CHAPTER 1

Introduction, Literature Review, and Basic Concepts

This chapter provides some related notions of the neutrosophic set (NS) and neutrosophic topology (NT) that is relevant and necessary to the research work.

1.1 Introduction

Topology is one of the main areas of abstract mathematics concerned with space and deformation. A non-empty set together with a topology is called a topological space. Generally, topological spaces are generalizations of Euclidean Spaces in which the idea of proximity, or limits, is described in terms of relationships between sets rather than distance. Starting from single topology it is extended to bitopology and tritopology etc.

The classical set theory allows the membership value of the elements in the set in binary terms. The fuzzy set (FS) theory is a tool for dealing with vagueness and incomplete data and is much more evolving and applied in different fields. The FS theory permits membership function valued in the interval [0,1]. The notion of the FS is widely used among both 'pure' and 'applied' mathematicians. It has also raised enthusiasm among many engineers, biologists, psychologists, economists, and experts in other areas who use or at least try to use mathematical ideas and methods in their research. Because of different options of human thinking, the FS with some conditions has been improved and extended to the intuitionistic fuzzy set (IFS). After that, the concept of a neutrosophic set (NS) was introduced by Professor F. Smarandache by extending the concepts of FS and IFS. He proposed NS to deal with and manage incomplete, indeterminant, and inconsistent information by utilizing the truth membership function, indeterminacy membership function, and falsity membership function (Smarandache, 1998, 2002, 2005). Whereas, a FS is used to control uncertainty by using the membership function only. To manage the unclear and inconsistent information seen in real-world situations, the NS is one of the most important sets.

One of the earliest areas of pure mathematics to which FS and NS were systematically applied was classical topology. Among several well-known challenges, one challenge is to find the number of topologies in the classical case. Even though no logical explicit and recursive counting formula is yet known, some results still exist. Another approach is the study of the topologies by their number of open sets. Some early researchers initiated the study of finding the number of fuzzy topological spaces (FTSs) and obtained some results. Salama and Alblowi (2012) introduced neutrosophic topological space (NTS). It is observed that the NTSs that satisfy some finiteness conditions have not been considered yet unlike the classical and fuzzy topologies (FTs), where this field is also active and attracting several researchers. The study of the number of NTSs, including neutrosophic clopen topological spaces, neutrosophic crisp topological spaces, and the corresponding number of neutrosophic bitopological spaces and tritopological spaces on a finite set, is the sole focus of this research work.

1.2 Literature Review

We all know how much classical set theory has influenced traditional mathematics. The two-valued logic states that an element can either belong to a set or not belong to that set. In other words, this logic is clear-cut and unambiguous. With the rapid development of science and technology, more and more scientists have gradually realized the vital importance of multi-valued logic. In 1965, Professor Lotfi A. Zadeh from Berkeley University put forward the concept of a FS (Zadeh, 1965) as a means of representing and handling data or information that is not precise. Atanassov (1986) introduced the idea of IFSs, which expands FSs by using a non-membership degree to deal with the existence of vagueness and uncertainty brought on by imperfect knowledge or information. Following that, a generalization of the FS was developed by several academics to describe numerous additional notions. In 1995, Smarandache found that some objects have indeterminacy or neutrality other than membership and non-membership, and so he coined the notion of neutrosophic theory (Smarandache, 1998, 2002, 2005). The theory is essential in many applications since indeterminacy is quantified explicitly, and truth membership, indeterminacy membership, and falsity membership are independent. This theory also generalizes the concept of classical sets, FSs, and IFSs. For the fundamental ideas of the neutrosophic crisp set and its operations, some potential definitions are taken into consideration (Hanafy et al., 2013; Salama, 2013). Smarandache (2016) extended the NS to neutrosophic overset, neutrosophic underset, and neutrosophic offset.

Kelly (1963) was one of the first to study bitopological space. A set $\mathscr{X}$ having two topologies, τ_1 and τ_2, on it is referred to as a bitopological space $(\mathscr{X}, \tau_1, \tau_2)$. The study of tritopological space was first initiated by Kovar (2000), where a non-empty set $\mathscr{X}$ with three topologies is called tritopological spaces. Later, Palaniammal (2011) investigated tritopological spaces.

In 1968, C. L. Chang developed the idea of FTSs (Chang, 1968). Later, the fundamental idea behind the so-called "IFTS" was coined by Çoker (1997). He introduced the definitions of fuzzy continuity, fuzzy compactness, fuzzy connectedness, and fuzzy Hausdorff space after providing the basic definitions and the necessary examples. Then, he worked to obtain several preservation properties and some characterizations of fuzzy compactness and fuzzy connectedness. The FTS is also extended to the fuzzy bitopological spaces, and fuzzy tritopological spaces. In 1989, Kandil introduced the concept of the fuzzy bitopological spaces (Kandil et al., 1995). Recently, many researchers discussed and studied fuzzy bitopological spaces (Das and Bhattacharya, 2018; Das et al., 2019, 2021).

After that, as a generalization of FT and intuitionistic fuzzy topology (IFT), Salama and Alblowi (2012) introduced the novel concept of NTS. They also introduced various forms of separation axioms in NTSs as a new tool for practical applications. Iswarya and Bageerathi (2016) introduced the neutrosophic semi-open sets and neutrosophic semi-closed sets in NTS and derived some of their characterizations. Numerous definitions, related properties, and characteristics have also been studied in addition to the notion of a generalized neutrosophic closed set (Dhavaseelan and Jafari, 2017). The concept of generalized semi-closed sets in NTS was coined by Shanthi et al. (2018). The connectedness of $\alpha\omega$-closed sets in NTS was studied by Parimala et al. (2020).

