Chapter 1

Introduction

1.1 Cosmology and cosmological models

The name "cosmology" comes from the Greek word "kosmos," which is translated as "the beauty of the sky." Cosmology is a branch of science that studies the large-scale structure of the universe. Cosmological rays, background radiation, pulsars and stars, star clusters and galaxies, often known as nebulae, are all components of the universe.

The dynamics of the system are the fundamental issue in cosmology. Gravity is the primary force holding solar systems, stars and galaxies together. Since galaxies, which make up a significant portion of the universe and the intergalactic medium, are known to be electrically neutral, other long-range interactions like electromagnetic forces may be ignored. Although it is well known that general relativity is a successful theory of gravity, accurately predicting the motion of test particles and photons in curved space-time, one must incorporate simplifying assumptions and approximations in order to apply it to the universe. The assumption of a continuous matter distribution is the initial approximation that is typically made, and it is true on a large scale.

The cosmological principle, which asserts that the universe is homogeneous and isotropic on sufficiently large scales, serves as the foundation for the study of cosmology. According to physical laws, the universe has no favoured period, direction, or position.

Cosmology is now a precision science with massive amounts of high-quality data that is constantly increasing our understanding of the physical universe, but paradoxically theoretical cosmology is increasingly proposing theories based on even more hypothetical physics, or concepts that are untestable even in principle (such as multiverse). We are also witnessing an increase in dogmatic claims about how scientific cosmology can also solve philosophical problems. Due to Einstein's creation of the theories of relativity, the special theory of relativity and the general theory of relativity in 1905 and 1915, respectively, two conceptual revolutions occur in science at the beginning of the 20th century. In 1925, quantum gravity was developed. Both from a scientific and philosophical perspective, this new concept had a major influence on how the current understanding of the universe developed. This impact is unquestionably greater than the available research medium.

Albert Einstein developed a model of the universe based on general relativity theory in 1917. According to this study, time is assumed to be the fourth dimension, and gravitational attraction is equivalent to the curvature of this four-dimensional space (Cotsakis and Papantonopoulos, 2002). Minkowski space, unlike Einstein's, has positive constant curvature. De-Sitter (1872-1937) (De-Sitter, 1917) discovered another solution to the Einstein equations with the cosmological term in which space is empty of matter. In three dimensions, an object travelling in a "straight line" in four dimensions appears to follow a curve. This is difficult to imagine, but consider the path of the shadow versus the path of the object. Because most scientists assumed the universe was static, Einstein proposed the existence of a gravitational force of repulsion between galaxies to balance the gravitational force of attraction. In his field equation, he included a term called cosmological constant Λ. The result of this "cosmological constant" was a static universe. As a result, he lost the opportunity to predict the expansion of the universe by introducing an arbitrary constant. Einstein later described this as his "biggest mistake" in life (Bergh, 2000). Willem De-Sitter, a Dutch astronomer, developed non-static models of the universe in 1917 by solving Einstein's equations with a cosmological constant for an empty universe (De-Sitter, 1917).

1.2 Friedmann-Lemaitre-Robertson-Walker Metric

The observable universe has an isotropic and homogeneous distribution of matter and radiation. Cosmological principles are motivated by observational evidence of the universe's large-scale uniformity and isotropy. The homogeneous model's beauty is that it may be explored locally. Since any component in a homogeneous model represents the universe as a whole. It is possible to remove the requirement for isotropy, although doing so complicates the creation of cosmological models. Therefore, no galaxy cluster is regarded to be preferred in any way according to the cosmological principle. The existence of the cosmic time *t*, which reduces to the correct time for each fundamental particle, is a crucial aspect of the universe's spatial homogeneity (SH). The universe's isotropy and SH are mathematically formulated to produce the following outcomes

(1) The maximally symmetric subspaces of the entire space-time are the hypersurfaces with constant cosmic time.

(2) All cosmic tensors, including the energy-momentum tensor (EMT) T_{ij}, are invariant with regard to the isometry of subspaces, not just the metric G_{ij}. Friedmann (Friedman, 1922, 1924), Lemaitre (Lemaitre, 1927), Robertson and Walker all provided the metric of space with homogeneous and isotropic sections that is maximally symmetric. This class is known as the FLRW metric.

The distribution of extragalactic nebular clusters in space is essentially isotropic around our own galaxy, and the number of clusters in a given region appears to be consistent throughout. To simplify the mathematical description, we assume that matter is dispersed homogeneously throughout the universe. Furthermore, we hope that the matter distribution will define the geometry of space. Mach's principle is the name given to this universal requirement. It was given this term by Einstein (Einstein, 1918) while generalising Mach's (Mach, 1883) idea that the inertia of one body is related to the presence of all other bodies in the universe. Einstein's equations are one example of this concept in action.

Einstein and De-Sitter presented the standard cosmological model of the universe in 1932, which remained the most popular among cosmologists until 1980. In his cosmic problem, Einstein first assumed homogeneity and isotropy. He choose a time coordinate *t* such that the line element of static space-time could be characterised as [Narlikar, An Introduction to Cosmology],

$$ds^2 = c^2dt^2 - g_{\mu\nu}dx^\mu dx^\nu \tag{1.1}$$

here $g_{\mu\nu}$ denotes the functions of space coordinates x^{μ} ($\mu = 1, 2, 3$ and $\nu = 1, 2, 3$).

We may now create the homogeneous and isotropic three-dimensional closed space that Einstein desired for his universe model. The equation for such a three-surface of a four-dimensional hypersphere of radius *a* is given in cartesian coordinates x_1, x_2, x_3, x_4 by

$x_1^2 + x_2^2 + x_3^2 + x_4^2 = a^2$

As a result, the spatial line element on the surface is given by

$$d\sigma^2 = d^2x_1^2 + d^2x_2^2 + d^2x_3^2 + d^2x_4^2 = a^2[dX^2 + \sin^2 X(d\theta^2 + \sin^2\theta d\phi^2)] \quad (1.2)$$

Here,

$x_1 = a\sin X\cos\theta$, $x_2 = a\sin X\sin\theta\cos\phi$, $x_3 = a\sin X\sin\theta\sin\phi$, $x_4 = a\cos X$, where $0 \le X \le \pi$, $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$.

We can also define $d\sigma^2$ using the coordinates , θ, ϕ with $r = \sin X$, $(0 \le r \le 1)$ is

$$d\sigma^2 = a^2[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \quad (1.3)$$

Therefore the Einstein universe's line element is define by

$$ds^2 = c^2dt^2 - d\sigma^2 = c^2dt^2 - a^2[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \quad (1.4)$$

which is only for positive curvature.

Hence, when $c = 1$, the FRW line element is reduced to,

$$ds^2 = dt^2 - a^2[\frac{dr^2}{1-r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \quad (1.5)$$

Here *a(t)* is an unknown function of time, this determines the scale of the space's geometry and is referred to as the scale-factor. A constant that can have a value of -1, 0 or +1 is the quantity of *k*. It is either negative (open and Bianchi type-V and VII_h), zero (flat and Bianchi type-I and VII_0), or positive (closed and Bianchi type-IX). As a result, the space is either negative, flat, or positive and hence the parameter *k* is known as the curvature index. The fundamental particles are at rest with regard to the coordinates (*r*, θ, ϕ, *t*), making them a moving coordinate system (*r*, θ, ϕ).

1.3 Bianchi Cosmologies

The most significant properties of space-time presented by Friedmann (1922) are homogeneity and isotropy. This resulted in the Friedmann Robertson Walker Metric being proposed.

