Chapter 5

Viscous Bianchi Type-III Cosmological Model with Bilinear Deceleration Parameter in Sen-Dunn Theory of Gravitation

5.1 Introduction

According to the universe's vast scale structure and CMB research, the current universe scenario is homogeneous, isotropic, and expanding, which is well reflected by the FRW model. However, understanding the importance of the anisotropic Bianchi model is required to comprehend several inconsistencies noted by WMAP [Jaffe et al.(2005),Hoftuft et al.(2009)]. Many authors have investigated Bianchi type-III space-time and obtained the dark energy models [Yadav & Yadav(2011),Reddy et al.(2013),Santhi et al.(2018),Korunur(2019)]. Viscosity is considered one of the fundamental factors that need to be examined while studying the evolution of the cosmos since it defines the physical properties and dynamics of the homogeneous and isotropic cosmological model. Some researchers [Bali et al.(2010), Sharif & Kausar(2011a), Amirhashchi(2014), Hatkar et al.(2019)] have investigated bulk viscous fluid with Bianchi type-III space-time with various cosmological contexts. In this chapter, the impact of bulk viscosity in Bianchi type-III space-time metric is analyzed with the bilinear form of deceleration parameter in the framework of Sen-Dunn theory, constraining its cosmological parameters and physical properties being discussed.

5.2 Line Element and Field Equations

Given is the metric of Bianchi Type-III space-time.

$$ds^2 = -dt^2 + A^2dx^2 + B^2e^{-2mx}dy^2 + C^2dz^2 \tag{5.1}$$

where the term A, B, and C represents the metric potential and are a function of cosmic time t. Here m is a positive constant. And Sen-Dunn's field equation for the combined scalar and tensor field (in natural units $c = 1$, 8πG = 1) is given as

$$R_{ij} - \frac{1}{2}Rg_{ij} = \omega\phi^{-2}(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}) - \phi^{-2}T_{ij} \tag{5.2}$$

The energy-momentum tensor with the presence of viscosity is given as

$$T_{ij} = (\rho + \bar{p})u_iu_j + \bar{p}g_{ij} \tag{5.3}$$

Where

$$\bar{p} = p - \xi\theta \tag{5.4}$$

Here p, ρ, $\bar{p}$, ξ and θ are isotropic pressure, energy density, effective pressure, coefficient of viscosity and expansion scalar, and $u^i = (0,0,0,1)$ is the four-velocity vector satisfying $g^{ij}u_iu_j = -1$.

The following equations are obtained from the field equation (5.2) for the space-time (5.1) using equation (5.3).

$$\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{B}\dot{C}}{BC}=\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)-\phi^2\bar{p} \tag{5.5}$$

$$\frac{\ddot{A}A}{A}+\frac{\ddot{C}}{C}+\frac{\ddot{A}\dot{C}}{AC}=\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)-\phi^2\bar{p} \tag{5.6}$$

$$\frac{\ddot{A}A}{A}+\frac{\ddot{B}}{B}+\frac{\ddot{A}\dot{B}}{AB}-\frac{m^2}{A^2}=\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)-\phi^2\bar{p} \tag{5.7}$$

$$\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{A}\dot{C}}{AC}+\frac{\dot{B}\dot{C}}{BC}-\frac{m^2}{A^2}=\phi^2\bar{p}-\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)-\phi^2\bar{p} \tag{5.8}$$

$$m\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)=0 \tag{5.9}$$

The physical and kinematical parameters : spatial volume (V), Hubble parameters(H), expansion scalar (θ), shear scalar (σ), mean anisotropy parameter (A_m), and decelerating parameter are expressed as

$$V=ABC \tag{5.10}$$

$$H=\frac{\dot{a}}{a} \tag{5.11}$$

$$\theta=3H \tag{5.12}$$

$$\sigma^2=\frac{1}{2}\left(\Sigma_{i=1}^{3}H_i^2-\frac{1}{3}\theta^2\right) \tag{5.13}$$

$$A_m=\frac{1}{3}\Sigma_{i=1}^{3}\left(\frac{H_i-H}{H}\right)^2=\frac{2\sigma^2}{3H^2} \tag{5.14}$$

$$q=-\frac{a\ddot{a}}{\dot{a}^2}=\frac{d}{dt}(\frac{1}{H})-1 \tag{5.15}$$

Where $i=x,y,z$ denotes the directional components along $x,y, and\ z$.

