Chapter 7

Bianchi Type-III Cosmological Model with Electromagnetic Field in Sen-Dunn Theory of Gravitation

7.1 Introduction

The magnetic field may be the source behind many astrophysical phenomena reflected in the universe's evolution, indicated that the magnetic field might have a cosmological origin [Zeldovich et al.(1983), Harrison(1973)]. The magnetic field in cosmology is vital for its essential role in describing the ionized behaviour conducting fluctuation in energy in the universe's expansion. The magnetic field may cause anisotropy in the accelerated expansion, and its presence impacts the expansion regardless of its strength, [Matravers & Tsagas(2000)]. The anisotropic universe becomes isotropic due to a decrease in a magnetic field, which affects the universe's geometry, demonstrating the interaction be-

The paper related to this chapter is published in Indian Journal of Science and Technology, 15(35) 1703-1711 (2022)

tween matter and geometry. Further noticeable is an interaction between anisotropy, magnetic field, and deceleration parameter, [Aktaş et al.(2018)]. In this chapter, the magnetized Bianchi type-III space-time with time-dependent deceleration parameter proposed by Banerjee and Das [Banerjee & Das(2005)] is investigated in the Sen-Dunn scalar-tensor theory of gravitation. The exact solutions are derived for the acquired model by amalgamating an electromagnetic field in an energy-momentum tensor and considering other plausible conditions. The cosmological parameters for the model signify its consistent with the recent observations.

7.2 Metric and Field Equation

Bianchi Type-III metric is given by

$$ds^2 = -dt^2 + A^2dx^2 + B^2e^{-2mx}dy^2 + C^2dz^2 \tag{7.1}$$

where A, B and C are function of cosmic time t.
The field equations given by Sen and Dunn (in natural units $c = 1$, 8πG = 1) are defined as

$$R_{ij} - \frac{1}{2}Rg_{ij} = \omega\phi^{-2}(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}) - \phi^{-2}T_{ij} \tag{7.2}$$

Here the function $\omega = \frac{3}{2}$, R_{ij} is the Ricci tensor, R is the Ricci scalar,T_{ij} is the energy stress tensor of the matter.
The energy-momentum tensor is as follows

$$T_{ij} = (\rho + p)u_iu_j + pg_{ij} + E_{ij} \tag{7.3}$$

where E_{ij} is the electromagnetic field tensor and is given as

$$E_{ij} = \frac{1}{4\pi}\left[g^{hl}F_{ih}F_{jl} - \frac{1}{4}F_{hl}F^{hl}g_{ij}\right] \tag{7.4}$$

From Maxwell's equation

$$\frac{\partial}{\partial x^j}\left(F^{ij}\sqrt{-g}\right)=0 \tag{7.5}$$

which leads to the following result

$$F_{12}=Ke^{-mx} \tag{7.6}$$

where K and m are constants.

The field equation (7.2) for line element (7.1) along with (7.3) is expressed as

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}-\frac{m^2}{A^2}=\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2-\phi^{-2}\left(p-\frac{K^2}{8\pi A^2B^2}\right) \tag{7.7}$$

$$\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{B}\dot{C}}{BC}=\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2+\phi^{-2}\left(p-\frac{K^2}{8\pi A^2B^2}\right) \tag{7.8}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{C}}{C}+\frac{\dot{A}\dot{C}}{AC}=\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2+\phi^{-2}\left(p-\frac{K^2}{8\pi A^2B^2}\right) \tag{7.9}$$

$$\frac{\ddot{A}}{A}+\frac{\dot{A}\dot{C}}{AC}+\frac{\dot{B}\dot{C}}{BC}-\frac{m^2}{A^2}=\phi^{-2}\left(\rho+\frac{K^2}{8\pi A^2B^2}\right)-\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 \tag{7.10}$$

$$m\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)=0 \tag{7.11}$$

The following equation can describe the average scaling component of Bianchi Type-III space-time

$$a(t)=(ABC)^{\frac{1}{3}} \tag{7.12}$$

7.3 Solution of the Field Equations

Integrating (7.11), the equation is obtained

$$A=nB \tag{7.13}$$

where n is the integration constant. Here considering $n = 1$ without the loss of generality. So the field equation (7.7)-(7.10) is transformed to