Many recent accurate experimental observations show hints of anisotropy (Boughn and Crittenden, 2004). This provides motivation to investigate Bianchi universes. Bianchi universes are homogenous but not necessarily isotropic cosmological models. They were named by Luigi Bianchi, who classified the three-dimensional spaces. They include the standard isotropic model, known as the FLRW universe, as a subclass.

Nucleosynthesis and microwave background anisotropies calculated in Bianchi models have been compared to evidence from the real universe, yielding null results that can be translated into upper bounds on anisotropy. Jaffe et al.'s (Jaffe and et al., 2005) tentative observation of non-zero anisotropic shear is thought to be incompatible with other known cosmological parameters (Collaboration and et al., 2016) and the polarisation of the microwave background in (Pontzen and Challinor, 2007). The models, on the other hand, are widely examined for their pedagogical value. Because the Einstein equations are homogeneous in space, they reduce from partial to ordinary differential equations in time, making them tractable for exact solutions.

As a result, nine classifications of spatially homogenous Bianchi models were established. The most basic type of Bianchi universe is known as type- I, which is an instantaneous generalisation of the FLRW flat metric. It is known that for a FRW generalisation, types I and VII were assumed, whereas types V and VII_h define open FRW generalisations. The non-flat universe is described by Bianchi types II, VI, VIII, and IX. FLRW in a flat metric is thought to be a subclass of Bianchi's classification, which has resulted in the Bianchi type-I universe as translation invariant in cartesian coordinate with line element as

$$ds^2 = -dt^2 + A^2dx^2 + B^2dy^2 + C^2dz^2 \tag{1.6}$$

Here, *A(t)*, $B(t)$ and $C(t)$ are the functions of cosmic time. In this case, $A(t) \neq B(t) \neq C(t)$ demonstrates that the universe is anisotropic, and these functions can vary independently in all directions. L. Bianchi (Bianchi, 1898) of Italy introduced the Bianchi classification and associated identities to identify these nine isometry classes. He organised his study around algebraic characteristics as well as a real and complex 3D lie algebra. Table A represents nine classifications of spatially homogenous Bianchi models.

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Bianchi Type	Line Element
Type-I	$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2$
Type-II	$ds^2 = -dt^2 + a^2(t)(d\theta^2 + d\phi^2) + S^2(t)(d\psi + \theta^2 d\phi)^2$
Type-III	$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)e^{-2\alpha x}dy^2 + C^2(t)dz^2$ where α is constant
Type-IV	$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)e^{2x}dy^2 - C^2(t)e^{2x}(xdy + dz)^2$
Type-V	$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)(dy^2 + dz^2)$
Type-VII_0	$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2)$ where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$
Type-VI_0	$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)e^{-2mx}dy^2 + C^2(t)e^{2mx}dz^2$ where m is constant
Type-VI_h	$ds^2 = -C^2(t)dt^2 + A^2(t)dx^2 + B^2(t)e^{2xy}dy^2 + C^2(t)e^{2hx}dz^2$ $where\; h = 0, \pm 1$ $when\; h = -1$, equivalent to type III
Type-VII_h	$ds = -dt^2 + a^2(t)(\frac{dr^2}{1+r^2} - r^2 d\Omega^2)$ where $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$
Type-VIII	$ds^2 = -dt^2 + a^2(t)(d\theta^2 + cosh^2\theta d\phi^2) + S^2(t)(d\phi + sinh\theta d\phi)^2$
Type-IX	$ds^2 = -dt^2 + a^2(t)(d\theta^2 + sin^2\theta d\phi^2) + S^2(t)(d\varphi + cos\theta d\phi)^2$

Table A: The metric for Bianchi types cosmological models

1.4 Cosmic Microwave Background (CMB)

In 1948, Cosmic Microwave Background (CMB) was predicted by Gamow(Alpher, 2014). The CMB is a leftover source of the early universe and a reliable observational probe that actively supports the Big Bang theory and the expansion of the universe. One of the most notable observational proofs of the evenly dispersed universe at large scales is the CMB. The Cosmic Background Explorer Satellite (COBE)(Clifton et al., 2012), on the other hand, investigated a minor amount of anisotropy.

The CMB is a second witness to the existence of an accelerating universe in addition to pointing to supernovae as the first direct observable evidence. Since the photons dissociate from the thermal bath, the CMB denotes the leftover radiation from the early universe that has reached our planet from the final scattering surface.

According to COBE, the CMB spectrum, which has a temperature of around 2.7 K, resembles the isotropic black body spectrum. The distribution of photons was not even, according to the COBE space probe. A well-known source of early universe physical data, the fluctuations of the distribution were on a scale of around one part in 10^5. Understanding how large-scale structure forms in a homogenous universe with small density disturbances is one of the most significant discoveries made by modern space probes. This raises the question of anisotropic universe models, which we will explore in this thesis.

1.5 Type Ia Supernova Observation

Walter Baade and Fritz Zwicky defined the phrase "supernova" in 1931. A supernova is a bright new star that is more potent than a nova. A supernova is an astronomical event that occurs in the final stages of the life of a massive star i.e., it is one final explosion that causes a bright star to briefly appear.

The Kepler's star of 1604 (SN 1604) is the most recent supernova discovered in our own galaxy i.e., the Milky Way (Reynolds and et al., 2008). With the help of a telescope, numerous others have been seen in distant galaxies. Further supernovae have been categorised into different categories as type I or type II. We concentrate on type Ia supernova among other subgroups.

Astronomical event of type Ia supernova occurs in a confined binary star system (two stars revolving round one another). Supernovae of type Ia, which are extremely brilliant objects, are used to calculate astronomical distances as standard candles (exploding stars with absolute luminosity). The apparent brightness of a few type Ia supernovae was originally studied in order to understand cosmic rapid expansion, and this characteristic of type Ia supernovae makes its study more important.

1.6 Energy-momentum tensor for Perfect fluid

The definition of a perfect fluid is a fluid that is frictionless, homogeneous, and incompressible and is not subject to tangential stress or shear action; instead, normal force acts between the fluid layers that are next to each other. Whether the fluid is in motion or at rest, the pressure at every place is constant in all directions. The energy stresses in the rest frame, viscosity and heat conduction can all be used to completely describe it.

For a perfect fluid with the properties of a suitable density ρ, a four-velocity u^i and a scalar pressure p, the full energy momentum tensor is given by

$$T_{ij} = (p+\rho)u_i u_j - p g_{ij} \tag{1.7}$$

where p is pressure, u_μ is the four velocity, $u_i = \frac{dx_i}{ds}$ of the individual particles, and ρ is the mass density.

1.7 Energy-momentum tensor for Viscous fluid

When an object tries to travel through a viscous fluid, the object is resisted. The viscous fluid's energy-momentum tensor is

$$T_{ij} = \rho u_i u_j + (p - \zeta\theta)H_{ij} - 2\eta\sigma_{ij} \tag{1.8}$$

where u_i = the four velocity vector, p = isotropic pressure, ρ = fluid density, and η, ζ are the coefficients of bulk viscosity and shear viscosity respectively.

Here, the volume expansion of fluid lines is denoted by $\theta = u^i_{;i}$

where the projection tensor $H_{ij} = g_{ij} - u_i u_j$ and the shear tensor $\sigma_{\mu\nu}$ are difined by

$$\sigma_{ij} = \frac{1}{2}(u^i_{,\mu}H^{\mu j} + u^j_{,\mu}H^{i\mu}) - \frac{1}{3}\theta H_{ij} \tag{1.9}$$

1.8 Energy-momentum tensor for Electromagnetic field

A physical field created by electrically charged objects is known as an electromagnetic field. It has an impact on how charged items behave when they are close to the field. The electromagnetic interaction is described by the electromagnetic field, which permeates all of space indefinitely.

