Chapter 2

Bianchi Type-VI_0 Dark Energy Cosmological Model with Variable Deceleration Parameter in Sen-Dunn Theory of Gravitation

2.1 Introduction

In recent decades, many authors have created an alternate hypothesis to Einstein's gravity theory. Brans-Dicke's theory [Brans & Dicke(1961)] is considered one of the most straightforward alternate scalar-tensor theory. However in Brans-Dicke theory, the metric tensor of Riemannian geometry identifies the tensor field, and the scalar field remains foreign to geometry. Sen and Dunn [Sen & Dunn(1971)] is considered a modified version of Brans-Dicke in which the significance of the scalar and tensor field is intrinsically ge-

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ometrized. Exciting explanations of the anisotropic history of the cosmos are provided by observations of the large-scale structure and the cosmic microwave background (CMB). Thus, studying anisotropic Bianchi-type cosmological models offers better knowledge of the isotropic universe. Tiwari et al. [Tiwari et al.(2016a)], Hegazy and Rahaman [Hegazy & Rahaman(2020)] are some researchers who have discussed the Bianchi type-VI_0 cosmic framework with varying deceleration parameters. Also, dark energy is considered one of the most important factors of the present accelerated universe scenario, becoming the major topic of investigation in modern cosmology. Also, several researchers discussed Bianchi type-VI_0 dark energy cosmological model with various cosmological contexts and practical constraints [Sharif & Zubair(2010), Adhav et al.(2011a), Santhi et al.(2017a), Reddy et al.(2016), Zia et al.(2018)]. In this chapter, the Sen-Dunn gravitational theory is used to examine the Bianchi type-VI_0 cosmological model with dark energy fluid while considering a defined scale factor that provides a variable deceleration parameter.

2.2 Metric and Field Equations

The line element for Bianchi type-VI_0 space-time is presented as

$$ds^2 = -dt^2 + A^2dx^2 + B^2e^{2x}dy^2 + C^2e^{-2x}dz^2 \tag{2.1}$$

where A, B, and C are the significance of cosmic evolution with time t.

The examination of the energy-momentum tensor's diagonal form while maintaining its diagonal form to generalise the EoS parameter of the ideal fluid independently on each spatial axis.

$$T_{ij} = diag[T_{00}, T_{11}, T_{22}, T_{33}] \tag{2.2}$$

Thus, on parameterize it as follows

$$
\begin{aligned}
T_{ij} &= [-\rho, p_x, p_y, p_z], \\
T_{ij} &= [-1, w_x, w_y, w_z]\rho, \\
T_{ij} &= [-1, w, w+\delta, w+\gamma]\rho.
\end{aligned} \tag{2.3}
$$

Here, p_x, p_y, and p_z represent the directional pressures, while w_x, w_y, and w_z represent the directional EoS parameters along the x, y, and z axes, respectively and ρ is the energy density of the fluid. The skewness parameters are δ and γ, while w is the so-called deviation-free EoS parameter. These parameters don't necessarily need to be constant; they might be a function of cosmic time t.

Sen and Dunn [Sen & Dunn(1971)] proposed the field equations for the combined scalar and tensor fields (in natural units, c = 1, 8πG = 1), which are as follows

$$R_{ij} - \frac{1}{2}Rg_{ij} = \omega\phi^{-2}(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}) - \phi^{-2}T_{ij} \tag{2.4}$$

The field equation (2.4) with (2.3) for the line element (2.1) gives rise to

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} + \frac{1}{A^2} = -\frac{w\rho}{\phi^2} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 \tag{2.5}$$

$$\frac{\ddot{C}}{C} + \frac{\ddot{A}}{A} + \frac{\dot{A}}{A}\frac{\dot{C}}{C} - \frac{1}{A^2} = -\frac{(w+\delta)\rho}{\phi^2} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 \tag{2.6}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} - \frac{1}{A^2} = -\frac{(w+\gamma)\rho}{\phi^2} + \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 \tag{2.7}$$

$$\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} + \frac{\dot{A}}{A}\frac{\dot{C}}{C} - \frac{1}{A^2} = \frac{\rho}{\phi^2} - \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 \tag{2.8}$$