$$2\frac{\ddot{B}}{B} + (\frac{\dot{B}}{B})^2 - \frac{m^2}{B^2} = \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 - \phi^{-2}\left(p - \frac{K^2}{8\pi B^4}\right) \tag{7.14}$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 + \phi^{-2}\left(p - \frac{K^2}{8\pi B^4}\right) \tag{7.15}$$

$$\frac{\ddot{B}}{B} + 2\frac{\dot{B}\dot{C}}{BC} - \frac{m^2}{B^2} = \phi^{-2}\left(\rho + \frac{K^2}{8\pi B^4}\right) - \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 \tag{7.16}$$

Here (7.14)-(7.16) equation contains five unknown parameters B, C, p, ρ, ϕ. To obtain the solutions of the unknown parameters considering the following mathematical conditions.
(i) Assuming the conditions that the shear scalar is proportional to the expansion scalar, which leads to the formulation

$$B = C^{\beta} \tag{7.17}$$

where β is an arbitrary constant.
(ii) The power-law relation as proposed by Johri and Desikan [Johri & Desikan(1994)] is given as

$$\phi = \phi_0 a^f(t) \tag{7.18}$$

where, ϕ_0; and f being an ordinary constant.
To obtain the solution of the field equation compaitable with the cosmological observation, considering the time-dependent deceleration parameter [Banerjee & Das(2005)]

$$q = -1 + \frac{k}{1+a^k} \tag{7.19}$$

which is the special form of deceleration parameter where $a(t)$ is the scale factor and $k(> 0)$ is a constant. As deceleration parameter plays a vital role in describing the transitional phase of expansion of the universe from decelerating to accelerating. Some researchers [Singh(2008b), Singha & Debnath(2009), Tiwari & Beesham(2018)] addressed different cosmological models in different frameworks considering the above said time-dependent deceleration parameter.

From the equation (7.19), the Hubble parameter is obtained as

$$H = \frac{\alpha e^{\alpha k t}}{e^{\alpha k t} - 1} = \alpha(1 + a^{-k}) \tag{7.20}$$

Further integrating the equation (7.20), the scale factor is given as

$$a(t) = \left(e^{\alpha k t} - 1\right)^{\frac{1}{k}} \tag{7.21}$$

where α and k are taken as positive constant.

Using equations (7.13), (7.17), and (7.21) in (7.12), the metric potential of the model is given as

$$A = B = \left(e^{\alpha k t} - 1\right)^{\frac{3\beta}{k(2\beta+1)}} \tag{7.22}$$

$$C = \left(e^{\alpha k t} - 1\right)^{\frac{3}{k(2\beta+1)}} \tag{7.23}$$

Thus the model (7.1) is reduced to the form

$$ds^2 = -dt^2 + \left(e^{\alpha k t} - 1\right)^{\frac{6\beta}{k(2\beta+1)}} dx^2 + \left(e^{\alpha k t} - 1\right)^{\frac{6\beta}{k(2\beta+1)}} dy^2 + \left(e^{\alpha k t} - 1\right)^{\frac{6}{k(2\beta+1)}} \tag{7.24}$$

7.4 Model's Cosmological parameters

The spatial volume (V), expansion scalar (θ), shear scalar (σ) and anisotropic parameter (A_m) is obtained as

$$V = \left(e^{\alpha k t} - 1\right)^{\frac{3}{k}} \tag{7.25}$$

$$\theta = \frac{3\alpha e^{\alpha k t}}{e^{\alpha k t} - 1} \tag{7.26}$$

$$\sigma = \frac{\sqrt{3}(\beta - 1)\alpha e^{\alpha k t}}{(2\beta + 1)(e^{\alpha k t} - 1)} \tag{7.27}$$

