Chapter 2

Five Dimensional Bianchi Type-I Anisotropic Cloud String Cosmological Model with Electromagnetic Field in S$\acute{a}$ez-Ballester Theory

2.1 Introduction

According to several forms of literature, our universe is expanding at an accelerated rate. Bull et al. (Bull and et al., 2016) investigated alternative cosmology and summarised the current status of ΛCDM as a physical theory in addition to the standard model ΛCDM in extending cosmology. On a massive scale, FRW space-time can only explain an isotropic and homogenous universe. Recent discoveries and reasoning, however, demonstrate that an anisotropic phase exists throughout the universe's cosmic expansion before it shifts to an isotropic phase. Bianchi type cosmological models depict the homogeneous and anisotropic worlds and their isotropy nature can be investigated across time. Furthermore, from a theoretical sense, anisotropic worlds are more general than isotropic model universes. Several studies (Akarsu and Kilinc, 2010; Amirhashchi et al., 2009; Sahoo and Mishra, 2015; Singh et al., 2021) examined the anisotropic Bianchi

The work presented in this chapter has been published in "Frontiers in Astronomy and Space Sciences", volume 9, pp.1-9, (2022)

type cosmological model from diverse perspectives. In his work "Topology of Cosmic Domains and strings," Kibble (Kibble, 1976) characterised the stable topological faults that occurred throughout the phase shift as strings. He also shown that the homogeneity group of the manifold of degenerate vacua impacts domain structure topological defects. Letelier (Letelier, 1983b) investigated string cosmology using the Bianchi type-I and Kantowski-Sachs space-time cosmological models.

Several authors have recently concentrated on string cosmological models due to the importance of string in characterising the early phases of our universe's evolution. At the same time, the string can describe the nature and essential configuration of the early universe. String theory is the most actively researched approach to quantum gravity and it can be used to analyse the mechanics of the early universe. String theory integrates all matter and forces into a single theoretical structure and portrays our universe's early phases in terms of vibrating strings rather than particles. According to Kibble (Kibble, 1976), cosmic strings are stable line-like topological objects/defects that emerge at some point during our universe's early history at the phase transition. According to GUT (grand unified theories), symmetry is broken during the phase transition in the early stages of the universe after the big bang (Everett, 1980; Kibble, 1976, 1980; Vilenkin, 1981a,c; Zel'dovich et al., 1974), and these strings appear when the cosmic temperature drops below. Strings can so play an essential role in researching the universe's early stages. Massive closed loops of strings generate massive scale structures such as galaxies and clusters of galaxies. The gravitational field interacts with the cosmic strings, which may contain stress-energy. As a result, one of the most fascinating projects is the study of gravitational forces caused by cosmic strings.

Letelier (Letelier, 1983b) was the first to obtain massive string cosmological models in Bianchi type-I and Kantowski- Sachs space-times. Following Letelier, a number of authors looked into string cosmological models in a variety of contexts.

Krori et al. (Krori et al., 1990) and Wang (Xing-Xiang, 2003) investigated the Letelier string cosmology model and obtained accurate solutions utilising Bianchi type-II, -VI0, -VIII, and -IX space-times. Using the LRS Bianchi type-I metric and the coefficient of bulk viscosity as a power function of energy density, Xing (Xing-Xiang, 2004) created an exact solution string cosmological model with bulk viscosity. Yavuz et al. (Yavuz et al., 2005) studied charged odd quark matter coupled to the string cloud in spherical symmetric space-time and discovered a one-parameter group of conformal motions. Yilmaz (Yilmaz, 2006) established the Kaluza-Klein cosmological solutions for quark matter coupled to the string cloud in the context of general relativity. Rao (Rao et al., 2008b) developed an accurate perfect fluid cosmological model based on the Lyra manifold with a constant displacement vector while studying a Bianchi type-V space-time in a scalar-tensor theory. However, if β is a function of cosmic time t, then this model only applies to radiation. Tripathy (Tripathy et al., 2009) studied an anisotropic and spatially homogeneous Bianchi type-VI0 space-time and produced cloud string cosmological models in S$\acute{a}$ez-Ballester scalar-tensor theory. Adhav et al. (Adhav et al., 2007b) created string cosmological models in the Brans-Dicke theory of gravitation by solving the field equations with the condition that the sum of tension density and energy density is zero. Pawar (Pawar et al., 2018) studied the Kaluza-Klein string cosmological model in the setting of the f(R, T) theory of gravity, solving the field equations utilising a power law link between scale factor and a time-varying deceleration parameter.

