CHAPTER 2

Cardinalities of the Neutrosophic Set and Neutrosophic Crisp Set on a Finite Set

In this chapter, the cardinalities of the NS for a non-empty finite set $\mathscr{X}$ whose neutrosophic values lie in $\mathscr{M}$ with $|\mathscr{M}| = m \geq 2$ and NCrS have been studied. Further, the NCs of cardinalities less than 4 in a NS have been computed. In addition, some interesting propositions have been studied.

2.1 Cardinalities of the Neutrosophic Set

This section presents the works concerned with the enumeration of neutrosophic subsets.

Let us assume that $\mathscr{X}$ and $\mathscr{M}$ are both finite, say $\mathscr{X} = \{v_1, v_2, \ldots, v_n\}$, and $\mathscr{M} = \{t_0, t_1, t_2, \ldots, t_{m-1}\}$ is a totally ordered set such that:
$(0, 1, 1) = t_0 < t_1 = (T_1, I_1, F_1) < t_2 = (T_2, I_2, F_2) < \ldots < t_{m-2} = (T_{m-2}, I_{m-2}, F_{m-2}) < t_{m-1} = (1, 0, 0)$, where $t_i = (T_i, I_i, F_i) < t_j = (T_j, I_j, F_j)$ iff $[T_i \leq T_j, I_i \geq I_j, F_i \geq F_j$ and at least one of $T_i < T_j$ or $I_i > I_j$ or $F_i > F_j]$ or $[T_i \leq T_j, I_i \geq I_j, F_i \leq F_j$ and at least one

The results discussed in this chapter have been communicated to Journal for Publication.

of $T_i < T_j$ or $I_i > I_j$ or $F_i < F_j$]. Also, let $\mathscr{N}_{\mathscr{X}}$ be the collection of neutrosophic subsets of $\mathscr{X}$ with neutrosophic values in $\mathscr{M}$.
$\mathscr{N}_{\mathscr{X}}$ is partially ordered by:
$A^{NT} \preceq B^{NT}$, if and only if $T_1(v_i) \leq T_2(v_i), I_1(v_i) \geq I_2(v_i), F_1(v_i) \geq F_2(v_i)$ or $T_1(v_i) \leq T_2(v_i), I_1(v_i) \geq I_2(v_i), F_1(v_i) \leq F_2(v_i)$ for each $i \in \{1, 2, \ldots, n\}$, where $A^{NT} = \langle \frac{x}{(T_1(x), I_1(x), F_1(x))} : x \in \mathscr{X} \rangle$ and $B^{NT} = \langle \frac{x}{(T_2(x), I_2(x), F_2(x))} : x \in \mathscr{X} \rangle$.
We also have,
$A^{NT} \prec B^{NT}$, if and only if $A^{NT} \preceq B^{NT}$ and [at least one of $T_1(v_i) < T_2(v_i)$ or $I_1(v_i) > I_2(v_i)$ or $F_1(v_i) > F_2(v_i)$] or [$T_1(v_i) < T_2(v_i)$ or $I_1(v_i) > I_2(v_i)$ or $F_1(v_i) < F_2(v_i)$], for some $i \in \{1, 2, \ldots, n\}$.

Then the first question that arises in our mind is "How many NSubs are there in a non-empty finite set $\mathscr{X}$ of n elements?" The study is based on a non-empty finite set because a NSub of the empty set does not have a conventional meaning since there are no elements to talk of in this set.

Definition 2.1.1

The set of all NSubs of a non-empty finite set $\mathscr{X}$ whose neutrosophic values lie in $\mathscr{M}$ with $|\mathscr{M}| = m \geq 2$ is called the neutrosophic power set of $\mathscr{X}$ with neutrosophic values in $\mathscr{M}$. The notation for the neutrosophic power set of $\mathscr{X}$ whose neutrosophic values lie in $\mathscr{M}$ is $\mathscr{P}_{\mathscr{M}}(\mathscr{X})$ and its cardinality is denoted by $|\mathscr{P}_{\mathscr{M}}(\mathscr{X})|$.

Proposition 2.1.1

A non-empty finite set $\mathscr{X}$ with $|\mathscr{X}| = n$ whose neutrosophic values lie in $\mathscr{M}$ with $|\mathscr{M}| = m \geq 2$ has m^n NSubs.

Proof:

Each element of $\mathscr{X}$ has m choices for neutrosophic values as $|\mathscr{M}| = m$. Hence, the total number of NSubs of $\mathscr{X}$ whose neutrosophic values lie in $\mathscr{M}$ is $\underbrace{m.m\ldots m}_{n \text{ times}} = m^n$.

Proposition 2.1.2 *If* $|\mathscr{X}| = n$ *and* $|\mathscr{M}| = m \geq 2$*, then the cardinality of the neutrosophic power set of* $\mathscr{X}$ *with neutrosophic values in* $\mathscr{M}$ *is* $|\mathscr{P}_{\mathscr{M}}(\mathscr{X})| = m^n$.

Proof:
By Definition 2.1.1 and Proposition 2.1.1, the proof is straightforward.

Example 2.1.1
Let $\mathscr{X} = \{u\}$ *and* $\mathscr{M} = \{(0,1,1), (\mathfrak{T}, \mathfrak{I}, \mathfrak{F}), (1,0,0)\}$.
It is seen that, $|\mathscr{X}| = n = 1$, $|\mathscr{M}| = m = 3$. *Then, the NSubs of* $\mathscr{X}$ *whose neutrosophic values lie in* $\mathscr{M}$ *are*

$$0^{NT} = \langle \tfrac{u}{(0,1,1)} \rangle, \qquad 1^{NT} = \langle \tfrac{u}{(1,0,0)} \rangle, \qquad A_1^{NT} = \langle \tfrac{u}{(\mathfrak{T},\mathfrak{I},\mathfrak{F})} \rangle.$$

Therefore, $|\mathscr{P}_{\mathscr{M}}(\mathscr{X})| = 3 = 3^1 = m^n$.