Also,

$$\frac{\sigma^2}{\theta^2} = \frac{(\beta-1)^2}{3(2\beta+1)^2} = constant(\neq 0, for \beta > 1) \tag{7.28}$$

$$A_m = \frac{2(\beta-1)^2}{(2\beta+1)^2} \tag{7.29}$$

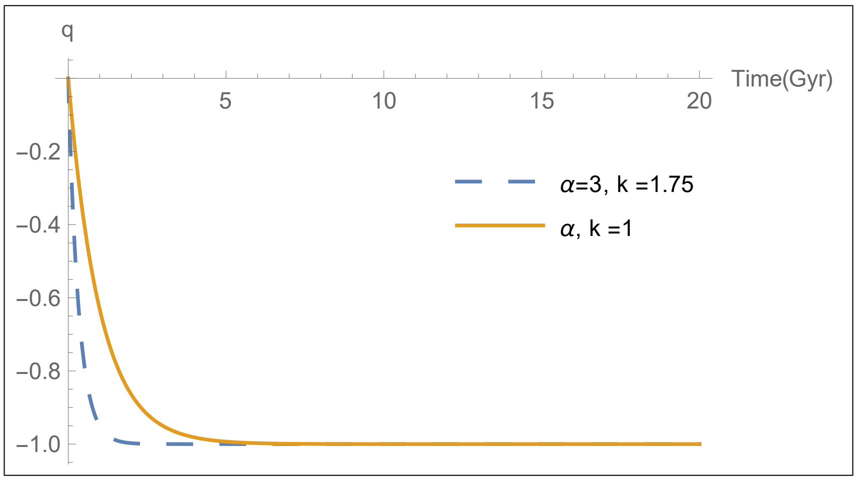

Figure 7.1: Variation of q $vs.$ t

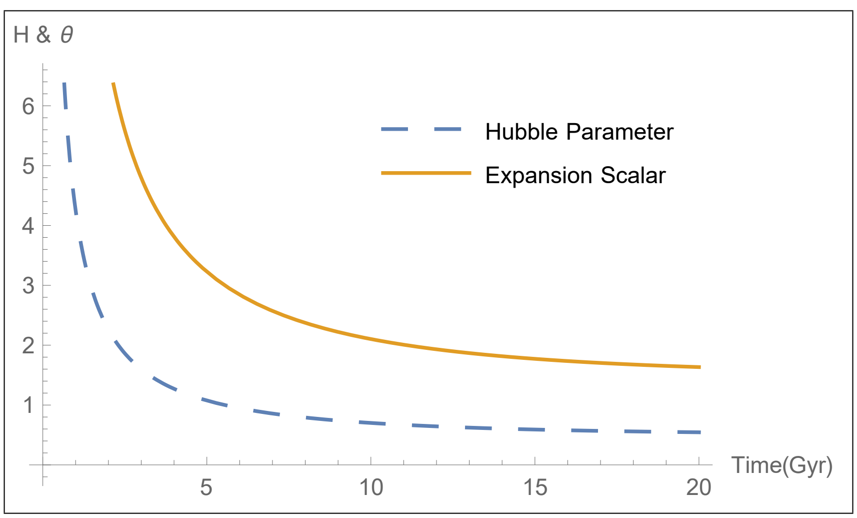

Figure 7.2: Variation of H and θ $vs.$ t, For α = 0.5, k = 1.75

From the equation (7.14)-(7.16), the following energy density and pressure are given as

$$\rho = \phi_0^2 \left(e^{\alpha kt}-1\right)^{\frac{f}{k}} \left[\frac{9(1+3\beta-\beta^2)\alpha^2 e^{2\alpha kt}}{(2\beta+1)^2(e^{\alpha kt}-1)^2} - \frac{3k\alpha^2 e^{\alpha kt}}{(2\beta+1)(e^{\alpha kt}-1)^2} + \frac{3\alpha^2 f^2 e^{2\alpha kt}}{4(e^{\alpha kt}-1)^2}\right] + \frac{K^2}{8\pi\left(e^{\alpha kt}-1\right)^{\frac{12\beta}{k(2\beta+1)}}} \tag{7.30}$$

$$-p = \frac{\phi_0^2}{2}\left(e^{\alpha kt}-1\right)^{\frac{f}{k}} \left[\frac{9(1+\beta+4\beta^2)\alpha^2 e^{2\alpha kt}}{(2\beta+1)^2(e^{\alpha kt}-1)^2} - \frac{3(3\beta+1)k\alpha^2 e^{\alpha kt}}{(2\beta+1)(e^{\alpha kt}-1)^2} - m^2\left(e^{\alpha kt}-1\right)^{\frac{-6\beta}{(2\beta+1)k}} - \frac{3\alpha^2 f^2 e^{2\alpha kt}}{2(e^{\alpha kt}-1)^2}\right] \tag{7.31}$$