$$\frac{\dot{C}}{C} - \frac{\dot{B}}{B} = 0 \tag{2.9}$$

It is specified that the typical scale factor for a space-time of Bianchi type-VI_0 is

$$a(t) = (ABC)^{\frac{1}{3}} \tag{2.10}$$

Also, other physical constraints such as the Hubble's parameter(H), expansion scalar (θ), shear scalar(σ), anisotropic parameter (A_m), and deceleration parameter (q) are defined as

$$H = \frac{\dot{a}}{a}, \tag{2.11}$$

$$\theta = 3H \tag{2.12}$$

$$\sigma^2 = \frac{1}{2}(\Sigma_{i=1}^{3} H_i^2 - \frac{1}{3}\theta^2) \tag{2.13}$$

$$A_m = \frac{1}{3}\Sigma_{i=1}^{3}(\frac{\Delta H_i}{H})^2 = \frac{2\sigma^2}{3H^2} \tag{2.14}$$

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = \frac{d}{dt}(\frac{1}{H}) - 1 \tag{2.15}$$

where the directional Hubble's parameter is represented by $\Delta H_i = H_i - H(i = x, y, z)$.

2.3 Solution of the Field Equations

Following the integration of the equation above (2.9), the following is given

$$C = mB \tag{2.16}$$

where m is a constant.

The equivalence in the skewness parameter along the y and z axis is observed for the simplification of the independent equations from (2.16), (2.7), and (2.6). Thus, the equations

(2.5) to (2.8) further reduces to as follows

$$2\frac{\ddot{B}}{B}+\frac{\dot{B}^2}{B^2}+\frac{1}{A^2}=-\frac{w\rho}{\phi^2}+\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 \tag{2.17}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}}{A}\frac{\dot{B}}{B}-\frac{1}{A^2}=-\frac{(w+\gamma)\rho}{\phi^2}+\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 \tag{2.18}$$

$$2\frac{\dot{A}}{A}\frac{\dot{B}}{B}+\frac{\dot{B}^2}{B^2}-\frac{1}{A^2}=\frac{\rho}{\phi^2}-\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2 \tag{2.19}$$

From the above three independent equations (2.17)-(2.19), six unknown parameters A, B,ρ,w, ϕ and γ are noticeable. More plausible relations between the variables are considered for explicit solutions for a more effective and predictable outcome.

First, using (2.16), and assuming that the expansion scalar (θ) is proportional to the shear scalar(σ) [Collins et al.(1980)], it produces

$$\frac{1}{\sqrt{3}}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)=\alpha_0(\frac{\dot{A}}{A}+2\frac{\dot{B}}{B}) \tag{2.20}$$

whereα_0 is an arbitrary constant, and the above equation yields to

$$\frac{\dot{A}}{A}=l\frac{\dot{B}}{B} \tag{2.21}$$

where $l=\frac{2\alpha_0\sqrt{3}+1}{1-\alpha_0\sqrt{3}}$

The following result is obtained from equation (2.21)

$$A=c_3B^l \tag{2.22}$$

where c_3 is constant, and choosing $c_3=1$ without sacrificing generality results as

$$A=B^l \tag{2.23}$$

Secondly, considering the power-law relation between the gauge function ϕ and the scale factor $a(t)$ as proposed by [Johri & Desikan(1994)]

$$\phi = \phi_0 a^{\alpha} = \phi_0 V^{\frac{\alpha}{3}} \tag{2.24}$$

In this chapter, a special characterized scale factor is applied

$$a(t) = c_2 e^{\frac{t}{c_1}} (c_1 t + c_1 - 1)^{\frac{1}{c_1^2}} \tag{2.25}$$

where $c_1$and c_2 are constant.

