x}{(0.2,\ 0.5,\ 0.7)} \rangle, \langle \frac{y}{(0.4,\ 0.5,\ 0.5)} \rangle \right\},$$
$$\mathcal{E} = \left\{ \langle \frac{x}{(0.2,\ 0.5,\ 0.5)} \rangle, \langle \frac{b}{(0.3,\ 0.5,\ 0.7)} \rangle \right\}.$$

Then $\urcorner_{X_{\mathcal{N}}} = \{0_{X_{\mathcal{N}}}, 1_{X_{\mathcal{N}}}, \mathcal{A}, \mathcal{B}\}$ *and* $\urcorner_{Y_{\mathcal{N}}} = \{0_{X_{\mathcal{N}}}, 1_{X_{\mathcal{N}}}, \mathcal{C}, \mathcal{D}, \mathcal{E}\}$ *are NTSs on* X *and* Y.
Now, let $\phi : (X, \urcorner_{X_{\mathcal{N}}}) \to (Y, \urcorner_{Y_{\mathcal{N}}})$ *be a mapping defined as* $\phi(a) = y, \phi(b) = x$ *and clearly,* ϕ *is NACM.*
Hence, $0_{X_{\mathcal{N}}}, 1_{X_{\mathcal{N}}}, \mathcal{C}, \mathcal{D}$ *are NOSs in* $\urcorner_{Y_{\mathcal{N}}}$ *but* $\phi^{-1}(\mathcal{E})$ *is not NOS in* $\urcorner_{X_{\mathcal{N}}}$ *and hence NACM is not NCM.*

Theorem 2.0.11

Neutrosophic semi-continuity and neutrosophic almost continuity are independent notions.

The proof is straightforward.

Definition 2.0.12

A NTS $(X, \urcorner_{X_{\mathcal{N}}})$ *is said to be a NSRS iff the collection of all NROSs of X forms a base for NT* $\urcorner_{X_{\mathcal{N}}}$.

Theorem 2.0.12

Let $\phi : (X, \urcorner_{X_{\mathcal{N}}}) \to (Y, \urcorner_{Y_{\mathcal{N}}})$ *be a mapping from NTS* X *to a NSRS* Y.
Then ϕ *is NACM iff* ϕ *is NCM.*

Proof

From Remark 2.0.6, it suffices to prove that if ϕ is NACM then it is NCM. Let $\mathcal{A} \in \daleth_{Y_{\mathcal{N}}}$, then $\mathcal{A} = \cup \mathcal{A}_\alpha$, where $\mathcal{A}_\alpha$'s are NROSs of Y. Now, from Lemma 2.0.1(i), 2.0.5 and Theorem 2.0.10 (c), we get

$$\begin{aligned}
\phi^{-1}(\mathcal{A}) &= \cup\, \phi^{-1}\mathcal{A}_\alpha \\
&\subseteq \cup\, \mathcal{N} \sim Int\Big(\phi^{-1}\big(\mathcal{N} \sim Cl(\mathcal{A}_\alpha)\big)\Big) \\
&= \cup\, \mathcal{N} \sim Int(\phi^{-1}(\mathcal{A}_\alpha)) \\
&\subseteq \mathcal{N} \sim Int \cup \big(\phi^{-1})(\mathcal{A}_\alpha)\big) \\
&= \mathcal{N} \sim Int\big(\phi^{-1}(\mathcal{A}_\alpha)\big).
\end{aligned}$$

which shows that $\phi^{-1}(\mathcal{A}_\alpha) \in \daleth_{X_{\mathcal{N}}}$.

Theorem 2.0.13

Let X_1, X_2, Y_1 and Y_2 be the NTSs such that Y_1 is product related to Y_2. Then the product $\phi_1 \times \phi_2 : X_1 \times X_2 \to Y_1 \times Y_2$ of NACMs $\phi_1 : X_1 \to Y_1$ and $\phi_2 : X_2 \to Y_2$ is NACM.