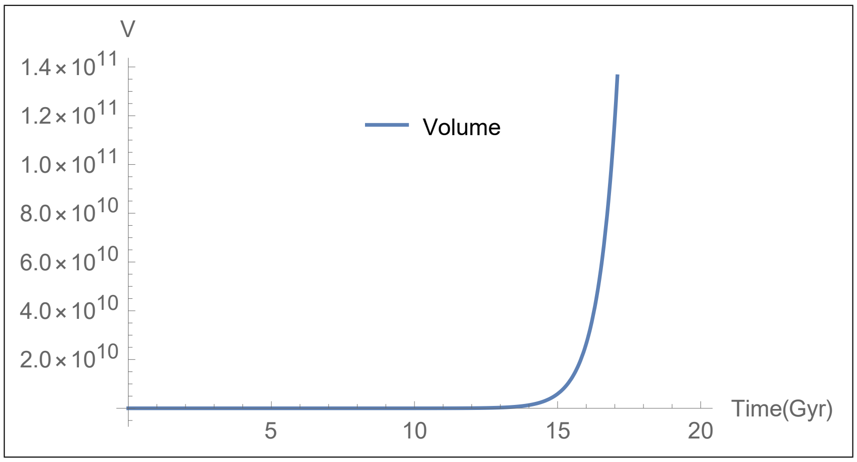

Figure 7.3: Variation of V *vs*. t, For α = 0.5, k = 1.75

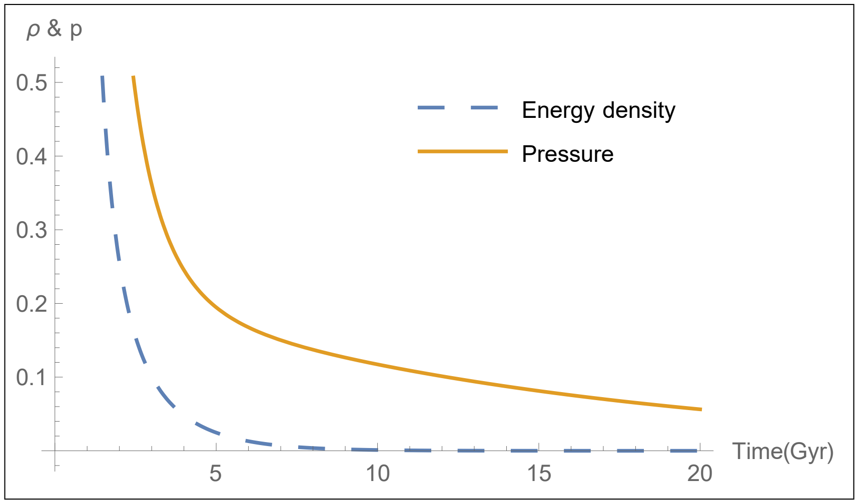

Figure 7.4: Variation of ρ and p $vs.\ t$, For α = 0.5, k = 1.75, β = 1.5, f = -0.62, ϕ_0 = 1

7.5 Energy condition and Statefinder parameter

From the cosmological parameters (7.30) and (7.31), the stability of the energy conditions for the model is identified.

$$\rho + p = \frac{\phi_0^2}{2}\left(e^{\alpha kt}-1\right)^{\frac{f}{k}}\left[\frac{9(1+5\beta-6\beta^2)\alpha^2 e^{2\alpha kt}}{(2\beta+1)^2(e^{\alpha kt}-1)^2} + \frac{3(3\beta-1)k\alpha^2 e^{\alpha kt}}{(2\beta+1)(e^{\alpha kt}-1)^2} + \frac{3\alpha^2 f^2 e^{2\alpha kt}}{(e^{\alpha kt}-1)^2} + m^2\left(e^{\alpha kt}-1\right)^{\frac{-6\beta}{(2\beta+1)k}}\right] + \frac{K^2}{8\pi\left(e^{\alpha kt}-1\right)^{\frac{12\beta}{k(2\beta+1)}}} \geq 0 \tag{7.32}$$