5.3 Solution of the Field Equations

Integrating equation (5.9) yields

$$A = nB \tag{5.16}$$

where "n" is the integration constant and taking $n = 1$ without the loss of generality. So, from (5.5)-(5.8), the following field equations are obtained with the help of equation (5.16) as

$$\frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{B}\dot{C}}{BC} = \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right) - \phi^2\bar{p} \tag{5.17}$$

$$2\frac{\ddot{B}}{B} + \left(\frac{\dot{B}}{B}\right)^2 - \frac{m^2}{B^2} = \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right) - \phi^2\bar{p} \tag{5.18}$$

$$\left(\frac{\dot{B}}{B}\right)^2 + 2\frac{\dot{B}\dot{C}}{BC} - \frac{m^2}{B^2} = \phi^2\rho - \frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right) \tag{5.19}$$

The following three independent eq. (5.17)-(5.19) comprise of five unknown parameters B, C, ρ, $\bar{p}$, ϕ. To determine the reasonable solution of the field equations considering the following conditions.

i). Taking the bilinear form of deceleration parameter [Mishra & Chand(2016)] expressed as:

$$q(t) = \frac{\alpha(1-t)}{1+t} \tag{5.20}$$

Here, it is noticeable that the decelration parameter for $0 < t < 1$, $\alpha > 0$ is $q > 0$ presenting the decelerated expansion of the cosmos in early time. For $\alpha > 0$ and t = 1, it is seen that $q = 0$ demonstrating continual expansion of the universe. Further, for $t > 1$ and $0 < \alpha < 1$ the deceleration parameter lies in $-1 < q < 0$ exhibiting an exponential expansion. But for $t > 1$ and $\alpha > 1$ the deceleration parameter $q < -1$ reflecting a super-exponential expansion of the universe. From the above equation, the expression of the Hubble parameter is derived as mentioned below

$$H(t) = \frac{1}{(1-\alpha)t + 2\alpha log(1+t)} \tag{5.21}$$

As, in general, the relation between the $q(t)$ and $H(t)$ is given by-

$$H = \frac{1}{\int[1+q(t)]dt + k_1} \tag{5.22}$$

Here, k_1 is an integration constant. It is observed that as $t \to \infty$, $H \to \infty$, for which $k_1 = 0$. From the above equation (5.22), the expression for H on expanding is given as

$$H = \frac{1}{(1+\alpha)t + 2\alpha\left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + ...\right)}$$

$$H = \frac{1}{(1+\alpha)t\left[1 - 2\alpha\left(\frac{1}{2} - \frac{t}{3} + \frac{t^2}{4} + ...\right)\right]}$$

$$H = \frac{1}{(1+\alpha)t}\left[1 - \frac{2\alpha t}{1+\alpha}\left[\frac{1}{2} - \frac{t}{3} + \frac{t^2}{4} - ...\right]\right]^{-1}$$

$$H = \frac{1}{(1+\alpha)t} + \frac{\alpha}{(1+\alpha)^2} + \frac{-2\alpha+\alpha^2}{3(1+\alpha)^3}t + \frac{3\alpha-2\alpha^2+\alpha^3}{6(1+\alpha)^4}t^2 + \frac{-18\alpha+11\alpha-14\alpha^3+2\alpha^4}{45(1+\alpha)^5}t^3 + K(t^4)$$