Proof

Let $\mathcal{A} = \cup(\mathcal{A}_\alpha \times \mathcal{B}_\beta)$, where $\mathcal{A}_\alpha$'s and $\mathcal{B}_\beta$'s are NOSs of Y_1 and Y_2 respectively, be a NOS of $Y_1 \times Y_2$. Following Lemma 2.0.3, for $(p_1, p_2) \in X_1 \times X_2$, we have

$$\begin{aligned}
(\phi_1 \times \phi_2)^{-1}(\mathcal{A})(p_1, p_2) &= (\phi_1 \times \phi_2)^{-1}\big\{ \cup (\mathcal{A}_\alpha \times \mathcal{B}_\beta)\big\}(p_1, p_2) \\
&= \cup \Big\{ (\mathcal{A}_\alpha \times \mathcal{B}_\beta)\big(\phi_1(p_1), \phi_2(p_2)\big)\Big\} \\
&= \cup \Big[min\big\{\mathcal{A}_\alpha \phi_1(p_1), \mathcal{B}_\beta \phi_2(p_2)\big\}\Big] \\
&= \cup \Big[min\big\{\phi_1^{-1}(\mathcal{A}_\alpha)(p_1), \phi_2^{-1}(\mathcal{B}_\beta)(p_2)\big\}\Big] \\
&= \cup \Big[\big(\phi_1^{-1}(\mathcal{A}_\alpha) \times \phi_2^{-1}(\mathcal{B}_\beta)\big)\Big](p_1, p_2)
\end{aligned}$$

$i.e.$, $(\phi_1 \times \phi_2)^{-1}(\mathcal{A}) = \cup\big\{\phi_1^{-1}(\mathcal{A}_\alpha)\phi_2^{-1}(\mathcal{B}_\beta)\big\}$

Now, $(\phi_1 \times \phi_2)^{-1}(\mathcal{A})$

$$
\begin{aligned}
&= \cup\big\{\phi_1^{-1}(\mathcal{A}_\alpha) \times \phi_2^{-1}(\mathcal{B}_\beta)\big\} \\
&\subseteq \cup\Big[\mathcal{N} \sim Int\Big(\phi_1^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A}_\alpha))\Big)\Big) \\
&\times \mathcal{N} \sim Int\Big(\phi_2^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{B}_\beta))\Big)\Big)\Big] \\
&\subseteq \cup\Big[\mathcal{N} \sim Int\Big\{\phi_1^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A}_\alpha))\Big) \\
&\times \phi_2^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{B}_\beta))\Big)\Big\}\Big] \\
&\subseteq \mathcal{N} \sim Int\Big[\cup\Big(\phi_1 \times \phi_2\Big)^{-1}\Big\{\mathcal{N} \sim Int\Big(\mathcal{N} \sim Cl(\mathcal{A}_\alpha)\Big) \\
&\times \mathcal{N} \sim Int\Big(\mathcal{N} \sim Cl(\mathcal{B}_\beta)\Big)\Big\}\Big] \\
&= \mathcal{N} \sim Int\Big[\cup(\phi_1 \times \phi_2)^{-1}\Big\{\mathcal{N} \sim Int\Big(N \sim Cl(\mathcal{A}_\alpha \times \mathcal{B}_\beta)\Big)\Big\}\Big] \\
&\subseteq \mathcal{N} \sim Int\Big[(\phi_1 \times \phi_2)^{-1}\Big\{\mathcal{N} \sim Int\Big(\mathcal{N} \sim Cl\big(\cup(\mathcal{A}_\alpha \times \mathcal{B}_\beta))\Big)\Big\}\Big] \\
&= \mathcal{N} \sim Int\Big[(\phi_1 \times \phi_2)^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A}))\Big)\Big]
\end{aligned}
$$

Thus, by Theorem 2.0.10 (c), $\phi_1 \times \phi_2$ is NACM.