$$\rho - p = \frac{\phi_0^2}{2}\left(e^{\alpha kt}-1\right)^{\frac{f}{k}}\left[\frac{9(2+4\beta+3\beta^2)\alpha^2 e^{2\alpha kt}}{(2\beta+1)^2(e^{\alpha kt}-1)^2} + \frac{9(\beta+1)k\alpha^2 e^{\alpha kt}}{(2\beta+1)(e^{\alpha kt}-1)^2} - m^2\left(e^{\alpha kt}-1\right)^{\frac{-6\beta}{(2\beta+1)k}}\right] + \frac{K^2}{8\pi\left(e^{\alpha kt}-1\right)^{\frac{12\beta}{k(2\beta+1)}}} \geq 0 \tag{7.33}$$

$$\rho+3p=\frac{\phi_0^2}{2}(e^{\alpha kt}-1)^{\frac{f}{k}}\left[\frac{9(3\beta-14\beta^2-1)\alpha^2e^{2\alpha kt}}{(2\beta+1)^2(e^{\alpha kt}-1)^2}+\frac{3(9\beta+1)k\alpha^2e^{\alpha kt}}{(2\beta+1)(e^{\alpha kt}-1)^2}+\frac{6\alpha^2f^2e^{2\alpha kt}}{(e^{\alpha kt}-1)^2}\right.$$
$$\left.+m^2\left(e^{\alpha kt}-1\right)^{\frac{-6\beta}{(2\beta+1)k}}\right]+\frac{K^2}{8\pi\left(e^{\alpha kt}-1\right)^{\frac{12\beta}{k(2\beta+1)}}}\geq 0 \tag{7.34}$$

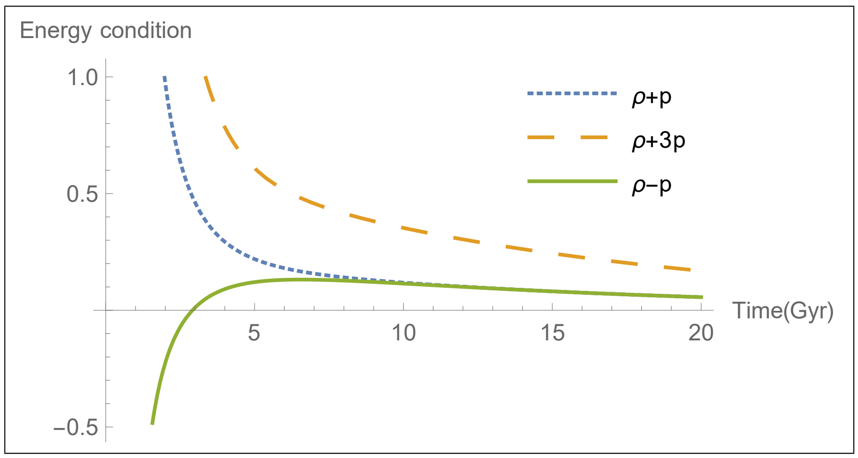

Figure 7.5: Variation of *energy condition vs. t*, For α = 0.5, k = 1.75, β = 1.5, f = -0.62, ϕ_0 = 1

The graphical representation of the energy conditions in Fig. 7.5 certifies that the model's Null energy condition (NEC) is satisfied. The WEC and SEC are also satisfied throughout the cosmic evolution. The DEC for the model is violated for some cosmic time, and later after time $t=0.3$, it satisfies the condition for the stability of the universe's expansion.