Since the relation between the Hubble parameter and scale factor $a(t)$ is taken as

$$H = \frac{\dot{a}}{a} \tag{5.23}$$

Thus the following expression can be obtained

$$\log\frac{a(t)}{a_0} = \frac{\log(t)}{(1+\alpha)} + \frac{\alpha}{(1+\alpha)^2}t + \frac{-2\alpha+\alpha^2}{3(1+\alpha)^3}t^2 + \frac{3\alpha-2\alpha^2+\alpha^3}{6(1+\alpha)^4}t^3 + \frac{-18\alpha+11\alpha-14\alpha^3+2\alpha^4}{45(1+\alpha)^5}t^4 + K(t^5)$$

$$a(t) = a_0 t^{\frac{1}{1+\alpha}} e^{K(t)} \tag{5.24}$$

Where

K(t) = $\frac{\alpha}{(1+t)^2}t$ + $\frac{-2\alpha+\alpha^2}{6(1+\alpha)^3}t^2$ + $\frac{3\alpha-2\alpha^2+\alpha^3}{18(1+\alpha)^4}t^3$ + $\frac{-18\alpha+11\alpha^2-14\alpha^3+2\alpha^4}{180(1+\alpha)^5}t^4$ + $K(t^5)$

ii). For the non-linear equation (5.17)-(5.19), assuming the condition that the shear scalar (σ) is proportional to the expansion scalar (θ) [Collins et al.(1980)], which leads to the following relation

$$B = C^f \tag{5.25}$$

where $f > 0$ is a positive constant.

iii). Considering the power-law relation of the given gauge function (ϕ) with scale factor $a(t)$ [Johri & Desikan(1994)] as

$$\phi = \phi_0 a^\beta(t) \tag{5.26}$$

Where β is an ordinary constant and ϕ_0 is a proportionality constant. The average scale factor that governs Bianchi Type-III space-time is given as

$$a(t) = (ABC)^{\frac{1}{3}} \tag{5.27}$$

From equations (5.16), (5.24), and (5.25) in (5.27), the metric potential is as follows

$$A = B = \left(a_0 t^{\frac{1}{1+\alpha}} e^{K(t)}\right)^{\frac{3f}{2f+1}} \tag{5.28}$$

$$C = \left(a_0 t^{\frac{1}{1+\alpha}} e^{K(t)}\right)^{\frac{3}{2f+1}} \tag{5.29}$$

Considering the value of $a_0 = 1$ and $m = 1$ leads the metric (5.1) to the following equation

$$ds^2 = -dt^2 + \left(t^{\frac{1}{1+\alpha}} e^{K(t)}\right)^{\frac{6f}{2f+1}} dx^2 + \left(t^{\frac{1}{1+\alpha}} e^{K(t)}\right)^{\frac{6f}{2f+1}} e^{-2x} dy^2 + \left(t^{\frac{1}{1+\alpha}} e^{K(t)}\right)^{\frac{6}{2f+1}} dz^2 \tag{5.30}$$

5.4 Model's Physical and Geometrical Properties

To examine the dynamical behaviour of the model, we obtained the physical and kinematical constraints of the model, such as Hubble parameters (H), spatial volume (V), expansion scalar (θ), shear scalar (σ), mean anisotropy parameter (A_m) as

$$H_1 = H_2 = \frac{3f}{2f+1}\left(\frac{1}{(1-\alpha)t + 2\alpha log(1+t)}\right) \tag{5.31}$$

$$H_3 = \frac{3}{2f+1}\left(\frac{1}{(1-\alpha)t + 2\alpha log(1+t)}\right) \tag{5.32}$$

$$V = a_0 t^{\frac{3}{1+\alpha}} e^{3K(t)} \tag{5.33}$$

$$\theta = \frac{3}{(1-\alpha)t + 2\alpha log(1+t)} \tag{5.34}$$

$$\sigma^2 = \frac{3(f-1)^2}{(2f+1)^2}\left(\frac{1}{(1-\alpha)t + 2\alpha log(1+t)}\right)^2 \tag{5.35}$$