Thus using this equation (2.25) in (2.15), the deceleration parameter is obtained as

$$q = -1 + \frac{1}{(t+1)^2} \tag{2.26}$$

Thus, the deceleration parameter can explain the nature of the model. The model decelerates if it is positive, whereas it accelerates if it is < 0. For $-1 \leq q < 0$, the accelerating nature of the universe is recognized. The exponential expansion of the universe is noted if $q = -1$ and the cosmos is observed to be expanding super-exponentially for $q < -1$. From the obtained equation (2.26), the nature of the model of the universe is in exponential expansion as $q = -1$. Using (2.16), (2.23), and (2.25) in (2.10), the directional scale factors are as follows

$$A = k_3 e^{\frac{3lt}{c_1(l+2)}} (c_1 t + c_1 - 1)^{\frac{3l}{c_1^2(l+2)}} \tag{2.27}$$

$$B = k_1 e^{\frac{3t}{c_1(l+2)}} (c_1 t + c_1 - 1)^{\frac{3}{c_1^2(l+2)}} \tag{2.28}$$

$$C = k_2 e^{\frac{3t}{c_1(l+2)}} (c_1 t + c_1 - 1)^{\frac{3}{c_1^2(l+2)}} \tag{2.29}$$

where $k_1 = m^{-\frac{1}{l+2}} c_2^{\frac{3}{l+2}}, k_2 = mk_1, k_3 = k_1^l$.

Hence, the metric (2.1) becomes

$$ds^2 = -dt^2 + k_3 e^{\frac{3lt}{c_1(l+2)}} (c_1 t + c_1 - 1)^{\frac{3l}{c_1^2(l+2)}} dx^2 + k_1 e^{\frac{3t}{c_1(l+2)}} (c_1 t + c_1 - 1)^{\frac{3}{c_1^2(l+2)}} e^{2x} dy^2$$
$$+ k_2 e^{\frac{3t}{c_1(l+2)}} (c_1 t + c_1 - 1)^{\frac{3}{c_1^2(l+2)}} e^{-2z} dz^2 \tag{2.30}$$

2.4 Model's Physical and Geometrical Properties

As a result, the spatial volume (V), Hubble's parameter (H), expansion scalar (θ), shear scalar(σ), and anisotropic parameter (A_m) are calculated.

$$V = c_2^3 e^{3\frac{t}{c_1}} (c_1 t + c_1 - 1)^{\frac{3}{c_1^2}} \tag{2.31}$$

$$H = \frac{1}{c_1} + \frac{1}{c_1} \frac{1}{(c_1 t + c_1 - 1)} \tag{2.32}$$

$$\phi = \phi_0 c_2^{\alpha} e^{\frac{\alpha t}{c_1}} (c_1 t + c_1 - 1)^{\frac{\alpha}{c_1^2}} \tag{2.33}$$

$$\theta = \frac{3(t+1)}{(c_1 t + c_1 - 1)} \tag{2.34}$$

$$\sigma^2 = 3\left(\frac{l-1}{l-2}\right)^2 \left(\frac{t+1}{c_1 t + c_1 - 1}\right)^2 \tag{2.35}$$

$$A_m = 2\left(\frac{l-1}{l+2}\right)^2 \tag{2.36}$$

The other physical parameters for the model are calculated using the field equations (2.17)-(2.19) with the equation (2.27), (2.28), and (2.33) as follows

$$\rho = \phi_0^2 c_2^{\alpha} e^{\frac{2\alpha t}{c_1}} (c_1 t + c_1 - 1)^{\frac{2\alpha}{c_1^2}} \left[\frac{9(2l+1)(t+1)^2}{(l+2)^2 (c_1 t + c_1 - 1)^2} - k_4 \frac{1}{e^{\frac{6lt}{c_1(l+2)}}} \frac{1}{(c_1 t + c_1 - 1)^{\frac{6l}{c_1^2(l+2)}}} + \frac{\alpha\omega}{2} \left(\frac{1}{c_1} + \frac{1}{c_1^2 (c_1 t + c_1 - 1)} \right)^2 \right] \tag{2.37}$$