Theorem 2.0.14

Let X, X_1 and X_2 be NTSs and $p_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection of $X_1 \times X_2$ onto X_i. Then if $\phi : X \to X_1 \times X_2$ is a NACM, $p_i\phi$ is also a NACM.

Proof

Since p_i is *NCM* Definition 2.0.5, for any NS $\mathcal{A}$ of X_i, we have $(i)\mathcal{N} \sim Cl\big(p_i^{-1}(\mathcal{A})\big) \subseteq p_i^{-1}\big(\mathcal{N} \sim Cl(\mathcal{A})\big)$ and $(ii)\mathcal{N} \sim Int\big(p_i^{-1}(\mathcal{A})\big) \supseteq p_i^{-1}\big(\mathcal{N} \sim Int(\mathcal{A})\big)$. Again, since (i) each p_i is a *NOM*, and (ii) for any NS $\mathcal{A}$ of X_i (a) $\mathcal{A} \subseteq p_i^{-1}p_i(\mathcal{A})$, and (b) $p_i^{-1}p_i(\mathcal{A}) \subseteq \mathcal{A}$, we have $p_i\Big(\mathcal{N} \sim Int\big(p_i^{-1}(\mathcal{A})\big)\Big) \subseteq p_ip_i^{-1}(\mathcal{A}) \subseteq \mathcal{A}$ and hence $p_i\Big(\mathcal{N} \sim Int\big(p_i^{-1}(\mathcal{A})\big)\Big) \subseteq \mathcal{N} \sim Int(\mathcal{A})$. Thus, $\mathcal{N} \sim Int\big(p_i^{-1}(\mathcal{A})\big) \subseteq p_i^{-1}p_i\Big(\mathcal{N} \sim Int\big(p_i^{-1}(\mathcal{A})\big)\Big) \subseteq (p_i^{-1}\big(\mathcal{N} \sim Int(\mathcal{A})\big)$ establishes that $\mathcal{N} \sim Int\big(p_i^{-1}$

$(\mathcal{A})\big) \subseteq p_i^{-1}\big(\mathcal{N} \sim Int(\mathcal{A})\big)$.

Now, for any *NOS* $\mathcal{A}$ of X_i,

$$\begin{aligned}(p_i\phi)^{-1}(\mathcal{A}) &= \phi^{-1}\big(p_i^{-1}(\mathcal{A})\big)\\ &\subseteq \mathcal{N} \sim Int\Big\{\phi^{-1}\Big(\mathcal{N} \sim Int\Big(\mathcal{N} \sim Cl\big(p_i^{-1}(\mathcal{A})\big)\Big)\Big)\Big\}\\ &\subseteq \mathcal{N} \sim Int\Big\{\phi^{-1}\Big(\mathcal{N} \sim Int\Big(p_i^{-1}\big(\mathcal{N} \sim Cl(\mathcal{A})\big)\Big)\Big)\Big\}\\ &= \mathcal{N} \sim Int\Big\{\phi^{-1}\Big(p_i^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A})\big)\Big)\Big)\Big\}\\ &= \mathcal{N} \sim Int\big((p_i\phi)^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A})\big)\Big).\end{aligned}$$

Theorem 2.0.15

Let X and Y be NTSs such that X is product related to Y and let $\phi : X \to Y$ be a mapping. Then, the graph $\psi : X \to X \times Y$ of ϕ is NACM iff ϕ is NACM.

