Chapter 3

Viscous Bianchi Type-IX Dark Energy Cosmological Model in Sen-Dunn Theory of Gravitation

3.1 Introduction

Sen and Dunn [Sen & Dunn(1971)] scalar-tensor theory of gravitation is an expansion of the well-known Brans-Dickie [Brans & Dicke(1961)] theory of gravitation. The expansion and keen interest in a scalar-tensor theory of gravitation are developed to study the cosmological model of the universe. In General Relativity, homogeneous and anisotropic models have been widely studied to bring out the pragmatic view of the universe. Generally, Bianchi type IX represents a non-flat model that admits expansion and rotation and shear being anisotropic. The bulk viscous matter and pressure produce the expansion and acceleration of the universe [Fabris et al.(2006), Singh(2008a), Avelino & Nucamendi(2009), Asgar & Ansari(2014)]. Some authors [Pradhan et al.(2005), Bali & Yadav(2005), Ghate & Sontakke(2013)] have discussed the Bianchi type-IX cosmological model with the presence of viscous fluid in different cosmological theories. Thus, motivated by the above literature, the Bianchi type-IX space-time is studied with bulk viscosity investigated in the Sen-Dunn theory of gravitation framework. The proportionality between expansion and shear scalar is taken, leading to the condition $A = B^m$. The chapter also discuss for the perfect fluid and the fluid obeying equation of state $p = \gamma\rho$ where $0 \leq \gamma \leq 1$. The physical and geometrical nature of the model is examined.

3.2 Metric and Solutions of field equations

Bianchi Type IX metric is given by

$$ds^2 = -dt^2 + A^2dx^2 + B^2dy^2 + (B^2sin^2y + A^2cos^2y)dz^2 - 2A^2cosy\,dx\,dz \tag{3.1}$$

where A and B are a function of cosmic time t.
The field equations for both scalar and tensor fields for the natural units, ($c = 1, 8\pi G = 1$) as proposed by Sen-Dunn, is as given below

$$R_{ij} - \frac{1}{2}Rg_{ij} = \omega\phi^{-2}(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}) - \phi^{-2}T_{ij} \tag{3.2}$$

For bulk viscous fluid's energy-momentum tensor is assumed to be

$$T_{ij} = (\rho + \bar{p})u_iu_j + \bar{p}g_{ij} \tag{3.3}$$

Where

$$\bar{p} = p - \xi\theta \tag{3.4}$$

Here p, ρ,$\bar{p}$, ξ and θ are isotropic pressure, energy density, effective pressure, coefficient of viscosity and expansion scalar, and $u^i = (0,0,0,1)$ is the four-velocity vector satisfying $g^{ij}u_iu_j = -1$.

The field equation (3.2) with the help of (3.3) for the metric (3.1) are as follows

$$2\frac{\ddot{B}}{B}+\frac{\dot{B}}{B^2}+\frac{1}{B^2}-\frac{3A^2}{4B^4}=\frac{\omega}{2}(\frac{\dot{\phi}}{\phi})^2-\phi^{-2}\bar{p} \tag{3.5}$$

$$\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\dot{A}\dot{B}}{AB}+\frac{A^2}{4B^4}=\frac{\omega}{2}(\frac{\dot{\phi}}{\phi})^2-\phi^{-2}\bar{p} \tag{3.6}$$

$$2\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{B}^2}{B^2}+\frac{1}{B^2}-\frac{A^2}{4B^4}=\phi^{-2}\rho-\frac{\omega}{2}(\frac{\dot{\phi}}{\phi})^2 \tag{3.7}$$

Where overdot over A and B represent ordinary differentiation concerning t.

Defining the average scale factor for a Bianchi type-IX space-time as

$$a(t)=(AB^2)^{\frac{1}{3}} \tag{3.8}$$

Also, the Hubble parameter, expansion scalar, shear scalar, anisotropy parameter, and deceleration parameter are defined as

$$H=\frac{\dot{a}}{a} \tag{3.9}$$

$$\theta=3H \tag{3.10}$$

$$\sigma^2=\frac{1}{2}(\Sigma_{i=1}^{3}H_i^2-\frac{1}{3}\theta^2) \tag{3.11}$$

$$A_m=\frac{1}{3}\Sigma_{i=1}^{3}(\frac{H_i-H}{H})^2=\frac{2\sigma^2}{3H^2} \tag{3.12}$$

$$q=-\frac{a\ddot{a}}{\dot{a}^2}=\frac{d}{dt}(\frac{1}{H})-1 \tag{3.13}$$

The directional Hubble's parameter is represented by H_i, where $i=x,y,z$.