Example 2.1.2
Let $\mathscr{X} = \{u, v\}$ *and* $\mathscr{M} = \{(0,1,1), (\mathfrak{T}, \mathfrak{I}, \mathfrak{F}), (1,0,0)\}$.
It is seen that, $|\mathscr{X}| = n = 2$, $|\mathscr{M}| = m = 3$. *Then, the NSubs of* $\mathscr{X}$ *whose neutrosophic values lie in* $\mathscr{M}$ *are*

$$\begin{array}{ll}
0^{NT} = \langle \tfrac{u}{(0,1,1)}, \tfrac{v}{(0,1,1)} \rangle, & 1^{NT} = \langle \tfrac{u}{(1,0,0)}, \tfrac{v}{(1,0,0)} \rangle, \\
A_1^{NT} = \langle \tfrac{u}{(0,1,1)}, \tfrac{v}{(\mathfrak{T},\mathfrak{I},\mathfrak{F})} \rangle, & A_2^{NT} = \langle \tfrac{u}{(0,1,1)}, \tfrac{v}{(1,0,0)} \rangle, \\
A_3^{NT} = \langle \tfrac{u}{(\mathfrak{T},\mathfrak{I},\mathfrak{F})}, \tfrac{v}{(0,1,1)} \rangle, & A_4^{NT} = \langle \tfrac{u}{(\mathfrak{T},\mathfrak{I},\mathfrak{F})}, \tfrac{v}{(\mathfrak{T},\mathfrak{I},\mathfrak{F})} \rangle, \\
A_5^{NT} = \langle \tfrac{u}{(\mathfrak{T},\mathfrak{I},\mathfrak{F})}, \tfrac{v}{(1,0,0)} \rangle, & A_6^{NT} = \langle \tfrac{u}{(1,0,0)}, \tfrac{v}{(0,1,1)} \rangle, \\
A_7^{NT} = \langle \tfrac{u}{(1,0,0)}, \tfrac{v}{(\mathfrak{T},\mathfrak{I},\mathfrak{F})} \rangle. &
\end{array}$$

Therefore, $|\mathscr{P}_{\mathscr{M}}(\mathscr{X})| = 9 = 3^2$.

Proposition 2.1.3
The NCs of length one in $\mathscr{P}_{\mathscr{M}}(\mathscr{X})$ *is* $\mathscr{C}_N(n, m, 1) = |\mathscr{P}_{\mathscr{M}}(\mathscr{X})| = m^n$.

Proof:
Every NSubs of $\mathscr{X}$ whose neutrosophic values lie in $\mathscr{M}$ are taken as the chain of length one, where $|\mathscr{X}| = n$ and $|\mathscr{M}| = m$. Then clearly,

$$\mathscr{C}_N(n, m, 1) = |\mathscr{P}_{\mathscr{M}}(\mathscr{X})| = m^n.$$

Example 2.1.3

Every NSub in $\mathscr{P}_{\mathscr{M}}(\mathscr{X})$ *is a chain of length one. So, if we consider example 2.1.2, then the chain of length one are*

$$0^{NT}, 1^{NT}, A_1^{NT}, A_2^{NT}, A_3^{NT}, A_4^{NT}, A_5^{NT}, A_6^{NT}, A_7^{NT}.$$

Proposition 2.1.4

The NCs of length two in $\hat{\mathscr{P}}_{\mathscr{M}}(\mathscr{X}) = \mathscr{P}_{\mathscr{M}}(\mathscr{X}) - \{0^{NT}, 1^{NT}\}$ *is*

$$\binom{m+1}{2}^n - 3m^n + 3.$$

Proof:

The existence of a chain of length two in $\hat{\mathscr{P}}_{\mathscr{M}}(\mathscr{X})$ is first subject to the condition that there are enough levels to contain such chains. That is we first assumed that $n(m-1) - 1 \geq 2$, or equivalently $n(m-1) \geq 3$. To compute the NCs of length two in $\hat{\mathscr{P}}_{\mathscr{M}}(\mathscr{X})$ we use algorithm 1.7.1.
We have $c_1(t_{i_1}, t_{i_2}, \ldots, t_{i_n}) = 1$, for each $(t_{i_1}, t_{i_2}, \ldots, t_{i_n}) \in \hat{\mathscr{P}}_{\mathscr{M}}(\mathscr{X})$. For each $(t_{j_1}, t_{j_2}, \ldots, t_{j_n})$ satisfying $(t_{j_1}, t_{j_2}, \ldots, t_{j_n}) \preceq (t_{i_1}, t_{i_2}, \ldots, t_{i_n})$, we have $0 \leq j_k \leq i_k$ for each k in $\{1, 2, \cdots, n\}$. That is there are exactly $(i_1+1)(i_2+1)\ldots(i_n+1) - 2$ such $(t_{j_1}, t_{j_2}, \ldots, t_{j_n})$'s. Removing $(i_1, i_2, \ldots i_n)$ and $(0, 0, \ldots, 0)$ from the list we obtain that the NCs containing two elements in $\hat{\mathscr{P}}_{\mathscr{M}}(\mathscr{X})$ and ending with $(t_{i_1}, t_{i_2}, \ldots, t_{i_n})$ is