Proof

Consider that ψ is a *NACM* and $\mathcal{A}$ is a NOS of Y. Then using Lemma 2.0.4 and Theorem 2.0.10 (c), we have

$$\begin{aligned}\phi^{-1}(\mathcal{A}) &= 1_{X_{\mathcal{N}}} \cap \phi^{-1}(\mathcal{A})\\ &= \psi^{-1}(1_{X_{\mathcal{N}}} \times \mathcal{A})\\ &\subseteq \mathcal{N} \sim Int\Big(\psi^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(1_{X_{\mathcal{N}}} \times \mathcal{A})\big)\Big)\Big)\\ &= \mathcal{N} \sim Int\Big(\psi^{-1}\Big(1_{X_{\mathcal{N}}} \times \mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A})\big)\Big)\Big)\\ &= \mathcal{N} \sim Int\Big(\psi^{-1}\Big(\mathcal{N} \sim Int\big(1 \times N \sim Cl(\mathcal{A})\big)\Big)\Big)\\ &= \mathcal{N} \sim Int\Big(\psi^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A})\big)\Big)\Big)\end{aligned}$$

Thus, by Theorem 2.0.10 (c), ϕ is *NACM*.

Conversely, let ϕ be a *NACM* and $\mathcal{B} = \cup(\mathcal{B}_\alpha \times \mathcal{A}_\beta)$, where $\mathcal{B}_\alpha$'s and $\mathcal{A}_\beta$'s are *NOS*s of X and Y respectively, be a *NOS* of $X \times Y$.

Since $\mathcal{B}_\alpha \cap \mathcal{N} \sim Int\Big(\phi^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A}_\beta)\big)\Big)\Big)$ is a *NOS*s of X contained in

$$\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{B}_\alpha)\big) \cap \phi^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A}_\beta)\big)\Big),$$

$$\mathcal{B}_\alpha \cap \mathcal{N} \sim Int\Big(\phi^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A}_\beta)\big)\Big)\Big)$$
$$\subseteq \mathcal{N} \sim Int\Big[\mathcal{N} \sim Int\Big(\mathcal{N} \sim Cl(\mathcal{B}_\alpha)\Big) \bigcap \phi^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A}_\beta)\big)\Big)\Big]$$

and hence using Lemmas 2.0.1 (i), 2.0.4 and 2.0.5 and Theorem 2.0.10(c), we have

$$\begin{aligned}
\phi^{-1}(\mathcal{B}) &= \phi^{-1}\big(\cup(\mathcal{B}_\alpha \times \mathcal{A}_\beta)\big) \\
&= \cup\big[\mathcal{B}_\alpha \cap \phi^{-1}(\mathcal{A}_\beta)\big] \\
&\subseteq \cup\Big[\mathcal{B}_\alpha \cap \mathcal{N} \sim Int\Big(\phi^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{A}_\beta)\big)\Big)\Big)\Big] \\
&\subseteq \cup\Big[\mathcal{N} \sim Int\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{B}_\alpha)\big)\Big) \cap \phi^{-1}\Big(\mathcal{N} \sim Int \\
&\big(\mathcal{N} \sim Cl(\mathcal{A}_\beta)\big)\Big)\Big] \\
&\subseteq \mathcal{N} \sim Int\Big[\cup\psi^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{B}_\alpha)\big)\Big) \times \mathcal{N} \sim Int \\
&\big(\mathcal{N} \sim Cl(\mathcal{A}_\beta)\big)\Big] \\
&= \mathcal{N} \sim Int\Big[\psi^{-1}\Big(\cup\Big(N \sim Int\big(\mathcal{N} \sim Cl(\mathcal{B}_\alpha \times \mathcal{A}_\beta)\big)\Big)\Big)\Big] \\
&\subseteq \mathcal{N} \sim Int\Big[\psi^{-1}\Big(\mathcal{N} \sim Int\Big(\mathcal{N} \sim Cl\big(\cup(\mathcal{B}_\alpha \times \mathcal{A}_\beta)\big)\Big)\Big)\Big] \\
&= \mathcal{N} \sim Int\Big[\psi^{-1}\Big(\mathcal{N} \sim Int\big(\mathcal{N} \sim Cl(\mathcal{B})\big)\Big)\Big].
\end{aligned}$$

Thus, by Theorem 2.0.10 (c), ψ is *NACM*.