The three independent field equations (3.5)-(3.7) contains A,B, $\bar{p}$, ρ, ϕ, unknown parameters. To obtain the solution, here introduce more conditions by assuming some physical condition or by the arbitrary mathematical notion.

Considering the shear scalar is proportional to the expansion scalar [Collins et al.(1980)]

to obtain the explicit solution, which leads to the relation of the metric potential as

$$A = B^m \tag{3.14}$$

where m is the constant.

Considering the power law relation between the gauge function ϕ and the scale factor a(t) [Johri & Desikan(1994)]

$$\phi = \phi_0 a^\alpha = \phi_0 V^{\frac{\alpha}{3}} \tag{3.15}$$

Here, considering the characterized scale factor [Sharma et al.(2019), Dixit et al.(2020)]

$$a(t) = exp\left[\frac{1}{\beta}\sqrt{2\beta t + k}\right] \tag{3.16}$$

where β and k are positive constant.

Using the equations (3.16),(3.14) with (3.8), the metric potential is obtained as

$$A = exp\left[\frac{3m}{\beta(m+2)}\sqrt{2\beta t + k}\right] \tag{3.17}$$

$$B = exp\left[\frac{3}{\beta(m+2)}\sqrt{2\beta t + k}\right] \tag{3.18}$$

Thus the metric (3.1) reduces to the form as

$$\begin{aligned} ds^2 = \Bigg[&-dt^2 + exp\left[\frac{6m}{\beta(m+2)}\sqrt{2\beta t + k}\right] dx^2 \\ &+ exp\left[\frac{6}{\beta(m+2)}\sqrt{2\beta t + k}\right] dy^2 + \Bigg(exp\left[\frac{6}{\beta(m+2)}\sqrt{2\beta t + k}\right] sin^2 y \\ &+ exp\left[\frac{6m}{\beta(m+2)}\sqrt{2\beta t + k}\right] cos^2 \Bigg) dz^2 \\ &- 2exp\left[\frac{6m}{\beta(m+2)}\sqrt{2\beta t + k}\right] cosy\, dx\, dz \Bigg] \end{aligned} \tag{3.19}$$

3.3 Physical and Geometrical properties of the model

The model's physical components as spatial volume V, Gauge function ϕ, Hubble parameter H, expansion scalar θ, mean anisotropy parameter A_m, shear scalar σ^2, and deceleration parameter q is calculated as follows:

Spatial Volume,

$$V = exp[\frac{3}{\beta}\sqrt{2\beta t + k}] \tag{3.20}$$

Gauge function,

$$\phi = \phi_0 \left(exp[\frac{\alpha}{\beta}\sqrt{2\beta t + k}] \right) \tag{3.21}$$

Hubble parameter,

$$H = \frac{1}{\sqrt{2\beta t + k}} \tag{3.22}$$

Expansion scalar,

$$\theta = 3H = \frac{3}{\sqrt{2\beta t + k}} \tag{3.23}$$

Mean Anisotropy Parameter,

$$A_m = \frac{2(m-1)^2}{(m+2)^2} = constant(\neq 0 \ \ for \ \ m \neq 1) \tag{3.24}$$

Shear scalar,

$$\sigma^2 = \frac{3(m-1)^2}{(m+2)^2(2\beta t + k)} \tag{3.25}$$

Also,

$$\frac{\sigma^2}{\theta^2} = \frac{1}{3}\frac{(m-1)^2}{(m+2)^2} = constant(\neq 0 \ \ for \ \ m \neq 1) \tag{3.26}$$

The rate of expansion H_i,

$$H_x = \frac{3m}{m+2}\frac{1}{\sqrt{2\beta t + k}}, H_y = H_z = \frac{3}{m+2}\frac{1}{\sqrt{2\beta t + k}} \tag{3.27}$$