If we take into account a field of charged particles that is moving and is described by a proper density ρ, a five velocity u_μ, and a current density vector J_μ. Therefore, the energy-momentum tensor for this field is given by

$$T_{ij} = -F_{il}F^l_j + \frac{1}{4}g_{ij}F_{lm}F^{lm} \tag{1.10}$$

1.9 Some Cosmological Parameters

Observations are utilised to establish a cosmological model under study's reliability. A few criteria are used to achieve this goal. In this part, we give a brief explanation of a few crucial cosmological parameters that are both theoretically and empirically significant.

1.9.1 Hubble's Parameter (*H*)

The universe is expanding at an accelerated rate currently, making it crucial to understand how fast the universe is expanding. The Hubble's parameter, represented by H, describes the rate of expansion of the universe. H is thus among the most important cosmological parameters. As defined, it is

$$H = \frac{\dot{a}}{a} \tag{1.11}$$

where a is the average cosmic scale factor. By way of Edwin Hubble's law, which he proposed in 1927, the Hubble's parameter is introduced. Since the scale factor determines how far apart galaxies are from one another as the universe expands, it follows that $l(t) = l_0 a(t)$ (where l_0 is the initial distance). Consequently, the observer's recession velocity *v(t)* is define by

$$v(t) = \frac{dl}{dt} = \frac{\dot{a}(t)}{a(t)} l(t) = H(t) l(t) \tag{1.12}$$

Hubble's law refers to the relationship in the above equation. It clearly shows that the time-varying Hubble's parameter impacts the rate of change of the scale factor *a(t)*, and so becomes an important parameter when applying observational data to a cosmological model under investigation. The Hubble's constant H_0 is the current value of the Hubble's parameter. Using equation (1.12) and assuming uniform expansion of the universe, the age of the universe in terms of time lapsed $t = \frac{l}{v}$ is determined as

$$t = H^{-1} \tag{1.13}$$

H^{-1}, also known as Hubble's time, provides a positive approximation of the time scale at which the universe came into existence. The age of the current universe starting from the Big Bang is determined using the constant H_0^{-1}.

1.9.2 Deceleration Parameter (*q*)

The deceleration parameter (DP) is represented by the letter q. It is a measurement of the acceleration of expansion of the universe. It is define as

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\ddot{a}}{aH^2} \tag{1.14}$$

The universal expansion's rate of change over time in terms of the scale factor is calculated using deceleration parameter. Therefore, $q < 0$ and $q > 0$ correspond to acceleration and deceleration, respectively, for acceleration $\ddot{a} > 0$ and for deceleration $\ddot{a} < 0$.

1.10 Some Kinematical Parameters

Here, we discuss a few of the variables that can be used to distinguish between the spherical motions of the cosmic fluid.

1.10.1 Shear Scalar (σ)

The shear scalar σ is given by

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} \tag{1.15}$$

where σ_{ij} and the shear tensor are defined as

$$\sigma_{ij} = \frac{1}{2}(u_{i;j} + u_{j;i}) + \frac{1}{2}(\dot{u}_i u_j - \dot{u}_j u_i) - \frac{1}{3}\theta h_{ij} \tag{1.16}$$

In the above equation, the terms $u_i, \theta = u^i_{;i}$ and $h_{ij} = (g_{ij} - u_i u_j)$ stand in for the fluid's five-velocity vector, expansion scalar and projection tensor respectively. The rate at which the matter flow is distorted is represents by the shear tensor.

1.10.2 Anisotropy Parameter ($\triangle$)

The measure of the departure from the isotropy is the anisotropy parameter $\triangle$.

For five dimensional space-time, the anisotropy parameter $\triangle$ is given by

$$\triangle = \frac{1}{4}\sum_{i=1}^{4}(\frac{H_i - H}{H})^2 \tag{1.17}$$

where $H_i(i = 1, 2, 3, 4)$ denotes the directional Hubble's Parameters in the x, y, z and ψ axis.

1.11 Scalar-Tensor Theories

General relativity remained an intriguing but largely untested theory for the first six decades of its existence, owing to a lack of technology capabilities to measure its perceptive non-Newtonian predictions for the solar system environment. However, theoreticians of that period did not sit back and wait for general relativity to be confirmed. Alternative gravity theories emerged as a result of their mathematical examination of the new fertile ground that Einstein had opened up (Ni, 1971). The majority of these used the fundamental components of general relativity but added details like new dynamical fields (scalars, vectors, additional tensor fields, and any combination there from) and occasionally earlier geometric elements (like fixed second metrics and non-dynamical scalars) (Will, 1994). The scalar-tensor theory of gravity is a type of modified gravity theory in which a scalar field is introduced in addition to the usual tensor field of general relativity. The scalar field modifies the strength of the gravitational interaction, and its dynamics are described by a scalar potential. This scalar field is what gives the theory its name, "scalar-tensor". In general, a scalar field is a field that has only one degree of freedom and can be described by a single scalar value at each point in space and time. In the scalar-tensor theory of gravity, the scalar field is responsible for controlling the strength of the gravitational interaction, and it couples to matter in a different way than the tensor field does. The scalar field can also evolve dynamically, which can affect the evolution of the universe. The scalar field theories of Jordan (Jordan, 1955), Brans and Dicke (Brans and Dicke, 1961), Barber(Barber, 1982) and

S$\acute{a}$ez-Ballester (Saez and Ballester, 1986) are the most well-known of these theories and are still in discussion today. Some of these alternative theories were later discovered to be in gross contradiction with experiment and are no longer taken into consideration, such as the conformally at theories of Nordstrom (Deruelle, 2011), which rule out the presence of gravitational lensing effects. Jordan (Jordan, 1955) began to embed a four-dimensional curved manifold in a five-dimensional space-time, which is how the scalar-tensor theory came to be. He demonstrated how a four-dimensional scalar field may be used as a constraint in the formulation of projective geometry, allowing one to express a gravitational constant that depends on space and time. The metric tensor field (g_{ij}), a tensor of rank two, is the basic building block of the general theory of relativity. Consequently, the theory might be referred to as a tensor theory. The so called scalar-tensor theories include a connected pair of dynamical scalar fields (ϕ) and a metric tensor field. Because Mach's principle, which asserts that inertia should result from acceleration with regard to the overall mass distribution of the universe, is not adequately explained by general relativity, new cosmological models have been developed. The natural generalisations of general relativity are provided by scalar-tensor theories of gravitation, which also offer a useful set of representations for the observational bounds on potential departures from general relativity.

1.12 S$\acute{a}$ez-Ballester Theory

In order to address the issue of dark matter, S$\acute{a}$ez-Ballester (Saez and Ballester, 1986) created a scalar-tensor theory of gravitation, in which the metric is simply coupled with a dimensionless scalar field ϕ. This coupling provides an accurate representation of weak fields. The missing-matter issue in non-flat FRW cosmologies is addressed by the S$\acute{a}$ez-Ballester theory of gravity. Thus, a scalar-metric formulation of gravitation is obtained, which has positive outcomes in its applications to cosmology.