Ozturk and Ozkan (2019) introduced the notion of neutrosophic bitopological space. Also, the concept of neutrosophic bitopological spaces was discussed by many researchers (Mwchahary and Basumatary, 2020;

Ali Abbas et al., 2021). Al-Hamido (2018) introduced neutrosophic crisp bitopological space and the concept of neutrosophic crisp tritopological space (Al-Hamido and Gharibah, 2018). Currently many authors (Al-Omeri, 2016; Al-Omeri and Smarandache, 2017; Babu and Rajasekhar, 2021a,b; Salama et al., 2014c; Salama, 2013; Salama et al., 2014a) also studied on NTSs. In this way, several researchers have given contributions to topological spaces based on crisp, fuzzy, and neutrosophic sense.

A simple finite set has a small number of subsets, so enumerating them is a straightforward problem. The challenges appear only when the set is unlimited. By introducing the concept of aleph and aleph-null, George Cantor solved the problem of infiniteness (Cantor, 1984). However, in a fuzzy environment and a neutrosophic environment, the problem of counting becomes complicated due to the lack of sharp boundaries. Many authors consider the counting problem in the case of fuzzy subsets (Yager, 1993; Mohapatra and Hong, 2022), which becomes more complicated even if the set is finite. This is considered a challenging problem due to the uncertainty in membership values. Murali (2006) showed that there exists a bijection between the collection of all equivalence classes of fuzzy subsets of $\mathscr{X}$ and the collection of all chains in the power set of $\mathscr{X}$.

Let $\mathscr{X}$ be a finite set having n elements. There is still no explicit formula for calculating $T(n)$, the number of topologies. If $|\mathscr{X}| = n$ (say $n = 1, 2, 3$), we can calculate it by hand. But when n becomes large, the difficulty in finding the number of topologies increases. Some results on the number of topologies on a finite set are computed, where the author obtained a sharper bound for the number of distinct topologies, namely, $2^{(n(n-1))}$ (Krishnamurthy, 1966). Then they pursued the argument and proved a theorem showing that $T(n)$ has precisely the number of certain $n \times n$ matrices of zeros and ones. Sharp Jr. (1968) studied quasi-orderings and topologies on finite sets. They showed that no topology, other than the discrete, has cardinal greater than $\frac{3}{4}2^n$ and some other bounds are derived on the cardinality of connected, non-T_0, connected and non-T_0, and non-connected topologies. Stephen (1968) proved two results connected with topologies on a finite set of elements. One of these two is the only topology on $\mathscr{X}$, having more than $3 \times 2^{n-2}$-open sets is the discrete topology. Several authors contributed to the counting problems of the number of topologies and obtained some results such as Shafaat (1968); Evans et al. (1967). Kleitman and Rothschild (1970) showed that the logarithm (base 2) of the number of distinct topologies on a set of n-elements is asymptotic to $\frac{n^2}{4}$ as n goes to infinity. Stanley (1971) found all non-homeomorphic topologies with n-points and $\geq \frac{7}{16}2^n$-open sets by using the correspondence between finite T_0-topologies and partial orders. He computed $T(n, k)$ for large values of k, viz.; $3.2^{(n-3)} \leq k \leq 2^n$ and also T_0 topologies on a set having either $n + 1, n + 2$ or $n + 3$-open sets. Butler and Markowsky (1973) have presented some formulae related to the number of T_0 topologies with the number of topologies on n points, the number of connected T_0 topologies with the number of connected topologies with n points, the number of isomorphism classes of T_0 topologies on n points with the number of isomorphism classes of connected T_0 topologies on n points, number of isomorphism classes of topologies on n points with the number of isomorphism classes of connected topologies on n points, and many more. Some research works on the number of topologies can be found in Borevich (1977), Borevich et al. (1979), Erné and Stege (1991). Borevich (1977) obtained the number of all the topologies on a fixed set of ten points and is equal to $8, 977, 053, 873, 043$. Out of these, $6, 611, 065, 248, 783$ topologies satisfy the separation axiom T_0. Erné (1981) has presented some combinatorial identities concerning the number $T_0(n, j)$ of all T_0 topologies on n points with j-open sets (which is also the number of all posets with n elements and j antichains). Also, the average cardinality of T_0 topologies on n points has shown to be $2^{\frac{n}{2}+O(\log n)}$. Erné and Stege (1991) obtained the numbers of finite topological spaces with n points and k-open sets for $n < 12$ and all k, by using the well-known one-to-one correspondence between finite quasi-ordered sets and finite topological spaces. Also, they obtained the numbers of all topologies on $n \leq 14$ points satisfying various degrees of separation and connectedness properties. Moreover, they found that the number of (connected) topologies on 14 points exceeds 10^{23}. Some results (Hartmanis, 1958; Schnare, 1967, 1968) have been improved by showing that, if $n \geq 4$, then any topological space on n points (equivalently, any preordered set on n points) which is not in a certain short 'forbidden' list has at least 2^n complements (Brown and Watson, 1996). Brinkmann and McKay (2005) enumerated isomorphism classes of several types of transitive relations (equivalently, finite topologies) up to 15 or 16 points. Benoumhani (2006) computed the number of topology on a set $\mathscr{X}$ having n elements and k-open sets for $2 \leq k \leq 12$, as well as other results concerning T_0 topologies on $\mathscr{X}$ having $n+4 \leq k \leq n+6$ open sets.