Deceleration parameter,

$$q = \frac{\beta}{\sqrt{2\beta t + k}} - 1 \tag{3.28}$$

The energy density for the model is obtained as

$$\rho = \phi_0 exp\Big(\frac{2\alpha\sqrt{(2\beta t+k)}}{\beta}\Big)\Bigg[\frac{36(2m+1)+3\alpha^2(m+2)^2}{4(m+2)^2(2\beta t+k)} + exp\Big(\frac{-6\sqrt{2\beta t+k}}{\beta(m+2)}\Big) - \frac{1}{4}exp\Big(\frac{6(m-2)\sqrt{2\beta t+k}}{\beta(m+2)}\Big)\Bigg] \tag{3.29}$$

And the effective pressure measures as

$$\bar{p} = \frac{\phi_0}{2}exp\Big(\frac{2\alpha\sqrt{(2\beta t+k)}}{\beta}\Big)\Bigg[\frac{9(m^2+m+4)}{(m+2)^2(2\beta t+k)} + \frac{1}{2}exp\Big(\frac{6(m-2)\sqrt{2\beta t+k}}{\beta(m+2)}\Big) - \frac{3\alpha^2}{2(\beta t+k)} - \frac{3\beta(m+3)}{(m+2)(2\beta t+k)^{\frac{3}{2}}}\Bigg] \tag{3.30}$$

For the identification of ξ, assuming the case that the fluid obeys the equation of state is given

$$p = \gamma\rho \tag{3.31}$$

where γ is a constant with $(0 \leq \gamma \leq 1)$. The value of γ determines three cases for the model:

(i) When $\gamma = 0$, the model tends to matter-dominated model.

(ii) When $\gamma = 1/3$, the model tends to radiation-dominated model.

(iii) When $\gamma = 1$, it follows $p = \rho$ as Zel'dovich fluid or stiff fluid model.

With the use of equation (3.31) further calculating, the isotropic pressure and bulk viscosity are given as

$$p = \gamma\phi_0 exp\Big(\frac{2\alpha\sqrt{(2\beta t+k)}}{\beta}\Big)\Bigg[\frac{36(2m+1)+3\alpha^2(m+2)^2}{4(m+2)^2(2\beta t+k)} + exp\Big(\frac{-6\sqrt{2\beta t+k}}{\beta(m+2)}\Big) - \frac{1}{4}exp\Big(\frac{6(m-2)\sqrt{2\beta t+k}}{\beta(m+2)}\Big)\Bigg] \tag{3.32}$$

$$\xi = \frac{3\phi_0 exp\Big(\frac{2\alpha\sqrt{(2\beta t+k)}}{\beta}\Big)}{\sqrt{(2\beta t+k)}}\Bigg[\frac{36\gamma(2m+1)+3\alpha^2(m+2)^2(\gamma-1)+18(m^2+m+4)}{4(m+2)^2(2\beta t+k)} + \gamma exp\Big(\frac{-6\sqrt{2\beta t+k}}{\beta(m+2)}\Big) - \frac{\gamma+2}{4}exp\Big(\frac{6(m-2)\sqrt{2\beta t+k}}{\beta(m+2)}\Big) - \frac{3\beta(m+3)}{2(m+2)(2\beta t+k)^{\frac{3}{2}}}\Bigg] \tag{3.33}$$

In the case of the fluid being perfect, the term ξ= 0 is considered, resulting that the effective pressure being equal to the isotropic pressure i.e

$$p = \bar{p} = \frac{\phi_0}{2}exp\Big(\frac{2\alpha\sqrt{(2\beta t+k)}}{\beta}\Big)\Bigg[\frac{9(m^2+m+4)}{(m+2)^2(2\beta t+k)} + \frac{1}{2}exp\Big(\frac{6(m-2)\sqrt{2\beta t+k}}{\beta(m+2)}\Big) - \frac{3\alpha^2}{2(\beta t+k)} - \frac{3\beta(m+3)}{(m+2)(2\beta t+k)^{\frac{3}{2}}}\Bigg] \tag{3.34}$$