S$\acute{a}$ez-Ballester (Saez and Ballester, 1986) considered the Lagrangian as follows

$$L = R - \omega\phi^n \phi_{,\beta}\phi^{,\beta} \tag{1.18}$$

ω and *n* are arbitrary dimensionless constants, *R* is the curvature, ϕ is the dimensionless scalar field, and $\phi^{,\beta} = g^{ij}\phi_{,\beta}$. The Lagrangian given by equation (1.18) has distinct dimensions for scalar fields with the dimension $\phi = G^{-1}$. However, in the case of a dimensionless scalar field, it is an appropriate Lagrangian. One can construct the action from the Lagrangian.

$$\xi = \int_{\Sigma} \sqrt{-g}(L + 8\pi L_m) d^4x \tag{1.19}$$

L_m is matter Lagrangian, $g =\mid g \mid$, Σ is an arbitrary region of integration. The following variational principle $\delta\xi = 0$, generates the S$\acute{a}$ez-Ballester scalar-tensor theory field equations (geometrized units with $G = c = 1$) by taking into account arbitrary independent fluctuations in the metric and the scalar field vanishing at the boundary of Σ

$$R_{ij} - \frac{1}{2}Rg_{ij} - \omega\phi^n(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,\alpha}\phi^{,\alpha}) = -8\pi T_{ij} \tag{1.20}$$

$$2\phi^n\phi^{;i}_{;i} + n\phi^{n-1}\phi_{,\alpha}\phi^{,\alpha} = 0 \tag{1.21}$$

Here ω and *n* are constants and R_{ij} is the Ricci tensor, *R* is the scalar curvature, T_{ij} is the energy momentum tensor, and ϕ is a dimensionless scalar field that is just a function of cosmic time *t*. Partial and covariant derivatives are indicated respectively, by a comma and a semicolon. The energy conservation equation, which results from equations (1.20) and (1.21), is also defined by

$$T^{ij}_{;j} = 0 \tag{1.22}$$

D. S$\acute{a}$ez discusses the initial singularity and inflationary universe in the S$\acute{a}$ez-Ballester scalar-tensor theory (Saez, 1987). D. S$\acute{a}$ez has demonstrated that there is an anti-gravity regime that could be in effect before the inflationary period or at its beginning. D. S$\acute{a}$ez also achieved a non-singular FRW model solution in flat space.

1.13 Isotropy of the Universe

When Hubble's (Hubble, 1929) was able to make these discoveries, the modern era of cosmology had begun. He developed methods for estimating the distances of the distant galaxies and studied their distributions in space, resulting in his famous law: All galaxies recede from us at velocities proportional to their distances from us. According to Hubble's findings, galaxies are distributed in homogeneous, isotropic ways on a large-scale. The first piece of observational proof for the cosmological models came from Hubble's discovery. The "Cosmological Principle"(CP) is the presumption that there is isotropy and large-scale homogeneity . The universe should be uniform on a grand scale, according to CP. However, inhomogeneities exist on smaller scalars. The supposition of the universe's homogeneity and isotropy aids in the idea that all spatial directions are equal and that there is no way to separate any area of the universe from another. The distribution of radio sources, cosmic x-ray background, cosmic microwave background and the distribution of galaxies in the sky along with their apparent magnitudes and redshifts all provide at least some circumstantial evidence. There is proof that the isotropy of the spread of these materials on a large-scale.

Any study of the universe should focus on developing a full grasp of its past, present and future states. The universe is currently thought to be homogenous and isotropic on a large scale. However, this belief of the universe's current state cannot be extrapolated or interpolated. Therefore, it is crucial to investigate the nature of the universe before and after its evolution, that is, after the Big Bang or before the recombination, both of which occurred. At that time, there is no observational evidence for an isotropic universe. Thus, early research of potential anisotropies becomes a common occurrence. Anisotropy exists in the universe, according to observations from the Cosmic Microwave Background radiation (CMBR) (Netterfield and et al., 2002) and Wilkinson Microwave Anisotropy Probe (WMAP) (Hinshaw and et al., 2013). It is assumed that the anisotropies in the universe are what led to the formation of the distinct structures that have been observed. The presence of an anisotropic universe, which eventually moves toward an isotropic phase is also supported by theoretical reasons. The study of homogeneous and anisotropic cosmological models is crucial in light of the findings outlined above.

A geometry that is more all-encompassing than isotropic and homogeneous geometry is needed to analyse anisotropic cosmological models. A set of homogeneous, anisotropic cosmological models with Bianchi type-I to Bianchi type-IX space-time line elements have been presented by the Italian geometrist L. Bianchi on the foundation of Lie algebra (Ellis and MacCallum, 1969). Among several other families of homogeneous and anisotropic geometries, these Bianchi type line elements are the most common. The simplest example of a spatially homogeneous and anisotropic cosmological model is known as the Bianchi type-I and it is stated as follows (in units of $c = 1$):

$$ds^2 = -dt^2 + A^2dx^2 + B^2dy^2 + C^2dz^2 \tag{1.23}$$

where *A*, *B* and *C* are the scale factors for the directions of *x*, *y* and *z* respectively.

1.14 Cosmic Strings

For a comprehensive explanation of the fundamental forces and particles in nature, including gravity, string theory is the most plausible contender. A fundamental physics model called string theory uses one-dimensional extended objects (strings) as its building blocks rather than the zero-dimensional points (particles) that make up the standard model of particle physics. Moreover, string theory may create new opportunities for the unification of the known natural forces (gravitational, electromagnetic, weak, and strong) by describing them with the same set of equations. In the late 1960s and early 1970s, string theory was initially developed and investigated as a possible explanation for several unusual aspects of hadron behaviour. Higher-dimensional objects as well as strings are predicted by string theory, according to studies. Ten or eleven (M-theory) space time dimensions are strongly predicted by string theory.

Yoichiro Nambu in particular realized in 1970 that the dual resonance model of strong interactions could be explained by a quantum mechanical model of strings. This strategy was dropped as an alternative theory. Physicists began to work on string theory as the most promising idea to unify theories of physics between 1984 and 1986 after realising that it could describe all fundamental particles and their interactions. A system of cosmic strings' energy momentum tensor is

$$T_{ij} = \rho u_i u_j - \lambda x_i x_j \tag{1.24}$$

where

$$\rho = \rho_p + \lambda \tag{1.25}$$

is the string cloud's rest energy density. λ is the string cloud's tension density, and ρ_p is the particle's rest energy density. The tension density λ may be positive or negative, while the energy densities for the coupled system ρ and ρ_p are constrained to be positive (Letelier, 1983a). u^i is the five-vector sum of the particle velocities, and x^i is the anisotropy direction. The string's direction satisfies the following condition

$$u_i u^i = 1, \quad -x_i x^j = 1, \quad u_i x^j = 0 \tag{1.26}$$

Cosmic strings may be one of the sources of density perturbations resulting in the formation of galaxies because they emerge during phase transitions when the universe approaches its critical temperature following the Big Bang ((Kibble, 1976),(Vilenkin, 1985)). Stachel string cloud model was constructed in a gauge-invariant form by Letelier (Letelier, 1979), who also studied various formal issues including the model's energy requirements. The spherically symmetric solution is used to build a double-layer model of a star. Zel'dovich (Zeldovich, 1980) and Vilenkin (Vilenkin, 1981b) investigated the evolution of the network of cosmic strings that were established in the early universe and proposed that galaxies and clusters are produced due to the gravitational impacts of cosmic strings. There is no conflict between the current observations of the universe and the existence of a large-scale network of strings in the early universe. According to Stachel (Stachel, 1980), phase transitions in the very early universe gave rise to topologically stable structures of great importance, such as the linear structure known as the geometric string. In Bianchi type-I and Kantowski-Sachs space periods, Letelier (Letelier, 1983a) was able to get cosmological solutions to the field equations describing clouds as heavy strings with particles connected along their extension. The generalisation of Takabaysi's p-string realistic string model is known as the cloud of strings (Letelier, 1978). This is the most basic model, which combines particles and strings. In this model, the strings can be removed and the particle cloud is left behind.