Kolli (2007) used a direct approach to compute the set of all labeled topologies on having k-open sets for all $n \geq 4$ and $k \geq 6.2^{n-4}$. Ragnarsson and Tenner (2010) studied the smaller possible number of points in a topological space having k-open sets. Iyer et al. (2013) introduced the concept of ϵ-chainability in topological spaces. Kolli (2014) studied the number of all labeled T_0- topologies having k-open sets and computed the number of labeled T_0- topologies having k-open sets on a set with n-points and also the number of those which are non-homeomorphic for $k \geq 5.2^{(n-4)}$ and arbitrary $n \geq 4$. Also, he computed numbers $t_0(n,k)$ of all unlabaled and non-T_0 topologies with k-open sets for $k \geq 2^{(n-2)}$. Kamel (2015) formulated a special case for computing the number of chain topological spaces and maximal elements with natural generalization. Beer and Bloomfield (2018) studied closure operators for clopen topologies. Recently, Obinna and Adeniji (2019) determined the number of k-element in open and clopen topological space for $1 \leq n \leq 4$, and the corresponding graph for $n \leq 3$.

Yager (1993) discussed the number of classes in a FS. Murali (2005) studied the number of k-level equivalence classes of fuzzy subsets of a finite set of n elements under a natural equivalence which is related to Stirling numbers. Viewing fuzzy subsets as functions from a set into the unit interval, he also associates a kernel partition with every equivalence class of fuzzy subsets. After some elementary properties of the equivalence, they provided a recurrence relation and a generating function concerning the number of k-level fuzzy subsets using Stirling numbers. Murali (2006) re-examined $F(n)$ through an interpretation of equivalence classes of fuzzy subsets as ordered partitions or chains in the boolean algebra of the power set of a set. Also, he derived some recurrence relations, an infinite series as a closed form, and a generating function for $F(n)$ for any natural number n. Jaballah (2000) worked on the effective generators for fuzzy ideals. Jaballah (2001) discussed the reduced fuzzy primary decomposition for fuzzy ideals and also studied the length of maximal chains and the number of ideals in commutative rings (Jaballah and Saidi, 2010). Benoumhani and Jaballah (2019) established several results concerning chains in $\mathscr{Y}^{\mathscr{X}}$, the lattice of mappings from a finite set $\mathscr{X}$ into a finite totally ordered set $\mathscr{Y}$. They computed total number of chains and the cardinalities of several collections of chains. As a result, they determined the total number of chained $\mathscr{Y}$-FTs defined on $\mathscr{X}$. Also, they obtained several related and other well-known results as corollaries. In addition, they presented some natural questions for further investigation. The issues of determining the number F_n of fuzzy subsets of a nonempty finite set $\mathscr{X}$ have been solved by Mohapatra and Hong (2022). They incorporate the equivalence relation on the collection of all fuzzy subsets of $\mathscr{X}$. Moreover, they derived two closed explicit formulae for F_n, which is the sum of a finite series in the product of binomial numbers or the sum of k-level fuzzy subsets $F_{n,k}$ by introducing a classification technique. The number of the maximal chains of crisp subsets of $\mathscr{X}$ can also be determined using these precise formulae. Additionally, they presented some fundamental properties of $F_{n,k}$, and F_n.

Shostak (1989) studied basic ideas and results of FT in his paper two decades of FT. Fora (2017) showed the number of fuzzy clopen sets in an arbitrary FTS can be any natural number greater than 1 if it is finite, and gives an upper bound for this number. He also proved that the number of all crisp fuzzy clopen sets in an arbitrary FTS is a power of 2 if it is finite. Benoumhani and Jaballah (2017) studied FTSs and computed some results for finding the number of FTs.

1.3 Aims and Objectives

The literature review of topological spaces reveals that the precise formula for calculating the number of topologies has not yet been discovered. Some work related to the number of neutrosophic topologies in the classical and fuzzy sense satisfying finiteness conditions have been observed. As the neutrosophic set is the extension of the fuzzy set and currently, researchers in several domains are mainly interested in neutrosophic environments. Among several fields in the neutrosophic environment, neutrosophic topology is the one that attracts researchers. Some works in neutrosophic topological spaces have been studied by several researchers, but so far, there is no work related to the number of neutrosophic topological spaces. By observing this, we have the following objectives for the research work.

The main objectives of the research work are:

(i) To find the cardinalities of the Neutrosophic Set and Neutrosophic Crisp Set.

To find the number of

(ii) Neutrosophic Topological Spaces.

(iii) Neutrosophic Clopen Topological Spaces.

(iv) Neutrosophic Bitopological Spaces and Neutrosophic Tritopological Spaces.

(v) Neutrosophic Clopen Bitopological Spaces and Neutrosophic Clopen Tritopological Spaces.

(vi) Neutrosophic Crisp Topological Spaces.

1.4 Research Methodology

In this research work, the definitions of NS, NCrS, NTS, and NCrTS have been taken as a base in computing the formulae for the number of topological spaces. Also, we proposed the definition of NCLTS and computed some results for finding the number of the NTSs and the number of NCrTSs. The concept of Combinatorics and the Stirling number of the second kind plays a vital role in the computation of the number of NTSs.

1.5 Importance of the Research Work

Topology is used to tell how elements of a set are related spatially to each other and it is observed that the same set can have different topologies. Studying in this particular area is also a very important part of the field of topology and this is one of the interesting and difficult research areas. It is observed that till now the explicit formula for finding the number of topologies is not obtained and many researchers are doing research on this particular area. In this study, the number of bitopological spaces and tritopological spaces is also included for the first time in the neutrosophic sense. Moreover, it is found that the results obtained in this research work are closely related to some known formulae in number theory.