$$c_2(t_{i_1}, t_{i_2}, \ldots, t_{i_n}) = (i_1+1)(i_2+1)\ldots(i_n+1) - 2.$$

Taking the sum over all elements of $\mathscr{P}_{\mathscr{M}}(\mathscr{X})$ lying on levels 2 to $n(m-1) - 1$, we obtained the number c_2 of chains of length two in $\hat{\mathscr{P}}_{\mathscr{M}}(\mathscr{X})$, which is given by

$$c_2(\hat{\mathscr{P}}_{\mathscr{M}}(\mathscr{X})) = \sum_{2 \leq i_1+i_2+\cdots+i_n \leq \, n(m-1)-1} c_2(t_{i_1}, t_{i_2}, \ldots, t_{i_n})$$

$$= \sum_{2 \le i_1+i_2+\cdots+i_n \le n(m-1)-1} ((i_1+1)(i_2+1)\dots(i_n+1)-2)$$

$$= \sum_{h=2}^{n(m-1)-1} \left(\sum_{\substack{i_1+i_2+\cdots+i_n=h,\\ t_0 \le t_i \le t_{m-1}}} (i_1+1)(i_2+1)\dots(i_n+1)-2 \right)$$

$$= \sum_{h=2}^{n(m-1)-1} \left(\sum_{\substack{y_1+y_2+\cdots+y_n=n+h,\\ 1 \le y_i \le m}} (y_1 y_2 \dots y_n - 2) \right)$$

(taking $y_k = i_k + 1$)

$$= \sum_{k=n+2}^{nm-1} \left(\sum_{\substack{y_1+y_2+\cdots+y_n=k,\\ 1 \le y_i \le m}} (y_1 y_2 \dots y_n - 2) \right)$$

$$= -2(m^n - 2 - n) + \sum_{k=n+2}^{nm-1} \left(\sum_{\substack{y_1+y_2+\cdots+y_n=k,\\ 1 \le y_i \le m}} (y_1 y_2 \dots y_n) \right).$$

Letting,

$$s = \sum_{\substack{k=n+2,\\ 1 \le y_i \le m}}^{nm-1} \left(\sum_{\substack{y_1+y_2+\cdots+y_n=k,\\ 1 \le y_i \le m}} (y_1 y_2 \dots y_n) \right),$$

and using Lemma 1.7.1, we have obtained,

$$\left(\frac{m(m+1)}{2}\right)^n = \sum_{k=n}^{nm}\left(\sum_{\substack{y_1+y_2+\cdots+y_n=k,\\ 1\le y_i\le m}} (y_1y_2\ldots y_n)\right)$$

$$= y_1^n + ny_1^{n-1}y_2 + s + y_m^n$$
$$= 1 + 2n + s + m^n$$

from which we deduced that

$$s = \left(\tfrac{m(m+1)}{2}\right)^n - m^n - 2n - 1.$$

Therefore,

$$\begin{aligned} c_2(\hat{\mathscr{P}}_{\mathscr{M}}(\mathscr{X})) &= -2(m^n - 2 - n) + \left(\tfrac{m(m+1)}{2}\right)^n - m^n - 2n - 1 \\ &= -3m^n + \left(\tfrac{m(m+1)}{2}\right)^n + 3 \\ &= \binom{m+1}{2}^n - 3m^n + 3. \end{aligned}$$

This proves the result in the case $n(m-1) \geq 3$.

Now, if n and m are positive integers such that $n(m-1) < 3$, and taking in account, we assumed that $n \geq 1$ and $m \geq 2$, we get

either (a) $m = 2$ and $n = 1$,

or (b) $m = 2$ and $n = 2$,

or (c) $m = 3$ and $n = 1$.

On one hand, these values are substituted in the above formula and gives the value 0. On the other hand, these values result in the ordered sets.

It is easy to check that we have no chain with length two in all cases, (a), (b), and (c), which is consistent with the obtained formula. This proves the proposition.

Corollary 2.1.1

In $\mathscr{P}_{\mathscr{M}}(\mathscr{X})$, the NCs of length four having both 0^{NT} and 1^{NT} is same as $c_2(\hat{\mathscr{P}}_{\mathscr{M}}(\mathscr{X}))$.

Proposition 2.1.5

The NCs of length two in $\mathscr{P}_{\mathscr{M}}(\mathscr{X})$ is

$$\mathscr{C}_N(n,m,2) = \binom{m+1}{2}^n - m^n.$$

Proof:

Let $|\mathscr{X}| = n$ and $|\mathscr{M}| = m$, then $|\mathscr{P}_{\mathscr{M}}(\mathscr{X})| = m^n$. To prove the result we have the following cases:

Case 1 Chain of 0^{NT} with 1^{NT}: The NC of length two of 0^{NT} with 1^{NT} is one i.e., $0^{NT} \subset 1^{NT}$.

Case 2 Chain of length one with 0^{NT} as subset: We know every neutrosophic subsets are chain of length one and 0^{NT} is a subset of every neutrosophic subsets. Therefore, in this case, the NCs of length two is $|\mathscr{P}_{\mathscr{M}}(\mathscr{X})| - 1 = m^n - 1$ as $0^{NT} \subseteq 0^{NT}$ which is a chain of length one.

Case 3 Chain of length one with 1^{NT} as super-subset: Since, 1^{NT} is a super-subset of every neutrosophic subsets. Therefore, in this case, the NCs of length two is $|\mathscr{P}_{\mathscr{M}}(\mathscr{X})| - 1 = m^n - 1$ as $1^{NT} \subseteq 1^{NT}$ which is a chain of length one.

Case 4 Chain of length two from $\hat{\mathscr{P}}_{\mathscr{M}}(\mathscr{X})$: Following Proposition 2.1.4, the NCs of length two is $\binom{m+1}{2}^n - 3m^n + 3$.