The cosmological model is crucial in the evolution of the universe and the formation of large scale structures such as galaxies and other celestial bodies in the presence of an electromagnetic field. The current period of accelerated expansion of the universe is caused by the presence of a cosmic electromagnetic field produced during inflation. Jimenez and Maroto (Jimenez and Maroto, 2009) demonstrated on cosmological scales that the presence of an electromagnetic field provides an effective cosmological constant that accounts for the universe's accelerated expansion. Tripathy et al. (Tripathi et al., 2017) studied an inhomogeneous string cosmological model with electromagnetic field in general relativity. Parikh (Parikh et al., 2018) recently examined a Bianchi type-II string dust cosmology model with an electromagnetic field in Lyra's Geometry. Grasso and Rubinstein (Grasso and Rubinstein, 2000) investigated a wide range of magnetic field properties in the early universe, as well as modern temporal fields and their evolution in galaxies and clusters. The magnetic field also helps to break down statistical homogeneity and isotropy. Large scale galaxies and galaxy clusters contain magnetic fields. Subramanian (Subramanian, 2016) proved in his work that primordial magnetic fields have a substantial impact on the formation of star formations, especially on dwarf galaxy scales.

2.2 The metric and field equations

We consider the spatially homogenous and anisotropic Bianchi type-I metric as given below

$$ds^2 = -dt^2 + A^2(dx^2 + dy^2) + B^2dz^2 + C^2d\psi^2 \quad (2.1)$$

where *A*, *B* and *C* are functions of cosmic time *t*.

S*á*ez-Ballester provides the Einstein field equations for the combined scalar and tensor fields as given below

$$R_{ij} - \frac{1}{2}Rg_{ij} - \omega\phi^m(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}) = -T_{ij} \quad (2.2)$$

also the scalar field fulfils the equation

$$2\phi^m\phi^{;i}_{;i} + m\phi^{m-1}\phi_{,k}\phi^{,k} = 0 \quad (2.3)$$

We make the following assumptions for the metric (2.1).

$x^1 = x,\ x^2 = y,\ x^3 = z,\ x^4 = \psi,\ and\, x^5 = t$

The energy-momentum tensor of a cloud string (Reddy, 2003) has the form

$$T_{ij} = \rho u_i u_j - \lambda x_i x_j + E_{ij} \quad (2.4)$$

where ρ_p denotes particle density, λ is string tension density and ρ is the rest energy density of the cloud of strings with particles connected to them by $\rho = \rho_p + \lambda$. x_i is a unit space-like vector that denotes string direction, with $x^2 = 0 = x^3 = x^4 = x^5$ and $x^1 \neq 0$ and u_i is the five velocity vector that satisfies the following conditions

$$u^i u_i = -x^i x_i = -1 \quad (2.5)$$

and

$$u^i x_i = 0 \quad (2.6)$$

The five velocity vectors u^i and the direction of the string x^i, are given by

$$u^i = (0, 0, 0, 0, 1) \tag{2.7}$$

and

$$x^i = (\frac{1}{A}, 0, 0, 0, 0) \tag{2.8}$$

where the direction of the strings is parallel to the x-axis.