1.6 Preliminaries

This section provides the basic definitions and operations related to the subsequent chapters.

Definition 1.6.1 *(Kelley, 2017)*
Let $\mathscr{X}$ be a set and let τ be a collection of subsets of $\mathscr{X}$ satisfying the following three conditions:

(i) $\phi, \mathscr{X} \in \tau$.

(ii) If $G_1, G_2 \in \tau$ *then* $G_1 \cap G_2 \in \tau$.

(iii) If $G_\lambda \in \tau$ *for every* $\lambda \in \Lambda$*, where* λ *is an arbitrary set then* $\cup\{G_\lambda : \lambda \in \Lambda\} \in \tau$.

Then τ is called a topology for $\mathscr{X}$, the members of τ are called τ-open or simply open sets, and the pair $(\mathscr{X}, \tau)$ is called a topological space. The elements of $\mathscr{X}$ will be called points of the space. In brief, a topology for $\mathscr{X}$ is a collection of subsets of $\mathscr{X}$, containing ϕ and $\mathscr{X}$ and closed under finite intersections and arbitrary unions.

Definition 1.6.2 *(Kelley, 2017)*
Let $(\mathscr{X}, \tau)$ be a topological space. A subset F of $\mathscr{X}$ is said to be τ closed if and only if its complement F' is open.

Definition 1.6.3 *(Kelley, 2017)*
A topology τ on a non-empty set $\mathscr{X}$ is called a clopen topology provided each member of τ is open and closed.

Definition 1.6.4 *(Kelly, 1963)*
Let τ_i and τ_j be two topologies on $\mathscr{X}$, then the triple $(\mathscr{X}, \tau_i, \tau_j)$ is said to be a bitopological space.

Definition 1.6.5 *(Kovar, 2000)*
Let τ_i, τ_j, and τ_k be three topologies on $\mathscr{X}$, then the quadruple $(\mathscr{X}, \tau_i, \tau_j, \tau_k)$ is said to be a tritopological space.

Remark 1.6.1 *(Palaniammal, 2011)*
It is easy to verify that every topological space can generate a bitopological space and that any bitopological space can generate a tritopological space.

Definition 1.6.6 *(Benoumhani, 2006)*
The number of partitions of a finite set with n elements into k blocks is the Stirling number of the second kind. It is denoted by $S(n,k)$ or $S_{n,k}$. The explicit formula for Stirling numbers of the second kind is

$$S(n,k) = S_{n,k} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n.$$

Definition 1.6.7 *(Stanley, 1971)*
A chain topology on $\mathscr{X}$, is a topology whose open sets are totally ordered by inclusion. The number of chain topologies on a set $\mathscr{X}$ with n elements, having k-open sets is usually denoted by $C(n,k)$.

Definition 1.6.8 *(Stanley, 1971)*
Let $C(n,k)$ be the number of chain topologies on $\mathscr{X}$ having k-open sets. Then,

$$C(n,k) = \sum_{l=1}^{n-1}\binom{n}{k}C(l,k-1) = (k-1)!S(n,k-1).$$

Theorem 1.6.1 *(Obinna and Adeniji, 2019)*
Let $(\mathscr{X}, \tau)$ be a topological space, then a topology having k-element is clopen if and only if $k = 2^n$ for $n \geq 1$.

1.7 Fuzzy Set and Fuzzy Topological Space

Definition 1.7.1 *(Zadeh, 1965)*
Let $\mathscr{X}$ be a universal set. Then a function $A : \mathscr{X} \to [0,1]$ define a FS on $\mathscr{X}$, where A is called the membership function and $\mu_A(x)$ is called membership grade of x.

The FS A is also written as $A = \{(x, \mu_A(x)) : x \in \mathscr{X}\}$, where each pair $(x, \mu_A(x))$ is called singleton.

Definition 1.7.2 *(Chang, 1968)*
A FT τ on a set $\mathscr{X}$ consists of a collection of fuzzy subsets of $\mathscr{X}$ called open set, satisfying the following three axioms:

(i) The fuzzy subsets $0_{\mathscr{F}}$ and $1_{\mathscr{F}}$ are in τ.

(ii) The union $\cup_{i \in I} U_i$ of any collection $\{U_i : i \in I\}$ of elements of τ is also in τ.

(iii) The intersection $U_1 \cap U_2 \in \tau$, for any two elements $U_1, U_2 \in \tau$.

The members of τ is called open set and the pair $(\mathscr{X}, \tau)$ is called FTS. The existence of a topology τ in $\mathscr{X}$ with membership values in M implies necessarily that $0_{\mathscr{F}}$ and $1_{\mathscr{F}}$ are open sets in τ.

Definition 1.7.3 *(Kandil et al., 1995)*
A fuzzy bitopological space is a triple a $(\mathscr{X}, \tau_1, \tau_2)$, where τ_1 and τ_2 are arbitrary FTs on $\mathscr{X}$.

Definition 1.7.4 *(Palaniammal, 2011)*
A fuzzy tritopological space is a quadruple $(\mathscr{X}, \tau_1, \tau_2, \tau_3)$, where τ_1, τ_2, and τ_3 are arbitrary FTs on $\mathscr{X}$.