$$A_m = \frac{2(f-1)^2}{(2f+1)^2} \tag{5.36}$$

The energy density and the effective pressure are obtained from eq. (5.17)-(5.19) using the eq. (5.26) and (5.27) as

$$\rho = \phi_0^2 \left(a_0 t^{\frac{1}{1+\alpha}} e^{K(t)}\right)^{2\beta} \left[\frac{9(3f - f^2 + 1)}{(2f+1)^2} H^2 - \frac{3(f-1)}{(2f+1)} \dot{H} + \frac{3\beta^2 H^2}{4}\right] \tag{5.37}$$

$$\bar{p} = \phi_0^2 \left(a_0 t^{\frac{1}{1+\alpha}} e^{K(t)}\right)^{2\beta} \left[\frac{-9(3f + f^2 + 1)}{(2f+1)^2} H^2 - \frac{3(f+1)}{(2f+1)} \dot{H} + \frac{3\beta^2 H^2}{4}\right] \tag{5.38}$$

For the fluid satisfying EoS, p = $\gamma\,\rho$, where $0 \leq \gamma \leq 1$, the isotropic pressure and viscosity coefficient are derived as.

$$p = \gamma \phi_0^2 \left(a_0 t^{\frac{1}{1+\alpha}} e^{K(t)}\right)^{2\beta} \left[\frac{9(3f - f^2 + 1)}{(2f+1)^2} H^2 - \frac{3(f-1)}{(2f+1)} \dot{H} + \frac{3\beta^2 H^2}{4}\right] \tag{5.39}$$

$$\xi = \frac{(1-\alpha)t + 2\alpha log(1+t)}{3}\left(a_0 t^{\frac{1}{1+\alpha}} e^{K(t)}\right)^{2\beta}\left[\frac{3(\gamma+1)-3f(\gamma-1)}{(2f+1)}\dot{H} + \frac{9f(3\gamma+1)-9f^2(\gamma-1)+9\gamma+1}{(2f+1)^2}H^2 + \frac{3\beta^2(\gamma-1)}{4}H^2\right] \tag{5.40}$$