The electromagnetic field E_{ij}, which is a component of the energy momentum tensor, is given as

$$E_{ij} = \frac{1}{4\pi}(g^{\alpha\beta} F_{i\alpha} F_{i\beta} - \frac{1}{4} g_{ij} F_{\alpha\beta} F^{\alpha\beta}) \tag{2.9}$$

$F_{\alpha\beta}$ denotes the electromagnetic field tensor. If the magnetic field is quantized along the x-axis, F_{15} is the only non-vanishing component of the electromagnetic field tensor F_{ij}. Assuming infinite electromagnetic conductivity, it can be defined that $F_{12} = F_{13} = F_{14} = F_{23} = F_{24} = F_{25} = F_{34} = F_{35} = 0$. (Singh et al., 2020).

As a result, the nontrivial components of the electromagnetic field E_{ij} can be determined using equation (2.9), as define below

$$E_1^1 = -E_2^2 = -E_3^3 = -E_4^4 = E_1^1 = -\frac{1}{2} g^{11} g^{55} F_{15}^2 = \frac{1}{2A^2} F_{15}^2 \tag{2.10}$$

Alternatively, if we take the magnetic field along the x-axis in co-moving coordinates, F_{34} is the only non-vanishing component of the electromagnetic field tensor F_{ij}. Assuming infinite electromagnetic conductivity, $F_{15} = F_{25} = F_{35} = F_{12} = F_{13} = F_{14} = 0$. (Singh et al., 2020).

As a result, the nontrivial components of the electromagnetic field E_{ij} can be determined using equation (2.9).

$$E_1^1 = -E_2^2 = -E_3^3 = -E_4^4 = E_1^1 = -\frac{1}{2A^2} F_{34}^2 \tag{2.11}$$

The physical quantities for the metric (2.1), the spatial volume *V*, the average scale factor *a*, the expansion scalar θ, the Hubble's parameter *H*, the deceleration parameter *q*, the shear scalar σ^2 and the mean anisotropy parameter $\triangle$ are given as follows

$$V = a^4 = A^2BC \tag{2.12}$$

$$\theta = u^i_{;i} = 2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \tag{2.13}$$

$$H = \frac{\dot{a}}{a} = \frac{1}{4}(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}) \tag{2.14}$$

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = \frac{d}{dt}(\frac{1}{H}) - 1 \tag{2.15}$$

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{2}(2\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2}) - \frac{\theta^2}{8} \tag{2.16}$$

$$\triangle = \frac{1}{4}\sum_{i=1}^{4}(\frac{H_i - H}{H})^2 \tag{2.17}$$

Here an over head dot indicates the first derivative of cosmic time *t*, whereas a double over head dot represents the second. Also, $H_i(i = 1, 2, 3, 4)$ represents the directional Hubble's Parameters in the *x*, *y*, *z* and ψ axis, which are determined for metric (2.1) as $H_1 = H_2 = \frac{\dot{A}}{A}$, $H_3 = \frac{\dot{B}}{B}$, and $H_4 = \frac{\dot{C}}{C}$.

When combined with equation (2.4) for the line element equation (2.1), the S$\acute{a}$ez-Ballester field equation (2.2), (2.3) are reduced to the following set of equations

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{C}}{C} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} = \lambda - \frac{1}{2A^2}F_{15}^2 + \omega\phi^m\frac{\dot{\phi}^2}{2} \tag{2.18}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{C}}{C} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} = \frac{1}{2A^2}F_{15}^2 + \omega\phi^m\frac{\dot{\phi}^2}{2} \tag{2.19}$$

$$2\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} + \frac{\ddot{C}}{C} + 2\frac{\dot{A}}{A}\frac{\dot{C}}{C} = \frac{1}{2A^2}F_{15}^2 + \omega\phi^m\frac{\dot{\phi}^2}{2} \tag{2.20}$$

$$2\frac{\ddot{A}}{A} + \frac{\dot{A}^2}{A^2} + \frac{\ddot{B}}{B} + 2\frac{\dot{A}}{A}\frac{\dot{B}}{B} = \frac{1}{2A^2}F_{15}^2 + \omega\phi^m\frac{\dot{\phi}^2}{2} \tag{2.21}$$

$$\frac{\dot{A}^2}{A^2} + 2\frac{\dot{A}}{A}\frac{\dot{B}}{B} + 2\frac{\dot{A}}{A}\frac{\dot{C}}{C} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} = \rho - \frac{1}{2A^2}F_{15}^2 - \omega\phi^m\frac{\dot{\phi}^2}{2} \tag{2.22}$$

$$\ddot{\phi} + \dot{\phi}(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}) + \frac{m}{2}\frac{\dot{\phi}^2}{\phi} = 0 \tag{2.23}$$