Definition 1.7.5 *(Chon, 2009)*
Let $\mathscr{X}$ be a set. A function $A : \mathscr{X} \times \mathscr{X} \to [0,1]$ is called a fuzzy relation in $\mathscr{X}$. The fuzzy relation A in $\mathscr{X}$ is reflexive iff $A(x,x) = 1$ for all

$x \in \mathscr{X}$, A is transitive iff $A(x,z) \geq sup\ min\,(A(x,y), A(y,z))$, and A is antisymmetric iff $A(x,y) > 0$ and $A(y,x) > 0$ implies $x = y$. A fuzzy relation A is a fuzzy partial order relation if A is reflexive, antisymmetric, and transitive. A fuzzy partial order relation A is a fuzzy total order relation iff $A(x,y) > 0$ or $A(y,x) > 0$ for all $x, y \in \mathscr{X}$. If A is a fuzzy partial order relation in a set $\mathscr{X}$, then $(\mathscr{X}, A)$ is called a fuzzy partially ordered set or a fuzzy poset. If B is a fuzzy total order relation in a set $\mathscr{X}$, then $(\mathscr{X}, B)$ is called a fuzzy totally ordered set or a fuzzy chain.

Definition 1.7.6 *(Chon, 2009)*
Let $(\mathscr{X}, A)$ be a fuzzy poset and let $B \subset \mathscr{X}$. An element $u \in \mathscr{X}$ is said to be an upper bound for a subset B iff $A(b,u) > 0$ for all $b \in B$. An upper bound u_0 for B is the least upper bound ($\vee$) of B iff $A(u_0, u) > 0$ for every upper bound u for B. An element $v \in \mathscr{X}$ is said to be a lower bound for a subset B iff $A(v,b) > 0$ for all $b \in B$. A lower bound v_0 for B is the greatest lower bound ($\wedge$) of B iff $A(v, v_0) > 0$ for every lower bound v for B.

Definition 1.7.7 *(Chon, 2009)*
Let $(\mathscr{X}, A)$ be a fuzzy poset. Then $(\mathscr{X}, A)$ is a fuzzy lattice iff $x \vee y$ and $x \wedge y$ exist for all $x, y \in \mathscr{X}$.

Proposition 1.7.1 *(Benoumhani and Jaballah, 2017)*
The number of FTs on $\mathscr{X}$, with membership values in M, is finite if and only if $\mathscr{X}$ and M are both finite.

Lemma 1.7.1 *(Benoumhani and Jaballah, 2017)*
Let m and n be positive integers, then for any numbers $y_1, y_2, \ldots, y_m$, we have $\sum_{(i_1, i_2, \ldots, i_n) \in \{1,2,\ldots,m\}^n} y_{i_1} y_{i_2} \cdots y_{i_n} = (\sum_{i=1}^{m} y_i)^n$.

Theorem 1.7.1 *(Benoumhani and Jaballah, 2017)*
The number $\tau_{\mathscr{F}}(n, m, 4)$ of FTs in $\mathscr{F}$ having exactly 4-open sets is given by

$$\tau_{\mathscr{F}}(n, m, 4) = \left(\frac{m(m+1)}{2}\right)^n - 3m^n + 2^{n-1} + 2.$$

Theorem 1.7.2 *(Benoumhani and Jaballah, 2017)*
The number $\tau_{\mathscr{F}}(n, m, 5)$ of FTs in $\mathscr{F}$ having exactly 5-open sets is given by

$$\tau_{\mathscr{F}}(n, m, 5) = \binom{m+2}{3}^n - 4\binom{m+1}{2}^n + 5m^n - (m-1)^n + (2m-1)^n - 2^{n+1}.$$

Theorem 1.7.3 *(Benoumhani and Jaballah, 2017)*
For $n \geq m \geq 2$, the number of FTs in $\mathscr{F}$ having k-open sets where $m^n - m^{n-2} < k < m^n$ and $k = m^n - m^{n-2}$ are

(i) $\tau_{\mathscr{F}}(n, m, k) = 0$ *for* $m^n - m^{n-2} < k < m^n$.

(ii) $\tau_{\mathscr{F}}(n, m, m^n - m^{n-2}) = n(n-1)$.

The following well known algorithm enables us to compute the number of chains of a certain length in a finite ordered set.

Algorithm 1.7.1 *(Benoumhani and Jaballah, 2017)*
Let $\mathscr{P}$ be a finite ordered set and $c_k(\mathscr{P})$ or c_k denotes the number of chains with k elements in the ordered set $\mathscr{P}$. Also, for each $u \in \mathscr{P}$, let $c_k(u)$ be the number of chains with k elements from $\mathscr{P}$ and with maximal element u. The numbers $c_1, c_2, \ldots, c_n$ are obtained recursively as follows:

(i) $c_1(u) = 1$, *for each* $u \in \mathscr{P}$.

(ii) $c_k(u) = \sum_{v<u} c_{k-1}(v)$, $2 \leq k \leq n$, *for each* $u \in \mathscr{P}$.

(iii) $c_k := c_k(\mathscr{P}) = \sum_{u \in P} c_k(u), 1 \leq k \leq n$.

1.8 Neutrosophic Set and Neutrosophic Topological Space

Definition 1.8.1 *(Smarandache, 2005)*
A NS A^{NT} on a universe of discourse $\mathscr{X}$ is defined as $A^{NT} = \langle \frac{x}{(T(x), I(x), F(x))} : x \in \mathscr{X} \rangle$, where $T, I, F : \mathscr{X} \to]^{-}0, 1^{+}[$. Note that $^{-}0 \leq T(x) + I(x) +$

$F(x) \leq 3^+$; $T(x), I(x)$, and $F(x)$ represents the degree of membership function, degree of indeterminacy, and degree of non-membership function respectively.