1.15 Higher dimensional space-time

The Kaluza-Klein theory is discussed in a five-dimensional space time. The extra space dimension is folded up into a tiny circle that exists everywhere but is too small to view with current instruments. The vibrations of the gravitational field in the rolled up extra space dimension would appear to viewers as vibrations in an electromagnetic field and a scalar field in the remaining three space and one time dimensions. As a result, the underlying five-dimensional space time has one type of force, namely gravitational force. However, the four-dimensional space time observed from a great distance appears to include three types of forces: gravitational, electromagnetic, and scalar forces. This is one approach to developing a unified theory that yields all of the other forces from higher-dimensional modes of the graviton.

Since Einstein stated that physicists consider time to be a fourth dimension, there has been speculations about a fifth and even higher dimension. To avoid extreme violations in experiments and ordinary experiences, it was assumed that any extra dimension would have to be compact, with unobservable impacts on current experiments. There was the idea that ordinary particles were somehow limited to a three-dimensional membrane embedded in a higher-dimensional space time. This possibility was not given much weight. However, new advancements in string theory reveal that such membranes are essential for string theory to be consistent.

The extra dimensions (six in string theory and seven in M-theory) are thought to be very tiny. The many types of particles in nature can be explained by the geometrical structure of these dimensions and the singularities inside these extra dimensions. Consistent quantum field theories in higher dimensions space-time are predicted by string theory. There are some particularly mysterious theories involving six dimensions. Any conventional quantum field theory is not like such theories. They have items that resemble strings and are electrically and magnetically charged at the same time, but they lack a coupling constant. The nonperturbative effects in four dimensional gauge theories will be better understood as a result of this. Instead of looking at a 4-dimensional universe, physicists looked at a 10-dimensional one in this theory. Not all aspects of the ten-dimensional universe are taken into account. The remaining six coordinates and the four related to general theory of relativity are grouped together. The 6-dimensional manifold is necessary for the interactions in 4-dimensions. Overduin and Wesson explain these in detail (Overduin and Wesson, 1997).

Gravitation and electromagnetism were theoretically attempted to be combined by Kaluza (Kaluza, 1921) and Klein (Klein, 1926a,b). The "Cosmological dimensional-reduction process," an interesting possibility, is predicated on the idea that, at a very early stage, all dimensions in the universe are comparable. Later, the extra dimensions' size shrunk to a point where they could no longer be seen through contraction. Forgacs and Horvath researched such cosmological theories (Forgacs and Horvath, 1979). Chodos and Detweiler (Chodos and Detweiler, 1980) demonstrated that, within the framework of a pure gravitational theory of Kaluza-Klein, the extra dimensions are unobservable due to dynamical contraction to a very small scale, while the three other spatial dimensions expand isotropically as a result of cosmological evolution. Freund (Freund, 1982) used the dynamical evolution of 10- and 11-dimensional super gravity models to explain the smallness of the extra dimensions of the universe. In contrast to the typical inflationary scenario, Guth (Guth, 1981) and Alvarez and Gavela (Alvarez and Gavela, 1983) found that extra dimensions produce a significant amount of entropy during the contraction process, offering a solution to the flatness and horizon problem. Gross and Perry (Gross and Perry, 1983) demonstrated that soliton solutions are permissible in the five-dimensional Kaluza-Klein theory of unified gravity and electromagnetism. Furthermore, they provided an explanation for the disparity between gravitational and inertial masses that arises from Kaluza-Klein theories' violation of Birkhoff's theorem, which is in line with the idea of equivalence. Through the use of the field equations, Appelquist and Chodos (Appelquist and Chodos, 1983) and Randjbar-Daemi et al. (Randjbar-Daemi et al., 1984) asserted that space time in the first four dimensions expands, while the fifth dimension either stays constant or contracts to the unobserved Planckian length scale, as required for the real universe. In light of the contemporary Kaluza-Klein theory, precise solutions to Einstein's field equations in dimensions beyond four are relevant in a variety of scenarios (Lee, 1984)(Lee, 1984). Several authors (Adhav et al., 2007a; Hong et al., 2008; Khadekar and Butey, 2009; Pahwa, 2014; Sahoo and Mahanta, 2021; Samanta and Dhal, 2013; Singh and Kotambkar, 2001; Singh et al., 2003; Tiwari and Singh, 2015; Xiu-Ju and Yong-Ji, 2004; Yilmaz and Yavuz, 2006) have investigated the higher dimension using the different types of cosmological models.

1.16 A brief review of various work done in Sáez-Ballester theory of gravitation

Numerous authors have looked at various elements of gravitational difficulties in this geometry since the development of Sáez-Ballester theory.

Reddy D.R.K. and Rao N.V. (Reddy and Rao, 2001) have obtained and investigated a few FRW cosmological models, a non-static plane symmetric stiff fluid model, and a spatially homogeneous and anisotropic Bianchi type-III model in the scalar-tensor theory presented by Sáez-Ballester (1985).

Reddy D.R.K. (Reddy, 2003) investigated a spatially homogeneous Bianchi type-I string cosmological model in the inflationary scalar-tensor theory of Sáez-Ballester. He discovered that the model, scalar field, and density are all free of initial singularity. In addition, the model is evolving over time.

Mohanty et al. (Mohanty and Sahu, 2003) solved the field equations for the spatially homogeneous and anisotropic Bianchi type-VI0 cosmological model with perfect fluid and obtained solutions for the barotropic fluid model in the first case, the perfect fluid model in the second case, and the stiff fluid model in the third case. It is also discovered that the models are non-rotating in nature, with no acceleration, and that they grow during the evolution process.

Singh et al. (Singh and Ram, 2003) investigated in Sáez-Ballester scalar-tensor theory homogeneous and isotropic zero-curvature Robertson-Walker models filled with ideal fluid. They devised a model in which the energy density and pressure decline as cosmic time increases, and as t approaches infinity, both vanish, leaving an empty universe for a long period.

Mohanty G. et al. (Mohanty and Sahu, 2004) studied the topic of viscous fluid distribution in Bianchi type-I space-time is explored here in the Sáez-Ballester theory of gravitation developed by Sáez-Ballester (1986), and they discover that the energy density, pressure, and bulk viscosity of the fluid decrease as the age of the universe increases.

Mohanty et al (Reddy et al., 2006a) studied a cosmological model with negative constant deceleration parameter in the framework of Sáez-Ballester scalar -tensor theory of gravitation using a particular law of variation for Hubble's parameter proposed by Bermann. The derived model features an initial singularity and extends endlessly with acceleration, despite the fact that all physical parameters diverge at the initial epoch.

Mohanty et al. (Mohanty et al., 2007) investigated the five-dimensional LRS Bianchi type-I string cosmological model using the Sáez-Ballester theory and discovered that our universe is still a four-dimensionally real space.

V.U.M. Rao (Rao et al., 2007) has presented a spatially homogeneous Bianchi type-V cosmological model in the context of Sáez-Ballester's (1986) scalar-tensor theory with perfect fluid.

Adhav et al. (Adhav et al., 2007c) investigated a spatially homogeneous Bianchi type-VI string cosmological model in the scalar-tensor theory given by Sáez-Ballester. They defined an inflationary model devoid of initial singularity. They also defined the scalar field and density, which are free of initial singularity. The model is anisotropic and expands with time.

Singh et al. (Singh et al., 2008) presented anisotropic Bianchi type-V cosmological models filled with ideal fluid in the scalar-tensor theory provided by Sáez-Ballester. They conclude that the universe decelerates for positive values of the deceleration parameter and accelerates for negative ones.