2.3 Solution of the field equation

Equations (2.18-2.22) are a set of five equations with seven unknown parameters (*A*, *B*, *C*, ρ, λ, ϕ and F_{15}). In order to achieve explicit system solutions, two additional constraints linking these parameters are required. These two relationships are considered as

(I) We use the fact that the shear scalar σ is proportional to the expansion scalar θ (Collins et al., 1983; Kiran et al., 2015).

$$B = C^n \tag{2.24}$$

where *n* is a constant.

(II) Berman (Berman, 1983b) and Ram et al. (Ram et al., 2010) established a relationship between the Hubble's parameter *H* and the average scale factor *R*. Berman (Berman, 1983b), Berman and Gomide (Berman and de Mello Gomide, 1988), and Ram et al. (Ram et al., 2010) solved FRW models using this type of connection, whereas Ram et al. (Ram et al., 2010) solved Bianchi Type-V cosmological models in Lyra's Geometry, which is defined as

$$H = a_1 a^{-k_1} \tag{2.25}$$

here $a_1 \geq 0$, $k_1 \geq 0$ are constants.

Equation (2.25) provides us

$$a = (a_1 k_1 t + k_2)^{\frac{1}{k_1}} \qquad if \qquad k_1 \neq 0 \tag{2.26}$$

and

$$a = k_3 e^{a_1 t} \qquad if \qquad k_1 = 0 \tag{2.27}$$

where k_2 and k_3 are constants.

Equation (2.15) gives the deceleration parameter q as given below

$$q = (k_1 - 1) \qquad if \qquad k_1 \neq 0 \tag{2.28}$$

$$q = -1 \qquad if \qquad k_1 = 0 \tag{2.29}$$

Case I: When $k_1 \neq 0$, we have

Equations (2.19), (2.20), (2.24) and (2.26) will provide us

$$C = k_5 e^{\frac{k_4 (a_1 k_1 t + k_2)^{\frac{k_1 - 4}{k_1}}}{a_1 (k_1 - 4)}} \tag{2.30}$$

In this equation, k_4 and k_5 are constants. We can make the assumption that $k_4 = k_5 = 1$ without losing generality

$$C = e^{\frac{(a_1 k_1 t + k_2)^{\frac{k_1 - 4}{k_1}}}{a_1 (k_1 - 4)}} \tag{2.31}$$

Equations (2.12), (2.24) and (2.26) will give us

$$A = (a_1 k_1 t + k_2)^{\frac{2}{k_1}} e^{-\frac{(a_1 k_1 t + k_2)^{\frac{k_1 - 4}{k_1}} (n+1)}{2 a_1 (k_1 - 4)}} \tag{2.32}$$

and

$$B = e^{\frac{n (a_1 k_1 t + k_2)^{\frac{k_1 - 4}{k_1}}}{a_1 (k_1 - 4)}} \tag{2.33}$$

Hence, the line element equation (2.1) becomes

$$ds^2 = -dt^2 + (a_1 k_1 t + k_2)^{\frac{4}{k_1}} e^{-\frac{(a_1 k_1 t + k_2)^{\frac{k_1 - 4}{k_1}} (n+1)}{a_1 (k_1 - 4)}} (dx^2 + dy^2) + e^{\frac{2n (a_1 k_1 t + k_2)^{\frac{k_1 - 4}{k_1}}}{a_1 (k_1 - 4)}} dz^2 + e^{\frac{2 (a_1 k_1 t + k_2)^{\frac{k_1 - 4}{k_1}}}{a_1 (k_1 - 4)}} d\psi^2 \tag{2.34}$$

Equation (2.34) is a Bianchi Type-I cosmological model universe with electromagnetic field and special law of Hubble's parameter that becomes singular for $k_1 = 4$.