Since the chain from case 1, i.e., $0^{NT} \subset 1^{NT}$ is present in case 2 and case 3, the total NCs of length two in the neutrosophic power set of $\mathscr{X}$ whose neutrosophic values lie in $\mathscr{M}$ is

$$1 + (m^n - 1 - 1) + (m^n - 1 - 1) + \binom{m+1}{2}^n - 3m^n + 3$$

i.e.,

$$\mathscr{C}_N(n,m,2) = \binom{m+1}{2}^n - m^n.$$

Example 2.1.4

From Example 2.1.2, we have $|\mathscr{X}| = n = 2$, $|\mathscr{M}| = m = 3$. *Then the NCs of length two is*

$$\binom{m+1}{2}^n - m^n = \binom{3+1}{2}^2 - 3^2 = 6^2 - 9 = 36 - 9 = 27.$$

These are

$$\begin{array}{llll}
0^{NT} \subset 1^{NT}, & 0^{NT} \subset A_1^{NT}, & 0^{NT} \subset A_2^{NT}, & 0^{NT} \subset A_3^{NT}, \\
0^{NT} \subset A_4^{NT}, & 0^{NT} \subset A_5^{NT}, & 0^{NT} \subset A_6^{NT}, & 0^{NT} \subset A_7^{NT}, \\
A_1^{NT} \subset 1^{NT}, & A_2^{NT} \subset 1^{NT}, & A_3^{NT} \subset 1^{NT}, & A_4^{NT} \subset 1^{NT}, \\
A_5^{NT} \subset 1^{NT}, & A_6^{NT} \subset 1^{NT}, & A_7^{NT} \subset 1^{NT}, & A_1^{NT} \subset A_2^{NT}, \\
A_1^{NT} \subset A_4^{NT}, & A_1^{NT} \subset A_5^{NT}, & A_1^{NT} \subset A_7^{NT}, & A_2^{NT} \subset A_5^{NT}, \\
A_3^{NT} \subset A_4^{NT}, & A_3^{NT} \subset A_5^{NT}, & A_3^{NT} \subset A_6^{NT}, & A_3^{NT} \subset A_7^{NT}, \\
A_4^{NT} \subset A_5^{NT}, & A_4^{NT} \subset A_7^{NT}, & A_6^{NT} \subset A_7^{NT}. &
\end{array}$$

Proposition 2.1.6

The NCs of length three in $\hat{\mathscr{P}}_{\mathscr{M}}(\mathscr{X})$ *is*

$$\frac{m^n(m+1)^n(m+2)^n}{6^n} - \frac{4m^n(m+1)^n}{2^n} + 6m^n - 4.$$

Proof:

Firstly, let us assume that $n(m-1)+1 \geq 5$. To compute the NCs of length three in $\hat{\mathscr{P}}_{\mathscr{M}}(\mathscr{X})$ we use Algorithm 1.7.1. For each $(t_{i_1}, t_{i_2}, \ldots, t_{i_n})$ in $\hat{\mathscr{P}}_{\mathscr{M}}(\mathscr{X})$, let $c_k(t_{i_1}, t_{i_2}, \ldots, t_{i_n})$ be the NCs with k elements from $\hat{\mathscr{P}}_{\mathscr{M}}(\mathscr{X})$ and with maximal element $(t_{i_1}, t_{i_2}, \ldots, t_{i_n})$. Then,

$c_1(t_{i_1}, t_{i_2}, \ldots, t_{i_n}) = 1$,

$c_2(t_{i_1}, t_{i_2}, \ldots, t_{i_n}) = (i_1+1)(i_2+1)\ldots(i_n+1) - 2$, [following the proof of proposition 2.1.4],

and

$$c_3(t_{i_1}, t_{i_2}, \ldots, t_{i_n}) = \sum_{0^{NT} \prec (t_{j_1}, t_{j_2}, \ldots, t_{j_n}) \prec (t_{i_1}, t_{i_2}, \ldots, t_{i_n})} c_2(t_{j_1}, t_{j_2}, \ldots, t_{j_n})$$

$$= -c_2(t_{i_1}, t_{i_2}, \dots, t_{i_n}) + \sum_{0^{NT} \prec (t_{j_1}, t_{j_2}, \dots, t_{j_n}) \preceq (t_{i_1}, t_{i_2}, \dots, t_{i_n})} c_2(t_{j_1}, t_{j_2}, \dots, t_{j_n}).$$

Hence,

$c_3(t_{i_1}, t_{i_2}, \dots, t_{i_n}) + c_2(t_{i_1}, t_{i_2}, \dots, t_{i_n})$

$$= \sum_{0^{NT} \prec (t_{j_1}, t_{j_2}, \dots, t_{j_n}) \preceq (t_{i_1}, t_{i_2}, \dots, t_{i_n})} c_2(t_{j_1}, t_{j_2}, \dots, t_{j_n})$$

$$= \sum_{0^{NT} \prec (j_1, j_2, \dots, j_n) \preceq (i_1, i_2, \dots, i_n)} ((j_1+1)(j_2+1)\dots(j_n+1)-2)$$

$$= -((0+1)(0+1)\dots(0+1)-2) + \sum_{\substack{0^{NT} \preceq (j_1, j_2, \dots, j_n) \\ \preceq (i_1, i_2, \dots, i_n)}} ((j_1+1)(j_2+1)\dots(j_n+1)-2)$$

$$= 1 + \sum_{\substack{0 \le j_k \le i_k, \\ 1 \le k \le n}} ((j_1+1)(j_2+1)\dots(j_n+1)-2)$$

$$= 1 + \sum_{\substack{0 \le y_k - 1 \le i_k, \\ 1 \le k \le n}} (y_1 y_2 \dots y_n - 2)$$

$$= 1 + \left(\sum_{\substack{0 \le y_k \le i_k + 1, \\ 1 \le k \le n}} y_1 y_2 \dots y_n \right) - 2 \prod_{k=1}^{n} (i_k + 1)$$

$$= 1 + \prod_{k=1}^{n} (1 + 2 + \dots + (i_k + 1)) - 2 \prod_{k=1}^{n} (i_k + 1)$$

$$= 1 + \prod_{k=1}^{n} \frac{(i_k+2)(i_k+1)}{2} - 2 \prod_{k=1}^{n} (i_k + 1).$$