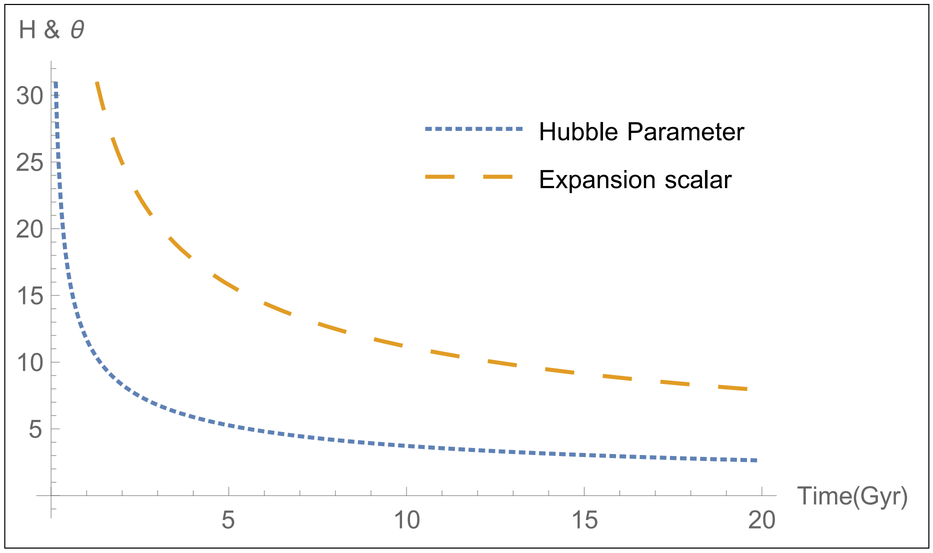

Figure 3.1: Variation of H and θ *vs*. t

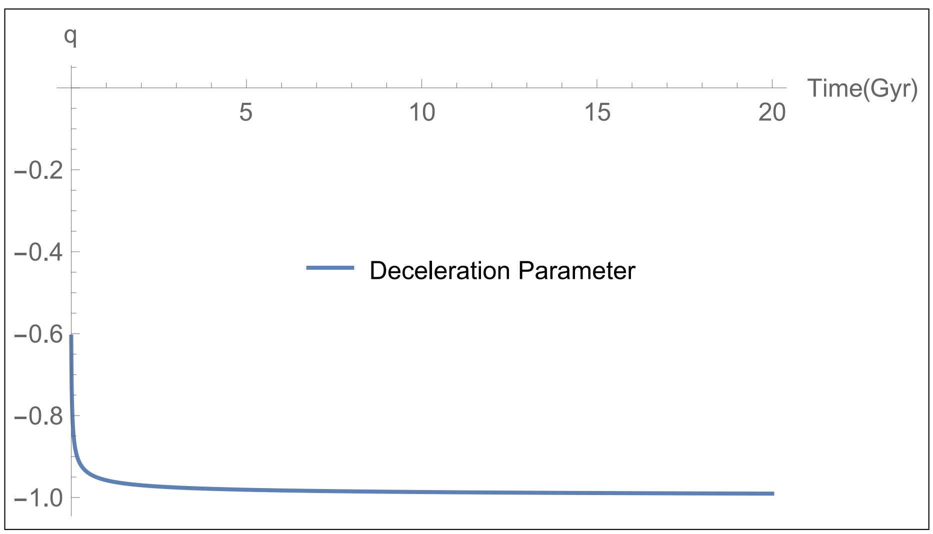

Figure 3.2: Variation of q *vs*. t

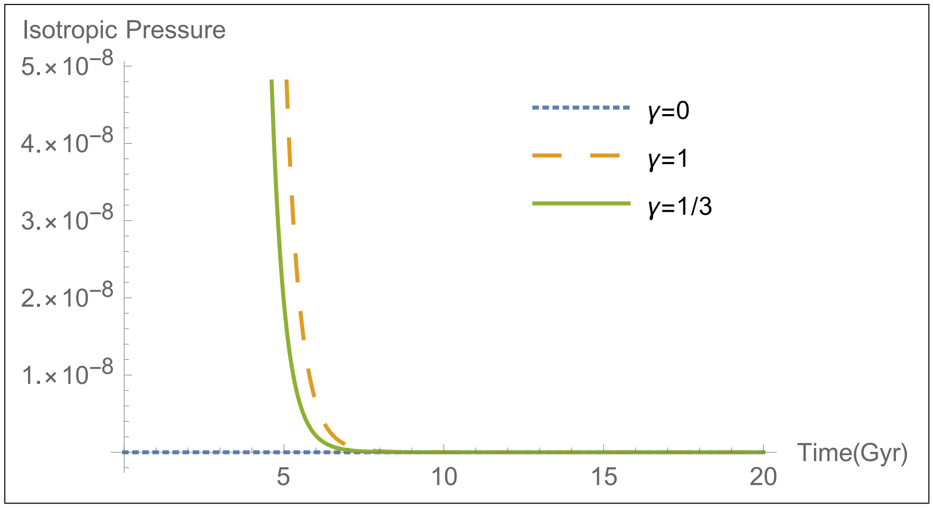

Figure 3.3: Variation of p $vs.$ t

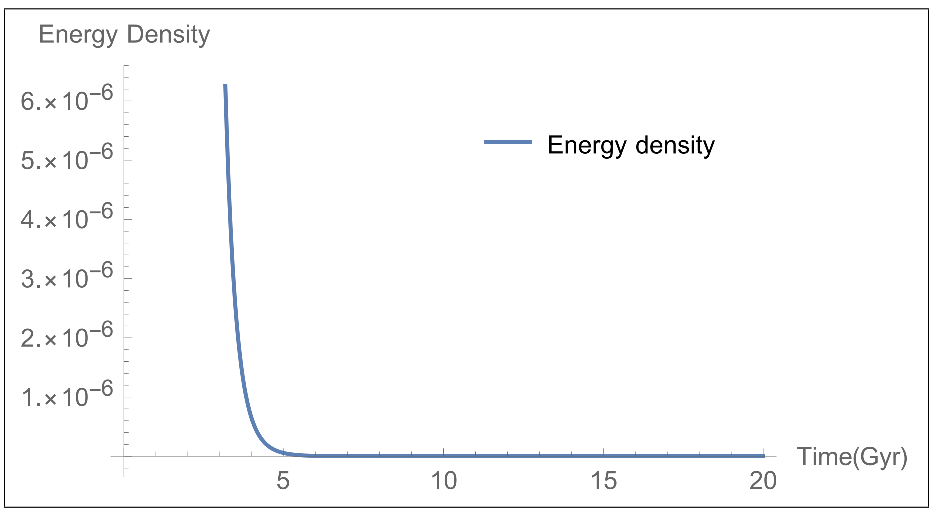

Figure 3.4: Variation of ρ $vs.$ t

3.4 Energy Condition and Statefinder Parameter

The energy conditions are respectively defined as:

(i) Null Energy Condition: $\rho + p \geq 0$

(ii) Weak Energy Condition: $\rho \geq 0$, $\rho + p \geq 0$

(iii) Dominant Energy Condition: $\rho \pm p \geq 0$, $\rho \geq 0$

(vi) Strong Energy Condition: $\rho + 3p \geq 0$

Thus, the energy condition from equations (3.29) and (3.32) is derived as