2.4 Some physical properties of the model

We can now calculate the values of energy density ρ, string tension density λ and electromagnetic field tensor F_{15} for the model universe provided by equation (2.34) using Equations (2.18)-(2.22).

$$\rho = -4a_1^2(k_1 - 4)(a_1k_1t + k_2)^{-2} \tag{2.35}$$

$$\begin{aligned}\lambda = \frac{1}{2}(3n^2 + 2n + 3)(a_1k_1t + k_2)^{\frac{-8}{k_1}} - 2a_1^2(3k_1 - 8)(a_1k_1t + k_2)^{-2} \\ - 4a_1(n + 1)(a_1k_1t + k_2)^{-(\frac{k_1+4}{k^1})} - \omega k_6^2(a_1k_1t + k_2)^{-\frac{8}{k_1}}\end{aligned} \tag{2.36}$$

$$\begin{aligned}F_{15} = \sqrt{2}[\frac{1}{4}(3n^2 + 2n + 3)(a_1k_1t + k_2)^{-\frac{4}{k_1}} - 4a_1^2(k_1 - 3)(a_1k_1t + k_2)^{-2(\frac{k_1-2}{k_1})} \\ - 2a_1(n + 1)(a_1k_1t + k_2)^{-1} - \omega\frac{k_6^2}{2}(a_1k_1t + k_2)^{-\frac{8}{k_1}}]^{\frac{1}{2}} e^{\frac{(a_1k_1t+k_2)^{\frac{k_1-4}{k_1}}(n+1)}{a_1(k_1-4)}}\end{aligned} \tag{2.37}$$

Using the relationship $\rho = \rho_p + \lambda$ we get

$$\begin{aligned}\rho_p = 2a_1^2k_1(a_1k_1t + k_2)^{-2} - \frac{1}{2}(3n^2 + 2n + 3)(a_1k_1t + k_2)^{-\frac{8}{k_1}} + \\ 4a_1(n + 1)(a_1k_1t + k_2)^{-(\frac{k_1+4}{k_1})} + \omega k_6^2(a_1k_1t + k_2)^{-\frac{8}{k_1}}\end{aligned} \tag{2.38}$$

The physical parameters viz. proper volume *V*, Hubble's parameter *H*, expansion scalar θ, scalar field ϕ, shear scalar σ^2 and the mean anisotropy parameter $\triangle$ are determined as follows:

$$V = a^4 = (a_1k_1t + k_2)^{\frac{4}{k_1}} \tag{2.39}$$

$$H = \frac{a_1}{(a_1k_1t + k_2)} \tag{2.40}$$

$$\theta = \frac{4a_1}{(a_1k_1t + k_2)} \tag{2.41}$$

$$\phi = [\frac{k_6(m+2)}{2a_1(k_1-4)}]^{\frac{2}{m+2}}[(a_1k_1t+k_2)^{(\frac{k_1-4}{k_1})}(\frac{2}{m+2})] \tag{2.42}$$

$$\sigma^2 = \frac{(3n^2+2n+3)}{4}(a_1k_1t+k_2)^{-\frac{8}{k_1}} - 2a_1(n+1)(a_1k_1t+k_2)^{-(\frac{k_1+4}{k_1})} + 2a_1^2(a_1k_1t+k_2)^{-2} \tag{2.43}$$

$$\triangle = 1 - \frac{(n+1)}{a_1}(a_1k_1t+k_2)^{(\frac{k_1-4}{k_1})} + \frac{(3n^2+2n+3)}{2a_1^2}(a_1k_1t+k_2)^{2(\frac{k_1-4}{k_1})} \tag{2.44}$$

Equations (2.41) and (2.43) gives

$$\lim_{t\to\infty}\frac{\sigma^2}{\theta^2} \neq 0 \tag{2.45}$$

The variation of some of the parameters is graphically represented using the values $a_1 = k_1 = k_6 = n = 1$ and $k_2 = 0$.