Taking the sum over all elements of $\hat{\mathscr{P}}_{\mathscr{M}}(\mathscr{X})$, we have,

$$c_3 + c_2 = \sum_{(t_{i_1}, t_{i_2}, \dots, t_{i_n}) \in \hat{\mathscr{P}}_{\mathscr{M}}(\mathscr{X})} c_3(t_{i_1}, t_{i_2}, \dots, t_{i_n}) + c_2(t_{i_1}, t_{i_2}, \dots, t_{i_n}).$$

$$= \sum_{0^{NT} \prec (t_{i_1}, t_{i_2}, \ldots, t_{i_n}) \prec 1^{NT}} \left(1 + \prod_{k=1}^{n} \frac{(i_k+2)(i_k+1)}{2} - 2 \prod_{k=1}^{n} (i_k+1) \right)$$

Therefore,

$c_2 + c_3 + 1 + \frac{(m+1)^n m^n}{2^n} - 2m^n$

$$= \sum_{0^{NT} \preceq (t_{i_1}, t_{i_2}, \ldots, t_{i_n}) \preceq 1^{NT}} \left(1 + \prod_{k=1}^{n} \frac{(i_k+2)(i_k+1)}{2} - 2 \prod_{k=1}^{n} (i_k+1) \right)$$

$$= m^n + \frac{1}{2^n} \left(\sum_{i_k=0}^{m-1} (i_k+2)(i_k+1) \right)^n - 2 \left(\sum_{i_k=0}^{m-1} (i_k+1) \right)^n$$

$$= m^n + \frac{1}{2^n} \left(\sum_{i_k=0}^{m-1} (i_k+2)(i_k+1) \right)^n - \frac{2m^n(m+1)^n}{2^n}.$$

Since,

$$c_2 = \left(\frac{m(m+1)}{2} \right)^n - 3m^n + 3$$

and

$$\begin{aligned} \left(\sum_{i_k=0}^{m-1} (i_k+2)(i_k+1) \right)^n &= \left(\sum_{j=1}^{m} (j+1)j \right)^n \\ &= \left(\sum_{j=1}^{m} (j^2+j) \right)^n \\ &= \left(\frac{m(m+1)(2m+1)}{6} + \frac{3m(m+1)}{6} \right)^n \\ &= \left(\frac{m(m+1)(2m+4)}{6} \right)^n \\ &= \left(\frac{m(m+1)(m+2)}{3} \right)^n. \end{aligned}$$

We have,

$$c_3 + \left(\frac{m(m+1)}{2}\right)^n - 3m^n + 3 + 1 + \frac{m^n(m+1)^n}{2^n} - 2m^n$$

$$= m^n + \frac{1}{2^n}\left(\frac{m(m+1)(m+2)}{3}\right)^n - \frac{2m^n(m+1)^n}{2^n}.$$

That is,

$$c_3 + 2\left(\frac{m(m+1)}{2}\right)^n - 5m^n + 4$$

$$= m^n + +\frac{1}{2^n}\left(\frac{m(m+1)(m+2)}{3}\right)^n - \frac{2m^n(m+1)^n}{2^n}.$$

Hence,

$$c_3 = -4 + 6m^n - \frac{4m^n(m+1)^n}{2^n} + \frac{m^n(m+1)^n(m+2)^n}{6^n}.$$

This proves the proposition.

Proposition 2.1.7

The NCs of length three in $\mathscr{P}_{\mathscr{M}}(\mathscr{X})$ is

$$\mathscr{C}_N(n, m, 3) = \frac{m^n(m+1)^n(m+2)^n}{6^n} - \frac{2m^n(m+1)^n}{2^n} + m^n$$

or

$$\mathscr{C}_N(n, m, 3) = \binom{m+2}{3}^n - 2\binom{m+1}{2}^n + m^n.$$

Proof:

Let $|\mathscr{X}| = n$ and $|\mathscr{M}| = m$, then $|\mathscr{P}_{\mathscr{M}}(\mathscr{X})| = m^n$. To prove the result we have the following cases:

Case 1 Chain of length one with both 0^{NT} and 1^{NT} as subset and super-subset: Every neutrosophic proper subsets of length one together with 0^{NT} and 1^{NT} as subset and super-subset forms a chain of length three. Then, the NCs of length three in the present case is $m^n - 2$.

Case 2 Chain of length two with 0^{NT} as subset: Chain of length two with 0^{NT} as subset forms a chain of length three. Therefore, the NCs of length three in this case is $\binom{m+1}{2}^n - 3m^n + 3$.

Case 3 Chain of length two with 1^{NT} as superset: Similar to case 2, the chain of length two with 1^{NT} as super-subset forms a chain of length three. Therefore, the NCs of length three in this case is $\binom{m+1}{2}^n - 3m^n + 3$.