Rao et al. (Rao et al., 2008a) examined the exact string cosmological models for the Bianchi type II, VIII, and IX space times in the Sáez-Ballester scalar tensor theory of gravitation. Their results are a generalisation of the general relativistic Bianchi type II, VIII, and IX string cosmological models with the dimensionless scalar field provided by the Sáez-Ballester. They obtained cosmological models with anisotropic time expansion and no singularities at $t = 0$.

Reddy et al. (Reddy et al., 2008) investigated a five-dimensional cosmic domain wall using the Sáez-Ballester scalar-tensor theory of gravitation. To obtain a definite solution, they assumed a stiff fluid state and a relationship between the metric coefficients.

Mohanty et al. (Mohanty and Sahoo, 2008) investigated five-dimensional LRS Bianchi type-I effective stiff fluid cosmological models in Sáez-Ballester's scalar-tensor theory of gravitation.

Ram et al. (Ram et al., 2009) investigated the variation law for Hubble's parameter with average scale factor, leading to a constant value for the deceleration parameter in a spatially homogeneous and anisotropic Bianchi type-V space-time model. Two laws-one in the shape of an exponential function and the other a power law-are derived by them to describe how the average scale factor varies across cosmic time. They discovered that the universe begins out at a constant volume where all physical variables function properly and expands at a steady rate.

Singh (Singh, 2009) investigated Sáez-Ballester's spatially homogeneous locally rotationally symmetric (LRS) Bianchi type-V perfect fluid model with heat conduction in scalar tensor theory. The field equations with and without heat conduction are solved using a rule of variation for the mean Hubble parameter, which is related to the average metric scale factor and produces a constant value for the deceleration parameter. He defined the constantly decelerating perfect fluid model with heat flow in an anisotropic geometry, which leads to a self consistent solution that may describe a well-defined phase of our universe's evolution.

Socorro et al. (Socorro et al., 2009) investigated the Sáez-Ballester theory's generalisation by introducing a dimensionless functional of the scalar field. They discovered that the evolution of the universe's scale factor is independent of the precise form of the functional F(t); in fact, the contribution of the scalar field in the Sáez-Ballester theory is that of a perfect fluid with a stiff (barotropic) equation of state. If it exists, its contribution to the universe's matter budget is only meaningful at the beginning.

Tripathy et al. (Tripathy et al., 2009) investigated string cloud cosmological models in Sáez-Ballester Scalar-Tensor theory of gravity utilising spatially homogeneous and anisotropic Bianchi type-VI0 metric.

Katore et al. (Katore et al., 2010) studied an exact higher-dimensional LRS Bianchi type-I cosmological model in the presence of thick domain walls in a scalar -tensor theory of gravitation proposed by S$\acute{a}$ez-Ballester. They discovered that the model is inflationary and does not approach isotropy. In this theory, the model obtained can be seen as a five-dimensional self-gravitating or stiff domain wall. Given the current increase in interest in scalar fields in general relativity and alternative theories of gravity in the setting of inflationary cosmology, this model obtained has significant astrophysical importance.

Pradhan et al. (Pradhan and Singh, 2010) investigated S$\acute{a}$ez-Ballester proposed a new class of spatially homogeneous and anisotropic Bianchi type-V cosmological models of the universe for perfect fluid distribution within the framework of scalar-tensor theory of gravitation by applying the law of variation for the generalised mean Hubble's parameter, which yields a constant value of deceleration parameter. They discovered that the model has an isotropic stage later in its history. The exponential solutions represent a universe without singularities.

Adhav et al. (Adhav et al., 2010) investigated the behaviour of cylindrically symmetric Einstein-Rosen thick domain walls in the S$\acute{a}$ez-Ballester scalar-tensor theory of gravitation. In this example, they obtained an Einstein-Rosen cylindrically symmetric static vacuum model in the presence of thick domain walls using the Seaz-Ballester scalar-tensor theory. In addition, a non-static, non-singular contracting cosmological model packed with stiff matter is constructed, which decelerates in the standard way.

Rao et al. (Rao et al., 2011) investigated the field equations for spatially homogeneous and anisotropic LRS Bianchi type-I metric in the framework of S$\acute{a}$ez-Ballester scalar-tensor theory of gravitation when the energy-momentum tensor is a viscous fluid comprising one dimensional strings. In the S$\acute{a}$ez-Ballester scalar-tensor theory of gravitation, the derived model reflects a bulk viscous inflationary cosmological model.

Pradhan et al. (Pradhan et al., 2012a) investigated a new class of spatially homogeneous and anisotropic Bianchi type-I cosmological models of the universe for perfect fluid distribution within the framework of S$\acute{a}$ez-Ballester's scalar-tensor theory of gravitation by considering time dependent deceleration parameter and obtained the models represent expanding, shearing, and non-rotating universe, which approach to isotropy for large values of t.

Pradhan et al. (Pradhan et al., 2012b) investigated a new class of spatially homogeneous and anisotropic Bianchi type-I cosmological models of the universe for ideal fluid distribution within the framework of S$\acute{a}$ez-Ballester's scalar-tensor theory of gravitation. They use distinct scale factors that provide time-dependent deceleration parameters (DP) indicating models that generate a transition of the universe from the early decelerated phase to the recent accelerated phase to prevail the deterministic solutions. For large values of t, they find an expanding, shearing, and non-rotating universe that approaches isotropy.

Rao et al. (Rao et al., 2012) investigated spatially homogeneous and anisotropic Bianchi type-II, VIII, and IX dark energy models with variable EoS parameters in a scalar-tensor theory of gravitation proposed by S$\acute{a}$ez-Ballester and found that our models are accelerating, more general, and represent not only the early stages of evolution but also the current stage of the universe.

Reddy et al. (Reddy et al., 2012) investigated a five-dimensional cosmological model in the presence of a perfect fluid with a variable equations of state (EoS) parameter in Kaluza-Klein space-time using the S$\acute{a}$ez-Ballester scalar-tensor theory of gravitation. They obtained a model that represents the KaluzaKlein dark energy cosmological model, which is expanding and free of starting singularity. It is discovered that the dark energy EoS parameter and the skewness parameter are time dependent.

Naidu et al. (Naidu et al., 2012) studied the LRS Bianchi type-II dark energy model in the scalar-tensor theory of gravitation proposed by S$\acute{a}$ez-Ballester.

Reddy et al. (Reddy and Kumar, 2012) investigated two fluid scenarios in the presence of a S$\acute{a}$ez-Ballester scalar field in spatially homogenous and isotropic FRW space-time. The EoS parameter is found to be an increasing function of cosmic time in open, closed, and flat FRW universes, which explains the universe's late temporal acceleration. It is also recognised that in both interacting and non-interacting cases, studies of the dynamics of scalar fields in association with an inflationary (accelerated) universe scenario are essential because they can solve some of the unresolved issues of standard 'Big Bang' cosmology.

Samanta et al. (Samanta et al., 2013) built five-dimensional LRS Bianchi type-I string cosmological models with bulk viscous fluid in a scalar-tensor theory of gravitation proposed by S$\acute{a}$ez-Ballester and discovered that cosmic strings do not survive in the presence of bulk viscous fluid for the equation of state $(\rho + \lambda) = 0$, but do survive for $\rho = (1 + \omega)\lambda$ and $\rho = \lambda$.

Pradhan et al. (Pradhan et al., 2013) investigated a spatially homogenous and anisotropic Bianchi type-V space-time in the context of S$\acute{a}$ez-Ballester's scalar-tensor theory of gravitation.