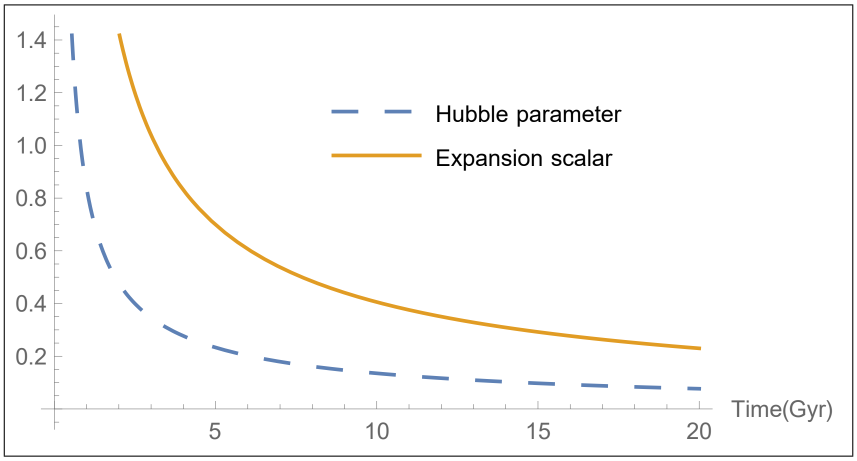

Figure 5.1: Variation of H and θ vs. t, For $\alpha = 0.5$

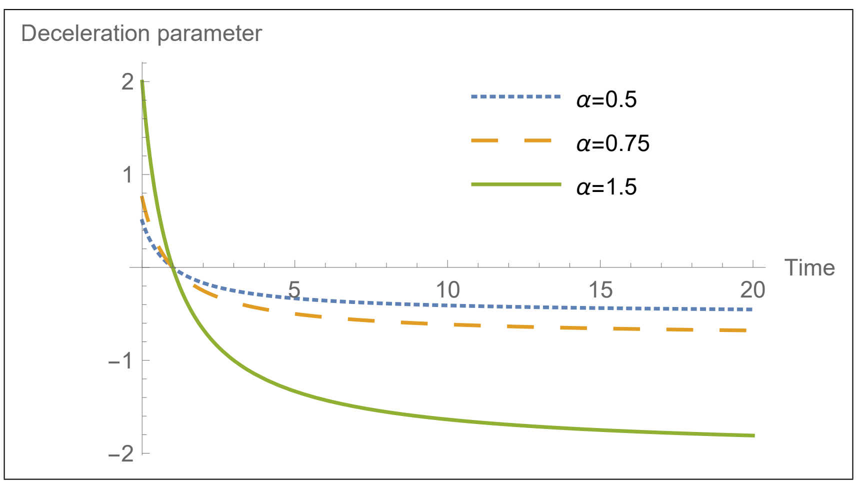

Figure 5.2: Variation of q vs. t

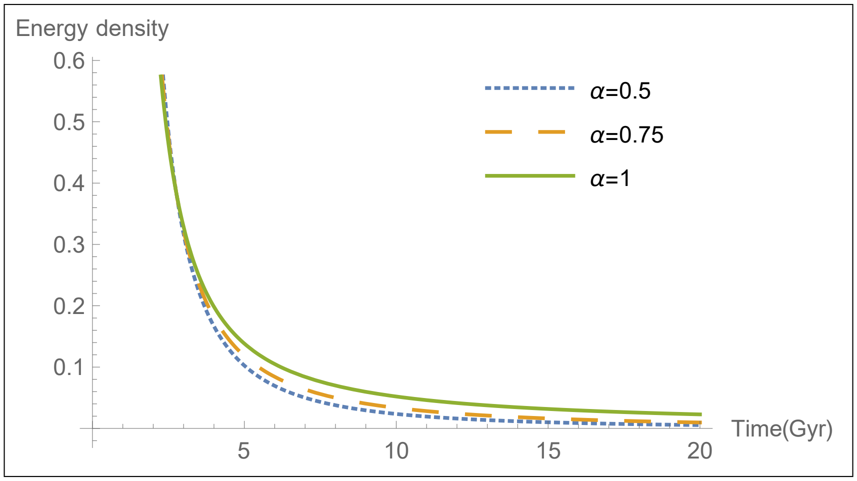

Figure 5.3: Variation of ρ vs. t, For a_0=1, ϕ_0=1,f=1.13

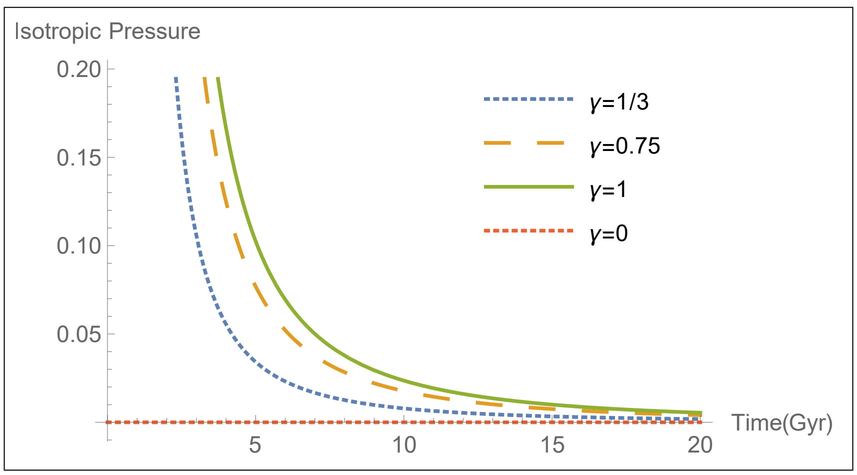

Figure 5.4: Variation of p vs. t, For a_0=1, ϕ_0=1,f=1.13

Further, studying the stability of the solution of the model also investigates the behaviour of the energy condition for the model. The null energy condition (NEC), the weak energy condition (WEC), dominant energy condition (DEC), and strong energy condition (SEC) with density and pressure are as follows.