$$\rho + p = (\gamma+1)\phi_0 exp\Big(\frac{2\alpha\sqrt{(2\beta t+k)}}{\beta}\Big)\Bigg[\frac{9(m^2+m+4)}{(m+2)^2(2\beta t+k)} + \frac{1}{2}exp\Big(\frac{6(m-2)\sqrt{2\beta t+k}}{\beta(m+2)}\Big) - \frac{3\alpha^2}{2(\beta t+k)} - \frac{3\beta(m+3)}{(m+2)(2\beta t+k)^{\frac{3}{2}}}\Bigg] \geq 0 \tag{3.35}$$

$$\rho - p = (1-\gamma)\phi_0 exp\Big(\frac{2\alpha\sqrt{(2\beta t+k)}}{\beta}\Big)\Bigg[\frac{9(m^2+m+4)}{(m+2)^2(2\beta t+k)} + \frac{1}{2}exp\Big(\frac{6(m-2)\sqrt{2\beta t+k}}{\beta(m+2)}\Big) - \frac{3\alpha^2}{2(\beta t+k)} - \frac{3\beta(m+3)}{(m+2)(2\beta t+k)^{\frac{3}{2}}}\Bigg] \geq 0 \tag{3.36}$$

$$\rho + 3p = (1+3\gamma)\phi_0 exp\Big(\frac{2\alpha\sqrt{(2\beta t+k)}}{\beta}\Big)\Bigg[\frac{9(m^2+m+4)}{(m+2)^2(2\beta t+k)} + \frac{1}{2}exp\Big(\frac{6(m-2)\sqrt{2\beta t+k}}{\beta(m+2)}\Big) - \frac{3\alpha^2}{2(\beta t+k)} - \frac{3\beta(m+3)}{(m+2)(2\beta t+k)^{\frac{3}{2}}}\Bigg] \geq 0 \tag{3.37}$$