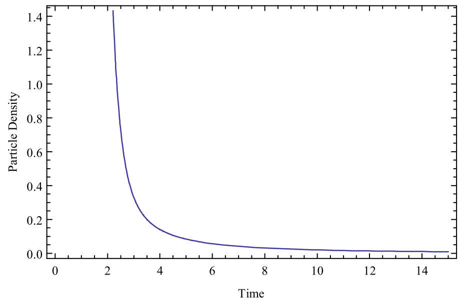

Figure 2.1: ρ_p vs. t (billion years)

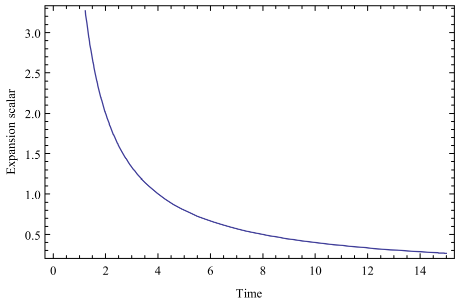

Figure 2.2: θ vs. t (billion years)

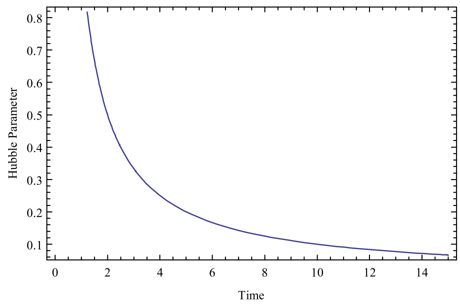

Figure 2.3: H vs. t (billion years)

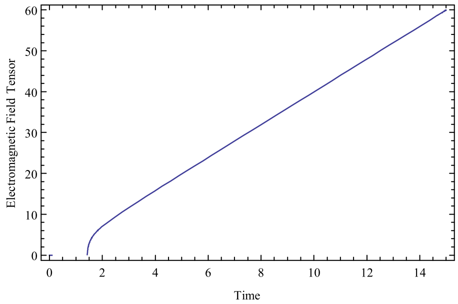

Figure 2.4: F_{15} vs. t (billion years)

.

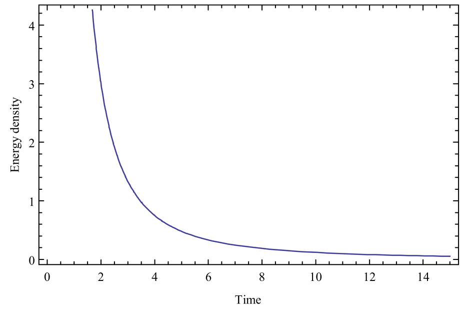

Figure 2.5: ρ vs. t (billion years)

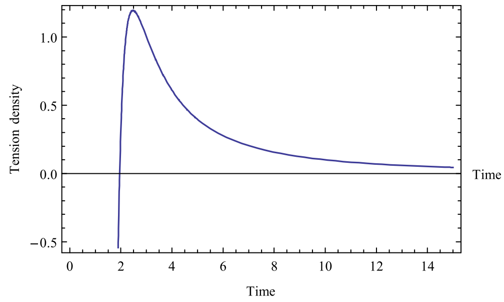

Figure 2.6: λ vs. t (billion years)

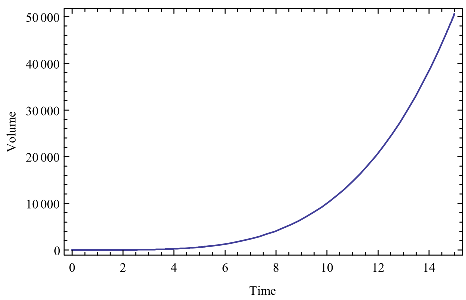

Figure 2.7: V vs. t (billion years)