Case 4 Chain of length three from $\hat{\mathscr{P}}_{\mathscr{M}}(\mathscr{X})$: Following Proposition 2.1.6, the NCs of length three in $\hat{\mathscr{P}}_{\mathscr{M}}(\mathscr{X})$ is

$$\frac{m^n(m+1)^n(m+2)^n}{6^n} - \frac{4m^n(m+1)^n}{2^n} + 6m^n - 4.$$

Hence, the total NCs of length three is

$$\mathscr{C}_N(n,m,3) = m^n - 2 + \binom{m+1}{2}^n - 3m^n + 3 + \binom{m+1}{2}^n - 3m^n + 3$$
$$+ \binom{m+2}{3}^n - 4\binom{m+1}{2}^n + 6m^n - 4$$

i.e., $\mathscr{C}_N(n,m,3) = m^n - 2\binom{m+1}{2}^n + \binom{m+2}{3}^n$

or

$$\mathscr{C}_N(n,m,3) = \frac{m^n(m+1)^n(m+2)^n}{6^n} - \frac{2m^n(m+1)^n}{2^n} + m^n.$$

Example 2.1.5

From Example 2.1.2, we have $|\mathscr{X}| = n = 2$, $|\mathscr{M}| = m = 3$. *Then the NCs of length three is*

$$\mathscr{C}_N(2,3,3) = \frac{3^2(3+1)^2(3+2)^2}{6^2} - \frac{2.3^2(3+1)^2}{2^2} + 3^2 = 37.$$

These are

$0^{NT} \subset A_1^{NT} \subset 1^{NT}, 0^{NT} \subset A_2^{NT} \subset 1^{NT}, 0^{NT} \subset A_3^{NT} \subset 1^{NT},$
$0^{NT} \subset A_4^{NT} \subset 1^{NT}, 0^{NT} \subset A_5^{NT} \subset 1^{NT}, 0^{NT} \subset A_6^{NT} \subset 1^{NT},$
$0^{NT} \subset A_7^{NT} \subset 1^{NT}, 0^{NT} \subset A_1^{NT} \subset A_2^{NT}, 0^{NT} \subset A_1^{NT} \subset A_4^{NT},$
$0^{NT} \subset A_1^{NT} \subset A_5^{NT}, 0^{NT} \subset A_1^{NT} \subset A_7^{NT}, 0^{NT} \subset A_2^{NT} \subset A_5^{NT},$
$0^{NT} \subset A_3^{NT} \subset A_4^{NT}, 0^{NT} \subset A_3^{NT} \subset A_5^{NT}, 0^{NT} \subset A_3^{NT} \subset A_6^{NT},$
$0^{NT} \subset A_3^{NT} \subset A_7^{NT}, 0^{NT} \subset A_4^{NT} \subset A_5^{NT}, 0^{NT} \subset A_4^{NT} \subset A_7^{NT},$
$0^{NT} \subset A_6^{NT} \subset A_7^{NT}, A_1^{NT} \subset A_2^{NT} \subset 1^{NT}, A_1^{NT} \subset A_4^{NT} \subset 1^{NT},$
$A_1^{NT} \subset A_5^{NT} \subset 1^{NT}, A_1^{NT} \subset A_7^{NT} \subset 1^{NT}, A_2^{NT} \subset A_5^{NT} \subset 1^{NT},$
$A_3^{NT} \subset A_4^{NT} \subset 1^{NT}, A_3^{NT} \subset A_5^{NT} \subset 1^{NT}, A_3^{NT} \subset A_6^{NT} \subset 1^{NT},$
$A_3^{NT} \subset A_7^{NT} \subset 1^{NT}, A_4^{NT} \subset A_5^{NT} \subset 1^{NT}, A_4^{NT} \subset A_7^{NT} \subset 1^{NT},$

$A_6^{NT} \subset A_7^{NT} \subset 1^{NT}, A_1^{NT} \subset A_2^{NT} \subset A_5^{NT}, A_1^{NT} \subset A_4^{NT} \subset A_5^{NT},$
$A_1^{NT} \subset A_4^{NT} \subset A_7^{NT}, A_3^{NT} \subset A_4^{NT} \subset A_5^{NT}, A_3^{NT} \subset A_4^{NT} \subset A_7^{NT},$
$A_3^{NT} \subset A_6^{NT} \subset A_7^{NT}.$

Lemma 2.1.1

In $\mathscr{P}_{\mathscr{M}}(\mathscr{X})$, the number of antichains (NACs) of size 2 (having two elements) with 1^{NT} as union and 0^{NT} as intersection is $2^{n-1}-1$.

Proof:

Let $(t_{i_1}, t_{i_2}, \ldots, t_{i_n})$ and $(t_{j_1}, t_{j_2}, \ldots, t_{j_n})$ form an antichain. These two NSubs are different from 0^{NT} and 1^{NT}, and satisfy
$t_{i_k} \cap t_{j_k} = (0,1,1)$ and $t_{i_k} \cup t_{j_k} = (1,0,0)$ for each $1 \leq k \leq n$.

That is $t_{i_k} = (0,1,1)$ if and only if $t_{j_k} = (1,0,0)$, and $t_{i_k} = (1,0,0)$ if and only if $t_{j_k} = (0,1,1)$. Thus, $(t_{j_1}, t_{j_2}, \ldots, t_{j_n})$ is automatically determined by $(t_{i_1}, t_{i_2}, \ldots, t_{i_n})$. There are exactly $\binom{n}{1}$ such $(t_{i_1}, t_{i_2}, \ldots, t_{i_n})$'s containing exactly one $(0,1,1)$, $\binom{n}{2}$ such $(t_{i_1}, t_{i_2}, \ldots, t_{i_n})$'s containing exactly two $(0,1,1)$, …, $\binom{n}{n-1}$ such $(t_{i_1}, t_{i_2}, \ldots, t_{i_n})$'s containing exactly $n-1$ $(0,1,1)$'s. That is, in total, we have, $\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} = 2^n - 2 = 2(2^{n-1}-1)$ different such $(t_{i_1}, t_{i_2}, \ldots, t_{i_n})$'s. Since, each pair of $(t_{i_1}, t_{i_2}, \ldots, t_{i_n})$ and corresponding $(t_{j_1}, t_{j_2}, \cdots, t_{j_n})$ is repeated twice by this process, we have exactly $2^{n-1}-1$ different antichains.