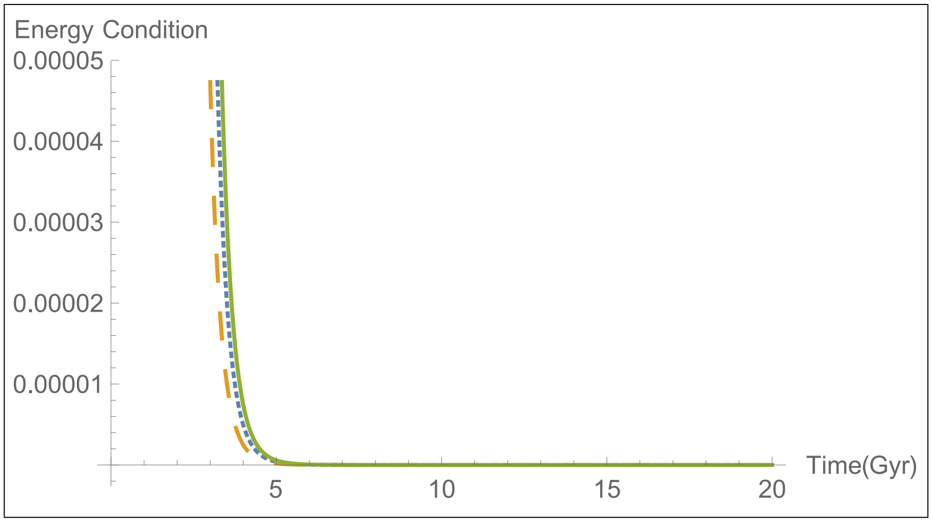

Figure 3.5: Variation of *energy condition vs. t*

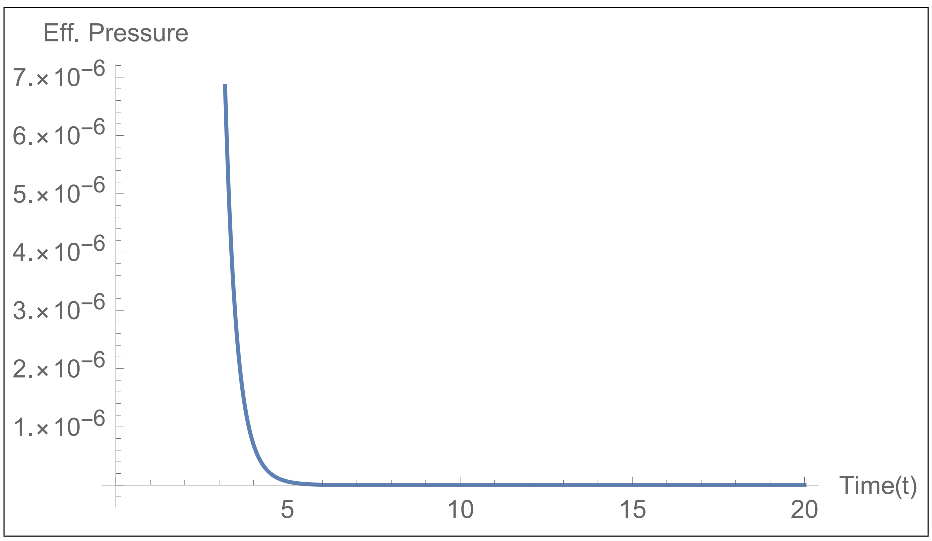

Figure 3.6: Variation of $\bar{p}$ *vs. t*

The statefinder parameter $\{$r,s$\}$ is defined as

$$r = \frac{\dddot{a}}{aH^3} = 1 + 3\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3} \tag{3.38}$$

$$s = \frac{(r-1)}{3(q-\frac{1}{2})} \tag{3.39}$$

where q represents the deceleration parameter and H represents the Hubble parameter.