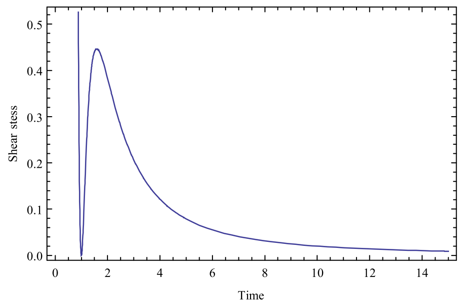

Figure 2.8: σ^2 vs. t (billion years)

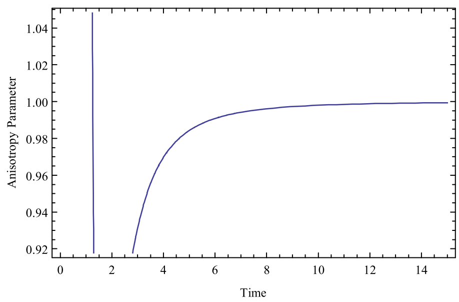

Figure 2.9: $\triangle$ vs. t (billion years)

2.5 Physical interpratation

As stated in equation (2.28), the deceleration parameter q is constant and negative. Therefore the model obtained in this chapter has been inflating and expanding at a consistent rate as cosmic time $t = 0$. The pace of expansion of the universe remains constant throughout the evolution. For $k_1 < 1$, $q = constant$ (negative value). Furthermore, the particle density is extremely positive at $t = 0$ during the big bang. As time passes, it approaches a positive value, finally attaining a finite constant value as $t \to \infty$

(shown in figure 2.1). This shows that the particle continues to dominate the universe over time and that the total number of particles in the universe remains constant. The expansion scalar and the Hubble's parameter, displays comparable variations throughout time. Both the Hubble's parameter *H* and the expansion scalar decreases with cosmic time *t* after developing from infinity at $t = 0$ to a finite value as $t \to \infty$ (shown in figures 2.2, 2.3). The expansion of the universe appears to be accelerating and the deceleration parameter is negative in this scenario, $\ddot{a} > 0$. The model defined in this chapter fulfils the energy conditions $\rho \geq 0$ and $\rho_p \geq 0$. The electromagnetic field modifies the model's behaviour as well as the physical parameter expressions. The rest energy density and string tension density decreases in the presence of an electromagnetic field (shown in figures 2.5, 2.6). String tension density and particle density are also comparable, however string tension density vanishes faster than particle density. This demonstrates that the model defined in this chapter displays a matter-dominated universe at late time scales, which matches current observational data. Equation (2.39) shows that the appropriate volume $V = k_2^{\frac{4}{k_1}}$ at $t = 0$ increases with cosmic time *t* (shown in Figure 2.7). This indicates that at $t = 0$, the universe has a finite volume and subsequently expands as cosmic time passes. As time $t \to \infty$, the corresponding volume *V* becomes infinite. The non-vanishing electromagnetic field tensor F_{15} rises exponentially as a function of cosmic time *t*, as shown by equation (2.37). (shown in figure 2.4). We can see that the electromagnetic field tensor F_{15} does not vanishes when $a_1 \neq 0$. It has a significant impact on the production of strings early in the universe's evolution. The string and the electromagnetic field coexist in this scenario.

The parameters shear scalar σ^2 and mean anisotropy parameter $\triangle$ diverges at the initial singularity (shown in figures 2.8, 2.9). The model portrays a shearing, non-rotating universe with the possibility of a big crunch at $t = 0$. From the equation (2.45), we obtained that the isotropic condition $\lim_{t\to\infty} \frac{\sigma^2}{\theta 2} \neq 0$, remains constant throughout the universe's evolution (from early to late time), indicating that the model does not achieve isotropy (Sahoo et al., 2017).