Definition 1.8.2 *(Smarandache, 2005)*
Let $\mathscr{X} \neq \phi$ and $A^{NT} = \langle \frac{x}{\left(T_{A^{NT}}(x), I_{A^{NT}}(x), F_{A^{NT}}\right)(x)} : x \in \mathscr{X} \rangle$, and $B^{NT} = \langle \frac{x}{\left(T_{B^{NT}}(x), I_{B^{NT}}(x), F_{B^{NT}}(x)\right)} : x \in \mathscr{X} \rangle$ are NS. Then,

(i) $A^{NT} \wedge B^{NT} = \langle \frac{x}{\left(min\left(T_{A^{NT}}(x), T_{B^{NT}}(x)\right), max\left(I_{A^{NT}}(x), I_{B^{NT}}(x)\right), max\left(F_{A^{NT}}(x), F_{B^{NT}}(x)\right)\right)} : x \in \mathscr{X} \rangle$.

(ii) $A^{NT} \vee B^{NT} = \langle \frac{x}{\left(max\left(T_{A^{NT}}(x), T_{B^{NT}}(x)\right), min\left(I_{A^{NT}}(x), I_{B^{NT}}(x)\right), min\left(F_{A^{NT}}(x), F_{B^{NT}}(x)\right)\right)} : x \in \mathscr{X} \rangle$.

In general, the intersection $\cap$ and union $\cup$ of a collection of NS $\{A_i^{NT} : i \in I\}$, are defined by

$$\cap_{i \in I} A_i^{NT} = \langle \frac{x}{\left(inf\{T_{A_i^{NT}}(x)\}, sup\{I_{A_i^{NT}}(x)\}, sup\{F_{A_i^{NT}}(x)\}\right)} : x \in \mathscr{X} \rangle,$$
$$\cup_{i \in I} A_i^{NT} = \langle \frac{x}{\left(sup\{T_{A_i^{NT}}(x)\}, inf\{I_{A_i^{NT}}(x)\}, inf\{F_{A_i^{NT}}(x)\}\right)} : x \in \mathscr{X} \rangle.$$

Definition 1.8.3 *(Salama and Alblowi, 2012)*
The neutrosophic subsets 0^{NT} and 1^{NT} in $\mathscr{X}$ are as follows:
0^{NT} may be defined as:

$$0^{NT} = \{\langle x, 0, 0, 1 \rangle : x \in \mathscr{X}\},$$
$$0^{NT} = \{\langle x, 0, 1, 1 \rangle : x \in \mathscr{X}\},$$
$$0^{NT} = \{\langle x, 0, 1, 0 \rangle : x \in \mathscr{X}\},$$
$$0^{NT} = \{\langle x, 0, 0, 0 \rangle : x \in \mathscr{X}\}.$$

1^{NT} may be defined as:

$$1^{NT} = \{\langle x, 1, 0, 0 \rangle : x \in \mathscr{X}\},$$
$$1^{NT} = \{\langle x, 1, 0, 1 \rangle : x \in \mathscr{X}\},$$
$$1^{NT} = \{\langle x, 1, 1, 0 \rangle : x \in \mathscr{X}\},$$
$$1^{NT} = \{\langle x, 1, 1, 1 \rangle : x \in \mathscr{X}\}.$$

Definition 1.8.4 *(Salama and Alblowi, 2012)*
Let $A = \langle \mu_A, \rho_A, \gamma_A \rangle$ be a NS on $\mathscr{X}$, then the complement of the set A, $C(A)$, for short, may be defined as three kinds of complements

(i) $C(A) = \{\langle x, 1-\mu_A(x), 1-\rho_A(x), 1-\gamma_A(x)\rangle : x \in \mathscr{X}\}$.

(ii) $C(A) = \{\langle x, \gamma_A(x), \rho_A(x), \mu_A(x)\rangle : x \in \mathscr{X}\}$.

(iii) $C(A) = \{\langle x, \gamma_A(x), 1-\rho_A(x), \mu_A(x)\rangle : x \in \mathscr{X}\}$.

Definition 1.8.5 *(Salama and Alblowi, 2012)*
Let $\mathscr{X}$ be a non-empty set, and neutrosophic subsets A^{NT} and B^{NT} in the form $A^{NT} = \langle \mu_{A^{NT}}, \rho_{A^{NT}}, \gamma_{A^{NT}} \rangle$, $B^{NT} = \langle \mu_{B^{NT}}, \rho_{B^{NT}}, \gamma_{B^{NT}} \rangle$, then the following two possible definitions may be considered for subsets.
$A^{NT} \subseteq B^{NT}$ may be defined as:

$$\begin{aligned} A^{NT} \subseteq B^{NT} &\iff \mu_{A^{NT}}(x) \leq \mu_{B^{NT}}(x), \rho_{A^{NT}}(x) \geq \rho_{B^{NT}(x)} \\ &\quad \textit{and } \gamma_{A^{NT}}(x) \leq \gamma_{B^{NT}}(x), \\ A^{NT} \subseteq B^{NT} &\iff \mu_{A^{NT}}(x) \leq \mu_{B^{NT}}(x), \rho_{A^{NT}}(x) \geq \rho_{B^{NT}(x)} \\ &\quad \textit{and } \gamma_{A^{NT}}(x) \geq \gamma_{B^{NT}}(x). \end{aligned}$$

Definition 1.8.6 *(Salama and Alblowi, 2012)*
A NT on a non-empty set $\mathscr{X}$ is a family τ^{NT} of neutrosophic subsets in $\mathscr{X}$ satisfying the following axioms

(i) $0^{NT}, 1^{NT} \in \tau^{NT}$.