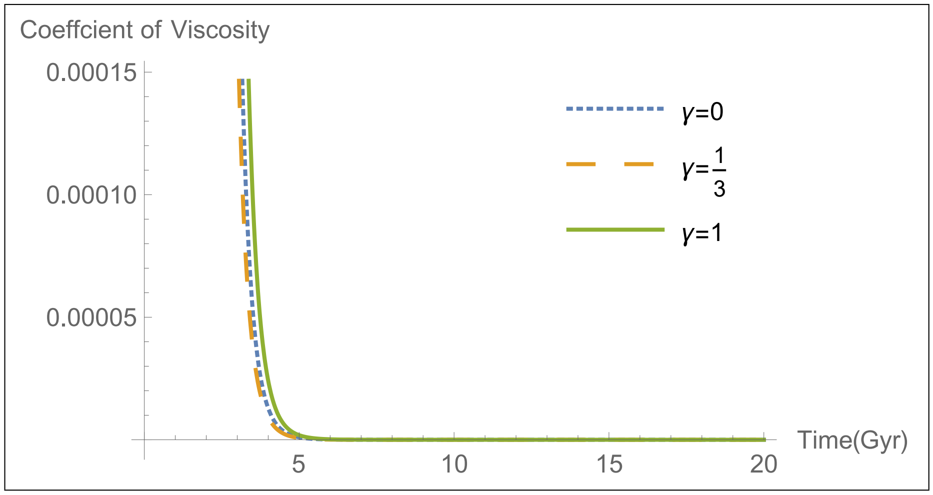

Figure 3.7: Variation of ξ *vs*. t

From the equation (3.22) and (3.28), the required statefinder parameter is as follows

$$r = 1 - \frac{3\beta}{\sqrt{2\beta t + k}} + \frac{3\beta^2}{2\beta t + k} \tag{3.40}$$

$$s = \frac{\beta^2\sqrt{2\beta t + k} - (2\beta t + k)\beta}{(2\beta t + k)(2\beta - 3\sqrt{2\beta t + k})} \tag{3.41}$$

From the above result, when $t \to \infty$ the pair $\{r, s\} \to \{1, 0\}$ defines the current ΛCDM model, starting with Einstein's static era.

3.5 Conclusion

The solutions of the anisotropic Bianchi type IX cosmological model with bulk viscosity are investigated in the Sen-Dunn theory. In this approach, the dynamical behaviors of the various physical and geometrical properties are plotted considering the constraints $\beta = 0.0036$, k = 0.000084 [Cunha(2009)], $\phi_0 = 0.07$, $\alpha = -0.20$ and $m = 0.5$. The Hubble parameter (H) and expansion scalar (θ) for the model (3.19) are positively decreasing and tend to zero as $t \to \infty$ as plotted in Fig. 3.1. The deceleration parameter for the model lies in the range $-1 \leq q < 0$ as plotted in Fig 3.2., which claims the universe is expanding exponentially, in line with recent observational findings [Perlmutter et al.(1999), Riess et al.(2001)], and that the universe is always accelerating. For the three cases of $\gamma = 0, 1, 1/3$, the isotropic pressure and energy density for the model (3.19) is decreasing function for cosmic time $t \to \infty$, as plotted in Fig 3.3 and Fig 3.4. Fig 3.5 and Fig 3.6 represents the statistical behaviour of the model's energy condition and effective pressure. For the fluid obeying the EoS $p = \gamma\rho$, the model's bulk viscosity is a positively decreasing time dependence and tends to zero as $t \to \infty$ as plotted in Fig. 3.7. Thus, claiming that the universe in the early universe was highly viscous but later decreases at the late time [Dixit et al.(2020)]. The model shows a positively decreasing pressure and tends to zero for the perfect fluid model as $t \to \infty$. Recently, Sharma and Poonia [Sharma & Poonia(2021)] investigated that the presence of bulk viscosity may generate cosmic inflation for the model obtained for the Bianchi type IX space-time. To study the characteristic feature of the accelerating model, the diagnostic pair $\{r, s\}$ is examined. The diagnostic pair measures that $\{r, s\} \to \{1, 0\}$ for $t \to \infty$ which represents that the model approaches the ΛCDM model. The model shows expanding, shearing, and non-rotating, and the model does not tends to isotropy for the large value of t, since $\frac{\sigma}{\theta} \neq 0$. Thus, bulk viscous plays an important role in studying the dynamic behaviour of the early universe and the late time universe.