Statefinder Parameters$\{$r,s$\}$:

The statefinder parameter $\{$r,s$\}$ is given as

$$r=\frac{\dddot{a}}{aH^3}=1+3\frac{\dot{H}}{H^2}+\frac{\ddot{H}}{H^3} \tag{7.35}$$

$$s=\frac{(r-1)}{3(q-\frac{1}{2})} \tag{7.36}$$

H represents the Hubble parameter, while q represents the deceleration parameter. The cosmological diagnostic pair {r,s} allows us to determine the characteristic properties of the dark energy in a model-independent approach. Thus with the help of equations (7.23) and (7.24), the diagnostic pair {r, s} is derived as

$$r = 1 - \frac{3k}{e^{\alpha kt}} + \frac{k^2(e^{\alpha kt}) + 1}{e^{2\alpha kt}} \tag{7.37}$$

$$s = \frac{2k(ke^{\alpha kt} + k - 3e^{\alpha kt})}{3e^{\alpha kt}(2k - 3e^{\alpha kt})} \tag{7.38}$$

It is observed from the above result that when $t \to \infty$ the cosmological diagnostic pair $\{r, s\} \to \{1, 0\}$, this demonstrates that the universe's model begins with the radiation period and progresses to the ΛCDM model.

7.6 Result and Discussion

The physical and geometrical properties of the model are :

- The relation $\frac{\sigma^2}{\theta^2} = constant$ shows that the model does not tend to isotropy for $\beta \neq 1$ at late times. This result contrast with the study of the Bianchi type I space-time model considering a similar deceleration parameter proclaims isotropy for large values of "t", [Singh(2008b)].

- The deceleration parameter explains the transition from the early deceleration to the late time acceleration. The deceleration parameter of the model here is always accelerating exponentially, as it lies in $-1 < q < 0$ as shown in Fig. 7.1 and it follows the recent cosmological observation. Some model also exhibits super-exponential expansion at the early phase with $q < -1$ and, at late times, tends to exponential expansion $q = -1$, [Santhi & Sobhanbabu(2020)].

- The energy density and the pressure for the model positively decreases. Also, it tends to be zero as $t \to \infty$; for a significant value of time t. A similar characteristic of the energy density, pressure, and Hubble parameter and expansion scalar is also observed in the string cosmological model for Bianchi type-III space-time in Lyra geometry, [Baro et al.(2021)].

- Initially, DEC gets violated for time $t < 0.3$; then rapidly, it satisfies the energy condition as positively decreasing with time. The energy condition depicts the universe's early deceleration to its current acceleration. The NEC, WEC, and SEC satisfy the energy condition throughout the time t.

- From the above result (7.37) and (7.38), the nature of the statefinder parameter is obtained as $\{r,s\} \to (1,0)$ as $t \to \infty$, which shows that the universe approaches the ΛCDM model at late times which match the cosmological model of the universe [Chand et al.(2016)].

- The model is expanding, shearing, and nonrotating with the spatial volume increasing from the finite volume at $t = 0$ and expanding to an infinite volume as $t \to \infty$. The model's expansion scalar and Hubble parameter positively decrease as time increases, which states that the universe's expansion rate is slower as time increases, and expansion gradually ends for $t \to \infty$.

7.7 Conclusion

The following chapter investigates the exact solutions of Bianchi type-III space-time in the presence of an electromagnetic field in the Sen-Dunn scalar-tensor theory of gravitation. A signature flip property is applied to deliver the solutions to the field equations, the time-dependent deceleration parameter proposed by Banerjee and Das [Banerjee & Das(2005)]. The model (7.24) expands with zero volume to infinite as time $t \to \infty$ and eventually, at late times, tends to the de-Sitter universe (as in Fig.7.3). For the significant value of t, the present model does not tend to isotropy, except for particular exceptional cases, $\beta = 1$. The energy density and pressure of the model positively decreases, which tend to zero as $t \to \infty$ (as in Fig. 7.4). The model is expanding, shearing, and anisotropic throughout the evolution admitting initial singularity at $t = 0$. The diagnostic pair $\{r, s\}$ represents that the universe model tends to ΛCDM model as time $t \to \infty$, which is consistent with the recent observational data. Thus, the study presents a better understanding of the Bianchi type-III cosmological model in the Sen-Dunn scalar-tensor theory of gravitation. The generated and given model represents an expanding and accelerating cosmological model, delivered by examining the above physical and geometrical factors. The model studied in this part is compatible with the recent cosmological observation. This study may further be executed in different Bianchi space-time in different frameworks to understand the universe's evolution scenario.