(ii) $A_1^{NT} \cap A_2^{NT} \in \tau^{NT}$, *for any* $A_1^{NT}, A_2^{NT} \in \tau^{NT}$.

(iii) $\cup A_i^{NT} \in \tau^{NT}$, *for arbitrary family* $\{A_i^{NT} : i \in I\} \in \tau^{NT}$.

The pair $\left(\mathscr{X}, \tau^{NT}\right)$ is called NTS and any NS in τ^{NT} is called neutrosophic open set (NOS) in $\mathscr{X}$.

Definition 1.8.7 *(Salama and Alblowi, 2012)*
Let $(\mathscr{X}, \tau^{NT})$ be a NTS over $\mathscr{X}$ and A^{NT} be a NS over $\mathscr{X}$. Then A^{NT} is said to be a neutrosophic closed set if its complement is a NOS.

Definition 1.8.8 *(Ozturk and Ozkan, 2019)*
Let τ_i^{NT} and τ_j^{NT} are any two NTs on $\mathscr{X}$, the triple $\left(\mathscr{X}, \tau_i^{NT}, \tau_j^{NT}\right)$ is said to be a neutrosophic bitopological space.

Definition 1.8.9 *(Ozturk and Ozkan, 2019)*
Let $(\mathscr{X}, \tau_1^{NT}, \tau_2^{NT})$ be a neutrosophic bitopological space. A NS $A^{NT} = \langle \frac{x}{(T_A^{NT}(x), I_A^{NT}(x), F_A^{NT}(x))} : x \in \mathscr{X} \rangle$ over $\mathscr{X}$ is called a pairwise NOS in $(\mathscr{X}, \tau_1^{NT}, \tau_2^{NT})$ if there exist a NS $A_1^{NT} = \langle \frac{x}{(T_{A_1^{NT}}(x), I_{A_{1NT}}(x), F_{A_1^{NT}}(x))} : x \in \mathscr{X} \rangle$ in τ_1^{NT} and a NS $A_2^{NT} = \langle \frac{x}{(T_{A_2^{NT}}(x), I_{A_{2NT}}(x), F_{A_2^{NT}}(x))} : x \in \mathscr{X} \rangle$ in τ_2^{NT} such that $A^{NT} = A_1^{NT} \cup A_2^{NT}$.

Definition 1.8.10 *(Das and Pramanik, 2021)*
Let τ_i^{NT}, τ_j^{NT}, and τ_k^{NT} be any three NTs on $\mathscr{X}$. Then the quadruple $(\mathscr{X}, \tau_i^{NT}, \tau_j^{NT}, \tau_k^{NT})$ is said to be a neutrosophic tritopological space.

Proposition 1.8.1 *(Salama and Alblowi, 2012)*
For any NS A^{NT} the following conditions are holds

(i) $0^{NT} \subseteq A^{NT}$, $0^{NT} \subseteq 0^{NT}$.

(ii) $A^{NT} \subseteq 1^{NT}$, $1^{NT} \subseteq 1^{NT}$.

1.9 Neutrosophic Crisp Set and Neutrosophic Crisp Topological Space

Definition 1.9.1 *(Salama, 2013)*
Let $\mathscr{X}$ be a non-empty fixed set. A neutrosophic crisp set (NCrS) A is an object having the form $A = \langle A_1, A_2, A_3 \rangle$, where A_1, A_2, and A_3 are subsets of $\mathscr{X}$ satisfying $A_1 \cap A_2 = \phi$, $A_1 \cap A_3 = \phi$, and $A_2 \cap A_3 = \phi$.

Remark 1.9.1 *(Salama, 2013)*
A NCrS $A = \langle A_1, A_2, A_3 \rangle$ can be identified as an ordered triple $\langle A_1, A_2, A_3 \rangle$, where A_1, A_2, and A_3 are subsets of $\mathscr{X}$.

Definition 1.9.2 *(Salama, 2013)*
$\phi_{\mathscr{N}}$ *may be defined in many ways as a NCrS as follows:*

(i) $\phi_{\mathscr{N}} = \langle \phi, \phi, \mathscr{X} \rangle$.

(ii) $\phi_{\mathscr{N}} = \langle \phi, \mathscr{X}, \mathscr{X} \rangle$.

(iii) $\phi_{\mathscr{N}} = \langle \phi, \mathscr{X}, \phi \rangle$.

(iv) $\phi_{\mathscr{N}} = \langle \phi, \phi, \phi \rangle$.

$\mathscr{X}_{\mathscr{N}}$ *may also be defined in many ways as a NCrS as follows:*

(i) $\mathscr{X}_{\mathscr{N}} = \langle \mathscr{X}, \phi, \phi \rangle$.

(ii) $\mathscr{X}_{\mathscr{N}} = \langle \mathscr{X}, \mathscr{X}, \phi \rangle$.

(iii) $\mathscr{X}_{\mathscr{N}} = \langle \mathscr{X}, \mathscr{X}, \mathscr{X} \rangle$.

Definition 1.9.3 *(Salama, 2013)*
Let $A = \langle A_1, A_2, A_3 \rangle$ *be a NCrS on* $\mathscr{X}$*, then the complement* A^c *of the set* A *may be defined in three different ways*

(i) $A^c = \langle A_1^c, A_2^c, A_3^c \rangle$.

(ii) $A^c = \langle A_3, A_2, A_1 \rangle$.

(iii) $A^c = \langle A_3, A_2^c, A_1 \rangle$.