If m and n are positive integers such that $n(m-1)+1 \leq 2$, or equivalently $n(m-1) \leq 1$, and since we assumed that $n \geq 1$ and $m \geq 2$, we obtain $m = 2$ and $n = 1$.

It is easy to check that there is no such antichain of size two in this case, which is consistent with the obtained formula. This completes the proof of the lemma.

Proposition 2.1.8

For a finite set $\mathscr{X}$ whose neutrosophic values lie in $\mathscr{M}$, $|\mathscr{M}| \geq 2$,

$$|P(\mathscr{X})| \leq |\mathscr{P}_{\mathscr{M}}(\mathscr{X})|.$$

Proof:
Let $|\mathscr{X}| = n$ and $|\mathscr{M}| = m, m \geq 2$. Then $|P(\mathscr{X})| = 2^n$ and $|\mathscr{P}_{\mathscr{M}}(\mathscr{X})| = m^n, m \geq 2$. This clearly shows that $|P(\mathscr{X})| \leq |\mathscr{P}_{\mathscr{M}}(\mathscr{X})|$.

2.2 Cardinalities of the Neutrosophic Crisp Set

This section discusses the work related to the counting of neutrosophic crisp subsets.

Definition 2.2.1
The set of all neutrosophic crisp subsets of a non-empty finite set $\mathscr{X}$ is called the neutrosophic crisp power set of $\mathscr{X}$. The notation for the neutrosophic crisp power set of $\mathscr{X}$ is $\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})$ and its cardinality is denoted by $|\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})|$.

Proposition 2.2.1
A set $\mathscr{X}$ with $|\mathscr{X}| = n$ has

$$(3.2^n - 4) + 3!\left\{\sum_{i=2}^{n} \mathcal{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n} \mathcal{S}(j,3)\binom{n}{j}\right\}$$

neutrosophic crisp subsets.

Proof:
Let $|\mathscr{X}| = n$, then $|P(\mathscr{X})| = 2^n = \sum_{i=0}^{n}\binom{n}{i}$. If $\mathscr{A} = \langle \mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3 \rangle$, where $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ are subsets of $\mathscr{X}$ such that $\mathscr{A}_1 \cap \mathscr{A}_2 = \mathscr{A}_1 \cap \mathscr{A}_3 = \mathscr{A}_2 \cap \mathscr{A}_3 = \emptyset$. Then $\mathscr{A}$ is a neutrosophic crisp subset of $\mathscr{X}$. Trivially, $\phi_{\mathscr{N}}$ and $\mathscr{X}_{\mathscr{N}}$ are always in the power set of the neutrosophic crisp subset of $\mathscr{X}$ as they are the smallest and the largest neutrosophic crisp subset of $\mathscr{X}$. Since, $\mathscr{A}$ has three components $\mathscr{A}_1, \mathscr{A}_2$, and $\mathscr{A}_3$, which are chosen in the following three ways.

Firstly, choose two components of $\mathscr{A}$ as $\emptyset$, and the other by any neutrosophic crisp proper subset of $\mathscr{X}$ say $\mathscr{A}_1$. Then $\mathscr{A}_1$ is chosen in $\binom{n}{i}, 1 \leq$

$i \leq n-1$ ways. Therefore, $\mathscr{A}_1$ is chosen in $\sum_{i=1}^{n-1} \binom{n}{i}$ different ways. We can place $\mathscr{A}_1$ in any of the three places in $3\sum_{i=1}^{n-1} \binom{n}{i} = 3(2^n - 2)$ different ways.

Secondly, choose one component of $\mathscr{A}$ as $\emptyset$ and then other two by two neutrosophic crisp proper subsets of $\mathscr{X}$ say $\mathscr{A}_1$ and $\mathscr{A}_2$ such that $\mathscr{A}_1 \cap \mathscr{A}_2 = \emptyset$. We can place $\mathscr{A}_k, k = 1, 2$ in any of the two places in 6 different ways. For a particular set $\mathscr{A}_k$, we have $\binom{n}{i}\mathcal{S}(i,2), 2 \leq i \leq n$ different ways. Therefore, the total number of ways to choose $\mathscr{A}$ is $6\sum_{i=2}^{n} \binom{n}{i}\mathcal{S}(i,2)$.

Thirdly, choose each three component of $\mathscr{A}$ as neutrosophic crisp proper subsets of $\mathscr{X}$ say $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{A}_3$ such that $\mathscr{A}_1 \cap \mathscr{A}_2 = \mathscr{A}_1 \cap \mathscr{A}_3 = \mathscr{A}_2 \cap \mathscr{A}_3 = \emptyset$. We can place $\mathscr{A}_k, k = 1, 2, 3$ in any of the three places in 6 different ways. For a particular set $\mathscr{A}_k$, we have, $\binom{n}{j}\mathcal{S}(j,3), 3 \leq j \leq n$ different ways.

Therefore, the total number of ways to choose $\mathscr{A}$ is $6\sum_{j=3}^{n} \binom{n}{j}\mathcal{S}(j,3)$.