Rahman et al. (Rahman and Ansari, 2013) investigated an anisotropic Bianchi type-III cosmological model in a S$\acute{a}$ez-Ballester scalar-tensor theory of gravitation. They defined exact solutions to the field equations under proper physical conditions that provide a time-dependent DP indicating a model that generates a transition from the universe's early decelerating phase to its current accelerating phase.

Reddy et al. (Reddy et al., 2013) investigated the field equations for a five-dimensional Kaluz-Klein model within the framework of S$\acute{a}$ez-Ballester's scalar-tensor theory of gravity, when the source of energy momentum tensor is a viscous fluid including one-dimensional strings.

Reddy et al. (Reddy et al., 2014b) investigated a spatially homogenous and anisotropic Bianchi type-V cosmological model in a scalar-tensor theory of gravity provided by S$\acute{a}$ez-Ballester where the source of the energy momentum tensor is a bulk viscous fluid comprising one-dimensional cosmic strings and obtained the model turns out to be singularity free, shearing, non rotating, and anisotropic throughout the universe's evolution.

Katore et al. (Katore and Shaikh, 2014) studied the Bianchi type-I magnetised cosmological model given by S$\acute{a}$ez-Ballester in scalar tensor theory with perfect fluid as a source and the behaviour of models having physical properties in the presence and absence of a magnetic field is discussed.

Sharma et al. (Sharma and Singh, 2014) investigated the spatially homogenous and fully anisotropic Bianchi type-II cosmological solutions of heavy strings in the presence of a magnetic field within the context of S$\acute{a}$ez-Ballester's scalar-tensor theory of gravitation. In this theory, a particular law of variation for Hubble's parameter provided by Berman string cosmological model is produced.

Vidyasagar et al. (Vidyasagar et al., 2014) studied a spatially homogeneous and anisotropic Bianchi type-III space-time in the presence of bulk viscous fluid containing one dimensional cosmic strings in the framework of S$\acute{a}$ez-Ballester's scalar-tensor theory of gravity. They established a determinate solution to this theory's field equations by employing I a barotropic equation of state for pressure and density and the bulk viscous pressure is proportional to the energy density.

Ghate et al. (Ghate and Sontakke, 2014) studied a Bianchi type-IX cosmological model with a changeable EoS parameter in S$\acute{a}$ez-Ballester theory of gravitation in the presence and absence of an energy density b magnetic field. In the presence and absence of a magnetic field, the values of the Hubble parameter, expansion scalar, mean anisotropic parameter of expansion, and shear scalar stay constant, however the component of magnetic field reduces the anisotropic fluid's energy density.

Kiran et al. (Kiran et al., 2014b) investigated stationary Spherically symmetric kink space-time in the presence of perfect fluid distribution using S$\acute{a}$ez-Ballester's scalar-tensor theory of gravitation.

Kiran et al. (Kiran et al., 2014a) studied a spatially homogenous and anisotropic Bianchi-V universe filled with two minimally interacting fluids; matter and holographic dark energy components in the Scalar-tensor theory given by S$\acute{a}$ez-Ballester. They derived an exact solution of the field equations, which depicts a minimally interacting matter and holographic dark energy model, by using the fact that the expansion scalar of space-time are proportional to the shear scalar. The model's cosmic jerk parameter is positive throughout the universe's history.

Reddy et al. (Reddy et al., 2014a) constructed a Kantowski-Sachs type cosmological model with matter source as bulk viscous fluid and one dimensional cosmic strings, in S$\acute{a}$ez-Ballester's scalar-tensor theory of gravitation. The resulting model is non-singular growing, non-rotating, and shearing in general. The model remains anisotropic throughout the universe's history. However, in a specific instance, the model becomes isotropic and shear free.

Rao et al. (Rao et al., 2015b) have studied a five dimensional FRW space-time in the scalar-tensor theory of gravitation provided by S$\acute{a}$ez-Ballester (1986) in the presence of perfect fluid distribution.

Mohurley et al. (Mohurley et al., 2016) explored a spatially homogeneous and anisotropic five-dimensional Bianchi type-I cosmological model filled with ideal fluid in the framework of Sáez-Ballester's scalar-tensor theory and determined the universe display exponential expansion and expand uniformly. The obtained models represent a shearing, non-rotating, expanding cosmos that approaches isotropy for large values of t.

Ramesh et al. (Ramesh and Umadevi, 2016) studied a minimally interacting holographic dark energy model in Bianchi type-II space time utilising a linearly variable deceleration parameter in the framework of a scalar-tensor theory developed by Sáez-Ballester. They discovered that the model is expanding, with all physical and kinematic characteristics diverging at the start epoch and becoming finite for infinitely large cosmic time values.

Santhi et al (Santhi et al., 2016a) investigated the spatially homogenous and anisotropic Bianchi type-V I0 metric in the presence of modified holographic Ricci dark energy and dark matter in the Sáez-Ballester scalar-tensor theory of gravitation and obtained an inflating model.

Raju et al. (Raju et al., 2016) studied a five-dimensional spherically symmetric space time populated with two minimally interacting fields, matter and holographic dark energy components in Sáez-Ballester's scalar tensor theory of gravitation. They discovered that the model spatially expands over time. Furthermore, the overall density parameter is found to be constant. They also observed that the cosmos initially decelerates and then accelerates in late time. and discovered that the universe is isotropic at all moments, implying that "cosmic re-collapse" will cause a transition from decelerated to accelerated phase.

Rao et al. (Rao et al., 2016b) studied the evolution of the dark energy parameter in the context of a spatially homogenous five-dimensional Kaluza-Klien universe filled with barotropic fluid and dark energy, using Sáez-Ballester's scalar-tensor theory of gravitation.

Reddy et al. (Reddy et al., 2016b) studied a spatially homogenous and anisotropic Bianchi type-VI0 world populated with minimally interacting dark matter and holographic dark energy, in the Sáez-Ballester scalar-tensor theory of gravitation. They obtain that the universe's spatial volume increases with time, indicating spatial expansion, and that it is zero at $t = 0$, indicating a point-type singularity. The behaviour of the dark energy EoS parameter demonstrates that the model traverses the dust and radiating regions, passes the phantom split line, and eventually attains a constant value in the phantom region.

Reedy et al. (Reddy et al., 2016a) examined such a holographic dark energy model in spatially homogenous anisotropic Bianchi type-VI0 space time. They discovered an exact solution to the S$\acute{a}$ez-Ballester field equations by utilising a relationship between the metric potentials and Bermann's special rule of variation for Hubble's parameter. The found solution provides an anisotropic Bianchi type-VI0 minimally interacting holographic dark energy model in this theory. In scalar-tensor cosmology, the model obtained reflects a non-singular and spatially expanding universe.

Rao et al. (Rao et al., 2016a) discussed a five-dimensional FRW model filled with dark matter and modified holographic Ricci dark energy, in the S$\acute{a}$ez-Ballester scalar-tensor theory of gravity. They used hybrid expansion law for the average scale factor to produce a determinate solution of the field equations. They discovered that the spatial volume of the models increases with cosmic time, indicating that the universe is expanding spatially. The matter energy density, modified Ricci dark energy density, scalar field, and scalar field are all examples of this. At late times, the dark energy pressure approaches zero, and they all differ at the beginning of time. The Hubble's value becomes constant in late times, indicating that the universe is expanding uniformly.

Santhi et al. (Santhi et al., 2016b) investigated the behaviour of a five-dimensional spherically symmetric cosmological model filled with minimally interacting matter and holographic dark energy in the S$\acute{a}$ez-Ballester scalar-tensor theory of gravitation.