Definition 1.9.4 *(Salama, 2013)*
Let $\mathscr{X}$ *be a non-empty set, and the NCrSs* A *and* B *be in the form* $A = \langle A_1, A_2, A_3 \rangle$, $B = \langle B_1, B_2, B_3 \rangle$ *respectively. Then the following two possible definitions may be considered for subsets (*$A \subseteq B$*):*

$A \subseteq B \iff A_1 \subseteq B_1$, $A_1 \subseteq B_2$, and $A_3 \supseteq B_3$, or
$A \subseteq B \iff A_1 \subseteq B_1$, $A_2 \supseteq B_2$, and $A_3 \supseteq B_3$.

Definition 1.9.5 *(Salama, 2013)*
Let $\mathscr{X}$ *is a non-empty set, and the NCrSs* A *and* B *be in the form* $A = \langle A_1, A_2, A_3 \rangle$ *and* $B = \langle B_1, B_2, B_3 \rangle$ *respectively. Then,*

(i) $A \cap B$ *may be defined in two ways:*

$$A \cap B = \langle A_1 \cap B_1, A_2 \cap B_2, A_3 \cup B_3 \rangle, \text{ or}$$
$$A \cap B = \langle A_1 \cap B_1, A_2 \cup B_2, A_3 \cup B_3 \rangle.$$

(ii) $A \cup B$ *may also be defined in two ways:*

$$A \cup B = \langle A_1 \cup B_1, A_2 \cap B_2, A_3 \cap B_3 \rangle, \text{ or}$$
$$A \cup B = \langle A_1 \cup B_1, A_2 \cup B_2, A_3 \cap B_3 \rangle.$$

Definition 1.9.6 *(Salama, 2013)*
A neutrosophic crisp topology (NCrT) on a non-empty set $\mathscr{X}$ *is a family* τ^{NC} *of neutrosophic crisp subsets in* $\mathscr{X}$ *satisfying the following axioms*

(i) $\phi_{\mathscr{N}}, \mathscr{X}_{\mathscr{N}} \in \tau^{NC}$.

(ii) $A_1 \cap A_2 \in \tau^{NC}$; *for any* $A_1, A_2 \in \tau^{NC}$.

(iii) $\cup A_j \in \tau^{NC}; \forall \{A_j : j \in J\} \subseteq \tau^{NC}$.

In this case, the pair $(\mathscr{X}, \tau^{NC})$ *is called a neutrosophic crisp topological space (NCrTS) in* $\mathscr{X}$. *The elements in* τ^{NC} *are called neutrosophic crisp open sets (NCrOSs) in* $\mathscr{X}$. *A NCrS* F *is closed if and only if its complement* F^c *is an open NCrS.*

Remark 1.9.2 *(Salama, 2013)*
The NCrTSs are very natural generalizations of topological spaces and intuitionistic topological spaces.

Proposition 1.9.1 *(Salama, 2013)*
For any NCrS A the following are hold:

(i) $\phi_{\mathscr{N}} \subseteq A$, $\phi_{\mathscr{N}} \subseteq \phi_{\mathscr{N}}$.

(ii) $A \subseteq \mathscr{X}_{\mathscr{N}}$, $\mathscr{X}_{\mathscr{N}} \subseteq \mathscr{X}_{\mathscr{N}}$.

Proposition 1.9.2 *(Salama, 2013)*
For any two NCrSs A *and* B *on* $\mathscr{X}$, *the followings are true*

(i) $(A \cap B)^c = A^c \cup B^c$.

(ii) $(A \cup B)^c = A^c \cap B^c$.

Proposition 1.9.3 *(Salama, 2013)*
Let $\{A_j : j \in J\}$ be arbitrary family of neutrosophic crisp subsets in $\mathscr{X}$. Then,

(i) $\cap A_j$ may be defined as the following types:

$$\cap A_j = \langle \cap A_{j_1}, \cap A_{j_2}, \cup A_{j_3} \rangle,$$
$$\cap A_j = \langle \cap A_{j_1}, \cup A_{j_2}, \cup A_{j_3} \rangle.$$

(ii) $\cup A_j$ may be defined as the following types:

$$\cup A_j = \langle \cup A_{j_1}, \cap A_{j_2}, \cap A_{j_3} \rangle,$$
$$\cup A_j = \langle \cup A_{j_1}, \cup A_{j_2}, \cap A_{j_3} \rangle.$$

1.10 Tools for Counting Methods

This section presents some known definitions and results in the area of combinatorics to deal with our main results.

Definition 1.10.1 *(Cameron, 1994)*
The number of m-element subsets of an n-element set (that is, the number of ways we could select m-distinct elements from an n-element set) is called a binomial number or a binomial coefficient, denoted by $\binom{n}{i}$.

Some of its the most important basic properties are as follows:

(i) $\binom{n}{0} = \binom{n}{n} = 1$.

(ii) $\binom{n}{1} = \binom{n}{n-1} = n$.

(iii) $\binom{n}{m} = \binom{n}{n-m}, 0 \leq m \leq n$.

(iv) $\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}, 1 \leq m < n$.

(v) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

(vi) $\sum_{i=1}^{n} \sum_{j=1}^{i} j = \frac{n(n+1)(n+2)}{6}$.

Lemma 1.10.1 *(Cameron, 1994)*

For $n \geq m \geq 0$*,*

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

with the convention that $\binom{n}{m} = 0$*, for any* $m > n$*.*

Theorem 1.10.1 *(Cameron, 1994)*

For any integer $n \geq 0$*,*

$$(a+b)^n = \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m}.$$