Hence, the total number of neutrosophic crisp subset of $\mathscr{X}$ is

$$2 + 3(2^n - 2) + 3!\left\{\sum_{i=2}^{n} \mathcal{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n} \mathcal{S}(j,3)\binom{n}{j}\right\}$$

i.e.,

$$(3.2^n - 4) + 3!\left\{\sum_{i=2}^{n} \mathcal{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n} \mathcal{S}(j,3)\binom{n}{j}\right\}.$$

Example 2.2.1

Let $\mathscr{X} = \{u\}$ and so $|\mathscr{X}| = n = 1$, then the neutrosophic crisp subsets of $\mathscr{X}$ are $\phi_{\mathscr{N}} = \langle \emptyset, \emptyset, \mathscr{X} \rangle$, $\mathscr{X}_{\mathscr{N}} = \langle \mathscr{X}, \emptyset, \emptyset \rangle$.

Therefore, the number of neutrosophic crisp subsets of $\mathscr{X} = 2 = (3.2^1 - 4) + 3!\left\{\sum_{i=2}^{1} \mathcal{S}(i,2)\binom{1}{i} + \sum_{j=3}^{1} \mathcal{S}(j,3)\binom{1}{j}\right\}$.

Corollary 2.2.1

If $|\mathscr{X}| = n$, then the cardinality of the power set of NCrS of $\mathscr{X}$ is

$$|\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})| = (3.2^n - 4) + 3!\left\{\sum_{i=2}^{n} \mathcal{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n} \mathcal{S}(j,3)\binom{n}{j}\right\}.$$

Proof:

By Definition 2.2.1 and Proposition 2.2.1, we can obtained the cardinality of the power set of NCrS of $\mathscr{X}$, which is

$$|\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})| = (3.2^n - 4) + 3!\left\{\sum_{i=2}^{n} \mathcal{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n} \mathcal{S}(j,3)\binom{n}{j}\right\}.$$

Example 2.2.2

Let $\mathscr{X} = \{u, v\}$ and so $|\mathscr{X}| = n = 2$, then the neutrosophic crisp subsets of $\mathscr{X}$ are

$\phi_{\mathscr{N}} = \langle \emptyset, \emptyset, \mathscr{X} \rangle$, $\mathscr{X}_{\mathscr{N}} = \langle \mathscr{X}, \emptyset, \emptyset \rangle$, $\mathscr{A}_1 = \langle \emptyset, \emptyset, \{u\} \rangle$,
$\mathscr{A}_2 = \langle \emptyset, \{u\}, \emptyset\} \rangle$, $\mathscr{A}_3 = \langle \{u\}, \emptyset, \emptyset \rangle$, $\mathscr{A}_4 = \langle \emptyset, \emptyset, \{v\} \rangle$,
$\mathscr{A}_5 = \langle \emptyset, \{v\}, \emptyset\} \rangle$, $\mathscr{A}_6 = \langle \{v\}, \emptyset, \emptyset \rangle$, $\mathscr{A}_7 = \langle \emptyset, \{u\}, \{v\} \rangle$,
$\mathscr{A}_8 = \langle \{u\}, \emptyset, \{v\} \rangle$, $\mathscr{A}_9 = \langle \{u\}, \{v\}, \emptyset \rangle$, $\mathscr{A}_{10} = \langle \emptyset, \{v\}, \{u\} \rangle$,
$\mathscr{A}_{11} = \langle \{v\}, \emptyset, \{u\} \rangle$, $\mathscr{A}_{12} = \langle \{v\}, \{u\}, \emptyset \rangle$.

Therefore,

$$\begin{aligned}|\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})| &= 14\\ &= (3.2^2 - 4) + 3!\left\{\textstyle\sum_{i=2}^{2} \mathcal{S}(i,2)\binom{2}{i} + \sum_{j=3}^{2} \mathcal{S}(j,3)\binom{2}{j}\right\}.\end{aligned}$$

Proposition 2.2.2

For a non-empty finite set $\mathscr{X}$,

$$|P(\mathscr{X})| \leq |\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})|.$$

Proof:

Let $|\mathscr{X}| = n$. Then, $|P(\mathscr{X})| = 2^n$ and $|\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})| = (3.2^n - 4) + 3!\{\sum_{i=2}^{n} \mathcal{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n} \mathcal{S}(j,3)\binom{n}{j}\}$.

Now, let $T = 3!\{\sum_{i=2}^{n} \mathscr{S}(i,2)\binom{n}{i} + \sum_{j=3}^{n} \mathscr{S}(j,3)\binom{n}{j}\}$ and clearly, $T \geq 0$ for $n \geq 1$.

Then,
$$\begin{aligned} |\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})| &= (3.2^n - 4) + T \\ &= 2^n + 4(2^{n-1} - 1) + T \\ &\geq 2^n \qquad \text{as } 4(2^{n-1} - 1) \geq 0 \text{ and } T \geq 0 \text{ for } n \geq 1 \\ &\geq |P(\mathscr{X})|. \end{aligned}$$

Hence, $|P(\mathscr{X})| \leq |\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})|$.

Example 2.2.3

(i) From Example 2.2.1, we have $|\mathscr{X}| = n = 1$ and using Corollary 2.2.1, $|\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})| = 2$. Also for $|\mathscr{X}| = n = 1$, we get $|P(\mathscr{X})| = 2^n = 2^1 = 2$. This clearly shows that $|P(\mathscr{X})| = |\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})|$.

(ii) From Example 2.2.2, we have $|\mathscr{X}| = n = 2$ and $|\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})| = 14$. Also for $|\mathscr{X}| = n = 2$, we get $|P(\mathscr{X})| = 2^n = 2^2 = 4$. This clearly shows that $|P(\mathscr{X})| < |\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})|$.