i) $\rho + p \geq 0$

ii) $\rho + p \geq 0, \rho \geq 0$

iii) $\rho \pm p \geq 0, \rho \geq 0$

iv) $\rho + 3p \geq 0$,

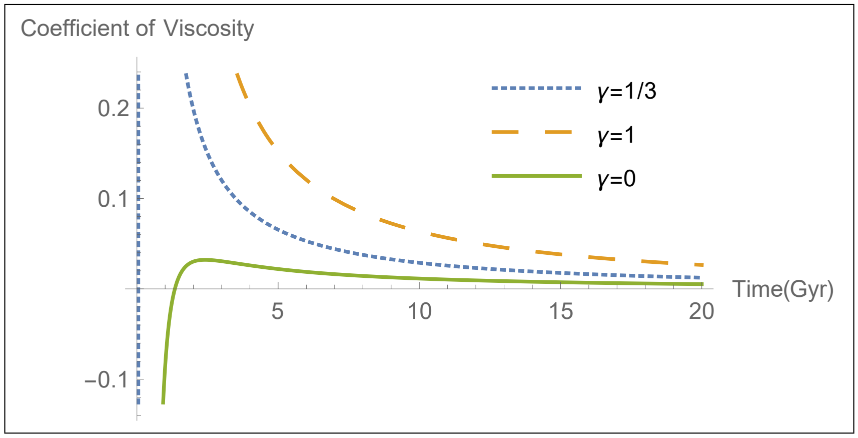

Figure 5.5: Variation of ξ vs t, For a_0=1, ϕ_0=1,f=1.13

From the equation (5.37) and (5.39), the energy condition are obtained as

$$\rho + p = (1+\gamma)\phi_0^2\left(a_0 t^{\frac{1}{1+\alpha}} e^{K(t)}\right)^{2\beta}\left[\frac{9(3f-f^2+1)}{(2f+1)^2}H^2 - \frac{3(f-1)}{(2f+1)}\dot{H} + \frac{3\beta^2H^2}{4}\right] \geq 0 \tag{5.41}$$

$$\rho - p = (1-\gamma)\phi_0^2\left(a_0 t^{\frac{1}{1+\alpha}} e^{K(t)}\right)^{2\beta}\left[\frac{9(3f-f^2+1)}{(2f+1)^2}H^2 - \frac{3(f-1)}{(2f+1)}\dot{H} + \frac{3\beta^2H^2}{4}\right] \geq 0 \tag{5.42}$$

$$\rho + 3p = (1+3\gamma)\phi_0^2\left(a_0 t^{\frac{1}{1+\alpha}} e^{K(t)}\right)^{2\beta}\left[\frac{9(3f-f^2+1)}{(2f+1)^2}H^2 - \frac{3(f-1)}{(2f+1)}\dot{H} + \frac{3\beta^2H^2}{4}\right] \geq 0 \tag{5.43}$$

The model satisfies the WEC, DEC, and SEC, which all hold throughout cosmic time.