where $k_4 = \frac{1}{k_1^{2l}}$

$$w=-\frac{\frac{27(t+1)^2-6}{(l+2)^2(c_1t+c_1-1)^2}+k_4\frac{1}{e^{\frac{6lt}{c_1(l+2)}}}\frac{1}{(c_1t-c_1-1)^{\frac{6l}{c_1^2(l+2)}}}+\frac{\alpha\omega}{2}\left(\frac{1}{c_1}+\frac{1}{c_1^2(c_1t+c_1-1)}\right)^2}{\frac{9(2l+1)(t+1)^2}{(l+2)^2(c_1t+c_1-1)^2}-k_4\frac{1}{e^{\frac{6lt}{c_1(l+2)}}}\frac{1}{(c_1t-c_1-1)^{\frac{6l}{c_1^2(l+2)}}}+\frac{\alpha\omega}{2}\left(\frac{1}{c_1}+\frac{1}{c_1^2(c_1t+c_1-1)}\right)^2} \tag{2.38}$$

$$\gamma=-\frac{\frac{9(t+1)^2(l^2+l+1)}{(l+2)^2(c_1t+c_1-1)^2}+\frac{3-3l(c_1t+c_1-1)}{(l+2)(c_1t+c_1-1)^2}-k_4\frac{1}{e^{\frac{6lt}{c_1(l+2)}}}\frac{1}{(c_1t-c_1-1)^{\frac{6l}{c_1^2(l+2)}}}+\frac{\alpha\omega}{2}\left(\frac{1}{c_1}+\frac{1}{c_1^2(c_1t+c_1-1)}\right)^2}{\frac{9(2l+1)(t+1)^2}{(l+2)^2(c_1t+c_1-1)^2}-k_4\frac{1}{e^{\frac{6lt}{c_1(l+2)}}}\frac{1}{(c_1t-c_1-1)^{\frac{6l}{c_1^2(l+2)}}}+\frac{\alpha\omega}{2}\left(\frac{1}{c_1}+\frac{1}{c_1^2(c_1t+c_1-1)}\right)^2} \tag{2.39}$$