Singh et al. (Singh and Tiwari, 2016) studied the importance of a novel class of Hypersurface-homogeneous cosmological models featuring bulk viscous matter as a source of cosmic fluid in the context of S$\acute{a}$ez-Ballester's scalar-tensor theory of gravitation. They discover that the universe begins in a finite-time big-bang single state and expands with cosmic time t.

Reddy (Reddy, 2017) examined a Bianchi type-V universe filled with matter and modified holographic Ricci dark energy in a scalar-tensor theory given by S$\acute{a}$ez-Ballester.

He observes that models with hybrid expansion laws and variable deceleration parameters invariably vary in the quintessence zone, but models with linearly varying deceleration parameters fluctuate in the phantom region.

Rao et al. (Rao and Prasanthi, 2017) explored the anisotropic and homogeneous LRS Bianchi type-I and Bianchi type-III models filled with matter and MHRDE in S$\acute{a}$ez-Ballester's tensor theory of gravitation. They characterised the spatial volume of the models increasing with cosmic time, implying that the universe is spatially expanding.

Kanakavalli et al. (Kanakavalli et al., 2017) investigated axially symmetric Bianchi type cosmic strings in a scalar-tensor theory of gravitation proposed by S$\acute{a}$ez-Ballester. They discovered that the geometric and massive strings perish while the Takabayasi string coexists with the S$\acute{a}$ez-Ballester scalar field.

Verma et al. (Verma et al., 2017) investigated exact solutions of field equations for hypersurface-homogeneous space-times in the presence of anisotropic dark energy in the S$\acute{a}$ez-Ballester theory of gravitation by using the variation law for generalised Hubble parameters, which yields a negative value of the deceleration parameter.

Reddy et al. (Reddy et al., 2017) investigated the LRS Bianchi type-II cosmological model in the context of the S$\acute{a}$ez-Ballester scalar-tensor theory of gravitation in the presence of a matter source and anisotropic MHRDE. They obtained an anisotropic universe model that approaches isotropy at late time.

Pawar et al. (Pawar and Shahare, 2018) investigated an anisotropic tilted Bianchi type-I cosmological model in Kasner form filled with non-viscous fluid within the context of the S$\acute{a}$ez-Ballester theory of gravitation. They discovered that the slanted Kasner universe expands anisotropically and that the universe is decelerating.

Rao et al. (Rao et al., 2018) investigated the non-static plane symmetric DE model in S$\acute{a}$ez-Ballester scalar-tensor theory of gravitation. They assumed the DE candidate was a freshly proposed modified version of holographic Ricci dark energy and built the model with a linearly increasing deceleration parameter.

Aditya et al. (Aditya and Reddy, 2018) investigated the rapid expansion by assuming a new holographic dark energy in an LRS Bianchi type universe using the S$\acute{a}$ez-Ballester scalar-tensor theory of gravity. They solved the S$\acute{a}$ez-Ballester field equations using the relationship between the metric potentials, which results in a variable deceleration parameter.

Santhi et al. (Santhi et al., 2018) studied an anisotropic Bianchi Type-III magnetised modified and holographic Ricci dark energy cosmological model in the context of the Sáez-Ballester theory of gravitation.

Vinutha et al. (Vinutha et al., 2018) investigated a spatially homogeneous anisotropic Kantowski-Sachs, locally rotationally symmetric (LRS) Bianchi type-I and LRS Bianchi type-III space time filled with dark energy and one dimensional string in the framework of Sáez-Ballester scalar tensor theory of gravitation. They obtained models that represent accelerating and expanding cosmological models of the universe.

Mishra et al (Mishra and Dua, 2019) examined bulk viscous LRS bianchi type-II cosmological models in Sáez-Ballaster theory and got universe models with accelerated expansion phases both now and in the future.

Sharma et al. (Sharma et al., 2019) studied the Bianchi-V universe with accurate solutions of Einstein's modified field equations in the setting of Sáez-Ballester theory with heat conduction and perfect fluid.

Vinutha et al. (Vinutha et al., 2019) investigated the Kantowski-Sachs (KS) space time in the Sáez-Ballester theory of gravitation filled with generalised ghost dark energy (GGDE) and dark matter. They discovered that our model is in a freezing zone and that the dark energy pressure is negative in the present and late times, which is the source of the universe's accelerated expansion. They obtain the model that portrays the universe's accelerating and expanding cosmological model.

Mishra et al. (Singh and et al., 2020) investigated Bianchi type-I cosmological models in Sáez-Ballester theory of gravity with bilinear variable DP. In this communication, they obtained an exact solution of modified EFEs by taking DP as an assumption, as well as the power law relationship between G and a(t).

Santhia et al. (Santhia and Sobhanbabub, 2020) examined Tsallis holographic dark energy (infrared cutoff being the Hubble radius) in homogeneous and anisotropic Bianchi type-III universes using the Sáez-Ballester scalar-tensor theory of gravitation. They built non-interaction and interaction dark energy models by solving the Sáez-Ballester field equations.

Raut (Raut, 2020) examined the cylindrically symmetric Einstein Rosen universe filled with two minimally interacting fluids, matter, and holographic dark energy components in S$\acute{a}$ez-Ballester's scalar-tensor theory of gravitation.

Manekar et al (Manekar et al., 2021) studied five-dimensional inflationary universes within the framework of the S$\acute{a}$ez-Ballester Theory of Gravitation and discovered that the Scalar field and the S$\acute{a}$ez-Ballester Scalar Field of the given inflationary model are straight lines that incense infinitely with infinite time intervals.

Mishra et al. (Mishra and Dua, 2021) have created a Bianchi type-I cosmological model with a changeable cosmic constant in the S$\acute{a}$ez-Ballester theory of gravity.

Naidu et al.(Naidu et al., 2021) explored FRW type Kaluza-Klein cosmological models in S$\acute{a}$ez-Ballester scalar tensor theory of gravitation and discovered that all models advocate for the universe's accelerated expansion.

Santhi et al. (Santhi and Sobhanbabu, 2021) examined interacting and non-interacting Tsallis holographic dark energy models in an anisotropic and homogeneous Bianchi type-VI0 space time using S$\acute{a}$ez-Ballester's scalar-tensor theory.

Vinutha et al. (Vinutha et al., 2021) proposed a spatially homogenous anisotropic Kantowski-Sachs space time filled with viscous string new holographic dark energy and dark matter within the context of the S$\acute{a}$ez-Ballester scalar-tensor theory of gravitation. They discovered that the spatial volume (V) increases with cosmic time and that the time-dependent DP (q) is always negative. They also discovered that the dark energy density has been increasing during the universe's development. They discovered that both dark matter and string tension density are positive and decreasing with regard to cosmic time.

Sobhanbabu et al. (Sobhanbabu and Santhi, 2021) studied the THDE in the spatially homogenous anisotropic KK universe using the S$\acute{a}$ez-Ballester scalar-tensor theory of gravity with DM and THDE. To reach the deterministic solution of the universe model, they consider various physically feasible conditions, which result in a variable deceleration parameter that depicts the universe's decelerating to accelerating expansion.

Santhi et al. (Santhi and Sobhanbabu, 2022) investigated the spatially homogenous and anisotropic Kantowski-Sachs (KS) space-time with viscous Ricci dark energy in the framework of the S$\acute{a}$ez-Ballester scalar-tensor theory of gravitation. They obtained the deterministic solution of the field equations, which leads to the variable deceleration parameter.

Singh et al. (Singh and Singh, 2022) investigated a spherically symmetric metric in 5D using Sáez-Ballester Theory, which allows for little dark energy-matter interaction. They notice that the model universe begins with the Big Bang and ends with the Big Freeze singularity.