Statefinder Parameter {r, s} is the diagnostic pair defined as the second and third derivative of the scale factor $a(t)$ and is given as

$$r = \frac{\dddot{a}}{aH^2} = 1 + 3\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3} \tag{5.44}$$

$$s = \frac{r-1}{3(q-\frac{1}{2})} \tag{5.45}$$

where H and a are Hubble parameter and scale factor. Using eq. (5.20), (5.21) in (5.44)

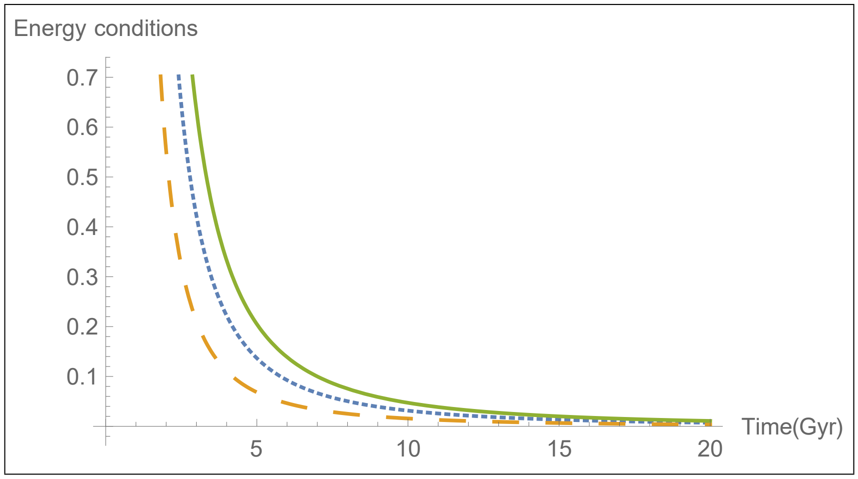

Figure 5.6: Variation of *energy condition vs t*, For $a_0 = 1$, $\phi_0 = 1$, f = 1.13, $\gamma = 0.75$

and (5.45), the satefinder parameter is expressed as

$$r = \frac{\alpha[1+2\alpha+4\alpha log(1+t)+(2-6\alpha)t+(2\alpha-1)t^2]}{(1+t)^2} \tag{5.46}$$

$$s = \frac{2[-1+\alpha+2\alpha^2-2(3\alpha^2+\alpha-1)t-(1+\alpha-2\alpha^2)t^2]}{3[2\alpha-1-(2\alpha+1)t](1+t)} \tag{5.47}$$

From eq. (5.46) and (5.47) the result for the diagnostic pair $(r,s) \to (1,0)$ as $t \to \infty$, which shows that the model of the universe approaches to standard ΛCDM model at late times.

5.5 Conclusion

In this chapter, the viscous Bianchi type-III cosmological model in the Sen-Dunn theory of gravity, along with the bilinear varying deceleration parameter (linear in both q and t), is carried out to obtain the model. The model is anisotropic except for the special case $f = 1$; the model remains isotropic. The spatial volume of the model is zero at $t = 0$, indicating the existence of a singularity at $t = 0$. As time $t \to \infty$, the spatial volume of the model grows exponentially from zero volume. The Hubble parameter (H) and the expansion scalar (θ) of the model are always positive, indicting an expanding universe and decrease as time increases (from Fig. 5.1). The deceleration parameter shows the transition from early deceleration to present accelerating universe (variation from positive to negative) depending on the value of α (from Fig. 5.2), indicating the exponential expansion for $\alpha < 1$ as well as for $\alpha > 1$ a super-exponential expansion. The cosmological parameters ρ, p, and ξ all are infinite at $t = 0$. As $t \to \infty$, the parameters ρ, p, θ, and ξ decrease and tend to zero, providing a space-time that is empty for the greater value of time. (from Fig. 5.3, Fig. 5.4, and Fig. 5.5). The energy conditions for the proposed model are all satisfied throughout the evolution of the universe (from Fig. 5.6) and $\{r,s\} \to \{1,0\}$ indicating that the universe is approaching ΛCDM model from Einstein static era. Since $\frac{\sigma}{\theta}$ tends to constant as $t \to \infty$, the model represents an anisotropic picture throughout the cosmic evolution.