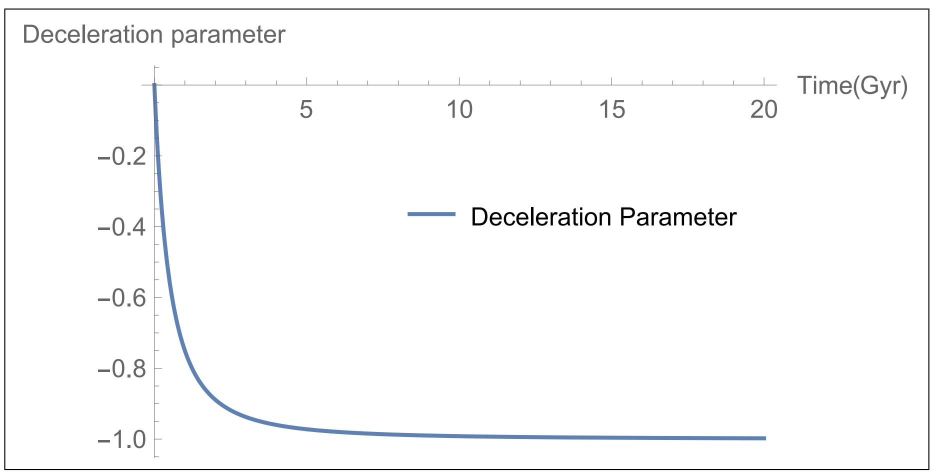

Figure 2.1: Variation of q $vs.$ t

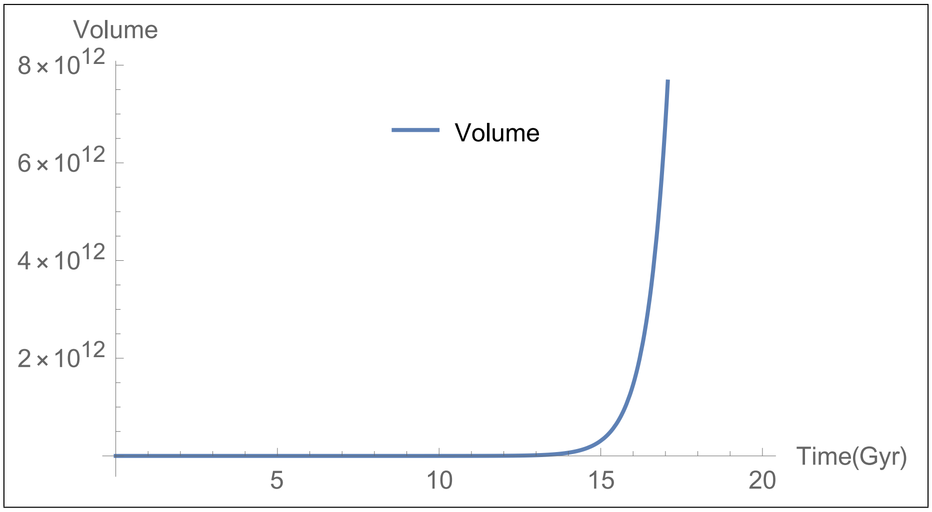

Figure 2.2: Variation of V $vs.$ t for c_2 =1.5, $c_1 = 2$

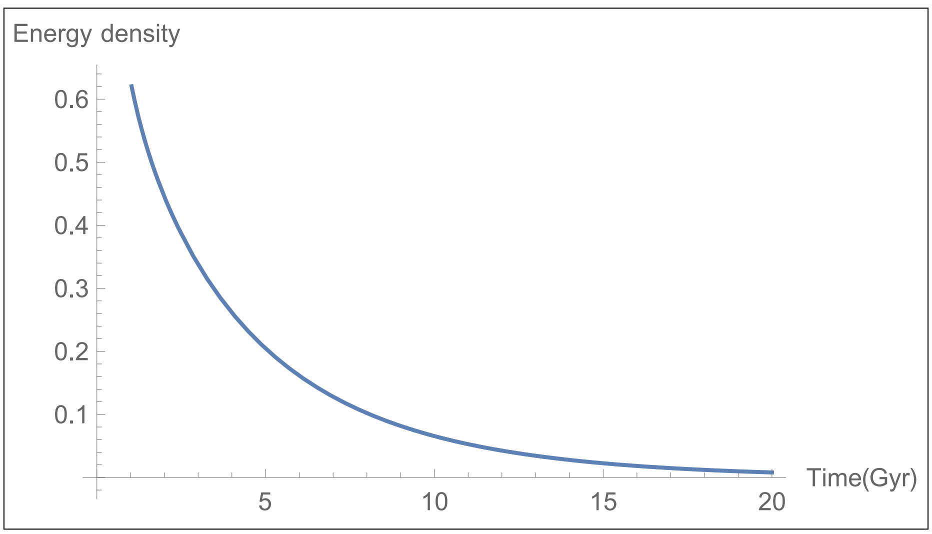

Figure 2.3: Variation of ρ $vs.$ t, c_2 =1.5, $c_1 = 2$, $\alpha = -0.2$, $l = 0.5$, $\phi_0 = 1$

2.5 The Jerk Parameter (j) and Statefinder Parameter $\{r, s\}$

The third derivative of the scale factor for cosmic time t is the jerk parameter (j), which has no dimensions. Blandford [Blandford et al.(2005)] addressed that the jerk parameterization offers an alternative and a practical way to define cosmological models close to the ΛCDM model. The deviation from $j = 1$ shows that the cosmological model does not

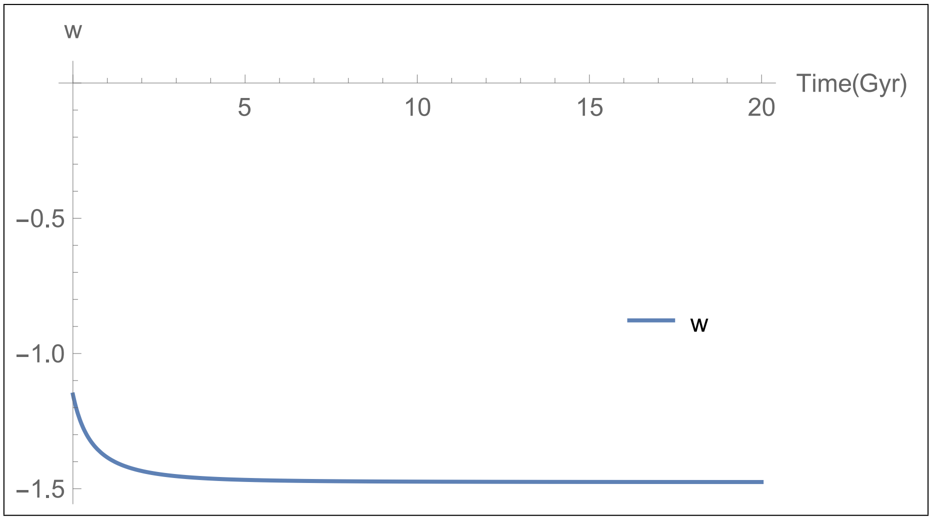

Figure 2.4: Variation of w $vs.$ t, c_2 =1.5, $c_1 = 2$, $\alpha = -0.2$, $l = 0.5$, $\phi_0 = 1$

represent the ΛCDM model. The following parameter is described as

$$j(t) = \frac{\dddot{a}}{aH^3} \tag{2.40}$$

The following equation (2.40) can be expressed as

$$j(t) = q + 2q^2 - \frac{\dot{q}}{H} \tag{2.41}$$

From equations (2.26), (2.32), and (2.41), the jerk parameter is given as

$$j(t) = 1 - \frac{3}{(t+1)^2} + \frac{2c_1}{(t+1)^3} \tag{2.42}$$

For the flat ΛCDM model, the value of jerk parameter (j) is $j = 1$. The statefinder $\{r, s\}$ as described by Sahni et al. [Sahni et al.(2003)], is as follows

$$r = \frac{\dddot{a}}{aH^3} = 1 + 3\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3} \tag{2.43}$$

and

$$s = \frac{r-1}{3(q-\frac{1}{2})} \tag{2.44}$$

From the above equations (2.26), (2.32), (2.43), and (2.44), the results obtained are

$$r = 1 - \frac{3}{(t+1)^2} + \frac{2c_1}{(t+1)^3} \tag{2.45}$$

and

$$s = -\frac{12(t+1) - 4c_1}{3(t+1)\{2-3(t+1)^2\}} \tag{2.46}$$

According to the results above, as t $\rightarrow\infty$, $\{r,s\} \rightarrow \{1,0\}$, implying that the universe's model progresses from Einstein's static age to the ΛCDM model as observed till this day. The transition of the model of the universe from Einstein static era to ΛCDM model that is well understood by the statefinder parameter [Amirhashchi et al.(2013), Ahmed & Pradhan(2014), Chand et al.(2016)].

2.5 Conclusion

In this chapter, a Bianchi Type-VI_0 dark energy cosmological model with variable deceleration parameter is investigated in Sen-Dunn theory and solved for exact solutions. The definitive solution for the model derived in this chapter, a characterized scale factor producing variable deceleration parameter, is considered, along with the shear scalar proportional to the expansion scalar. According to the model (2.30), the gauge function ϕ and the spatial volume grow as the time approaches infinity. The metric potential of the model rises from zero initially and then to infinite along with the increase in time, thus representing that the universe begins from an initial singularity with high energy density. The Hubble parameter decreases and approaches zero as time tends to infinity and the model (2.30) is anisotropic and homogeneous since $\frac{\sigma^2}{\theta^2} = constant$. The model obtained isotropy for the value $l = 1$ since it observed that the anisotropy parameter and shear scalar vanish for the specific value. Thus the model here is shearing and anisotropic throughout the time and is always accelerating as shown in Fig 2.1. The EoS parameter is at the range $-1.5 \leq w < -1$, passing the quintessence region to the phantom region, and is negative at the late times, which might be momentarily plausible for the model simulating the dynamics of the DE model and is in good accordance with the recent day observations [Carroll et al.(2003)]. Also, Knop et al. [Knop et al.(2003)] from the observational results such as SN Ia data govern the limit $-1.67 < w < -0.62$ and Tegmark et al. [Tegmark et al.(2004)] collaborated SN Ia data with CMBR anisotropy as well as galaxy clustering statistics defining the limit $-1.33 < w < -0.79$. The model experiences a transition from deceleration to acceleration as time passes, according to equations (2.42), (2.45), and (2.46). This result provides a DE scenario progressing towards the ΛCDM model.