CHAPTER 3

Number of Neutrosophic Topological Spaces on a Finite Set

In this chapter, for a finite set $\mathscr{X}$ with neutrosophic values in $\mathscr{M}$, the number of neutrosophic topological spaces with two, three, and four neutrosophic open sets have been computed. Also, the number of neutrosophic bitopological spaces and the number of neutrosophic tritopological spaces with k ($k = 2, 3, 4$) neutrosophic open sets on finite sets have been calculated. Moreover, it has been found that the formulae for NNTS, NNBTS, and NNTRS are related to one another. The formulae for the number of chains presented in Chapter 2 have been helpful in this chapter.

3.1 Number of Neutrosophic Topological Spaces

Proposition 3.1.1
The NNTs on $\mathscr{X}$, whose neutrosophic values lies in $\mathscr{M}$, is finite if and only if both $\mathscr{X}$ and $\mathscr{M}$ are finite.

The results discussed in this chapter have been published in the journal, Basumatary, B. and Basumatary, J. (2023). Number of Neutrosophic Topological Spaces on Finite Set with $\mathscr{k} \leq 4$ Open Sets. *Neutrosophic Sets and Systems*, 53:508-518.

Proof:

If $\mathscr{X}$ and $\mathscr{M}$ are both finite of cardinalities n and m respectively, then a NT on $\mathscr{X}$ is a collection of elements of $\mathscr{N}_{\mathscr{X}}$. Therefore, the number of NTs cannot exceed 2^{m^n}.

On the other hand, if $\mathscr{X}$ is infinite, let $v_1, v_2, \ldots, v_n, \ldots$ be an infinite sequence of elements of $\mathscr{X}$, and let $\mathscr{U}_v^1$ be the NSub defined by $\mathscr{U}_v^1(v) = (1, 0, 0)$ and $\mathscr{U}_v^1(x) = (0, 1, 1)$ for every $x \neq v$. Let $\tau_i = \{0^{NT}, \mathscr{U}_{v_i}^1, 1^{NT}\}$. Then, τ_i, $i \geq 1$, is an infinite collection of NTs of $\mathscr{N}_{\mathscr{X}}$. Finally if $\mathscr{M}$ is infinite, let $t_1, t_2, \cdots, t_m, \cdots$ be an infinite sequence of elements of $\mathscr{M}$, and let $\mathscr{U}^t$ be the NSub defined by $\mathscr{U}^t(v) = t$ for every $v \in \mathscr{X}$. Let $\sigma_i = \{0^{NT}, \mathscr{U}^{t_i}, 1^{NT}\}$. Then $\sigma_i, i \geq 1$ is an infinite collection of NTs of $\mathscr{N}_{\mathscr{X}}$. This proves the proposition.

3.2 Number of NTSs having 2-Neutrosophic Open Sets

Proposition 3.2.1

In $\mathscr{N}_{\mathscr{X}}$, the NNTSs having 2-NOSs is one i.e., $\mathscr{T}_{\mathscr{X}}^{NT}(n, m, 2) = 1$.

Proof:

The NT having 2-NOSs is the indiscrete NT which is $\mathscr{T}_1^{NT} = \{0^{NT}, 1^{NT}\}$. Hence, $\mathscr{T}_{\mathscr{X}}^{NT}(n, m, 2) = 1$.

3.3 Number of NTSs having 3-Neutrosophic Open Sets

Proposition 3.3.1

In $\mathscr{N}_{\mathscr{X}}$, the NNTs having 3-NOSs is $m^n - 2$ i.e., $\mathscr{T}_{\mathscr{X}}^{NT}(n, m, 3) = m^n - 2$.

Proof:

The NTs with exactly 3-NOSs necessarily consists of a chain containing $0^{NT}, 1^{NT}$ and any NSub of $\mathscr{X}$. In this case, the NOSs in NTs are in the chain of the form $0^{NT} \subset A_1^{NT} \subset 1^{NT}$, where A_1^{NT} is any NSub of $\mathscr{X}$. Hence, $\mathscr{T}_{\mathscr{X}}^{NT}(n, m, 3) = m^n - 2$.

Example 3.3.1

Let $\mathscr{X} = \{u, v\}$ *and* $\mathscr{M} = \{(0,1,1), (0.6,0.1,0.2), (1,0,0)\}$.

It is seen that, $|\mathscr{X}| = n = 2$, $|\mathscr{M}| = m = 3$. *Then, the number of elements in* $\mathscr{N}_{\mathscr{X}}$ *i.e.,* $|\mathscr{N}_{\mathscr{X}}| = 3^2 = 9$. *These are*

$$0^{NT}, 1^{NT}, A_1^{NT} = \langle \tfrac{u}{(0,1,1)}, \tfrac{v}{(0.6,0.1,0.2)} \rangle,$$
$$A_2^{NT} = \langle \tfrac{u}{(0,1,1)}, \tfrac{v}{(1,0,0)} \rangle, \qquad A_3^{NT} = \langle \tfrac{u}{(0.6,0.1,0.2)}, \tfrac{v}{(0,1,1)} \rangle,$$
$$A_4^{NT} = \langle \tfrac{u}{(0.6,0.1,0.2)}, \tfrac{v}{(0.6,0.1,0.2)} \rangle, \qquad A_5^{NT} = \langle \tfrac{u}{(0.6,0.1,0.2)}, \tfrac{v}{(1,0,0)} \rangle,$$
$$A_6^{NT} = \langle \tfrac{u}{(1,0,0)}, \tfrac{v}{(0,1,1)} \rangle, \qquad A_7^{NT} = \langle \tfrac{u}{(1,0,0)}, \tfrac{v}{(0.6,0.1,0.2)} \rangle.$$

Therefore, $\mathscr{T}_{\mathscr{X}}^{NT}(2,3,3) = 3^2 - 2 = 7$.

The NTs having 3*-NOSs are*

$$\tau_1^{NT} = \{0^{NT}, 1^{NT}, A_1^{NT}\}, \qquad \tau_2^{NT} = \{0^{NT}, 1^{NT}, A_2^{NT}\},$$
$$\tau_3^{NT} = \{0^{NT}, 1^{NT}, A_3^{NT}\}, \qquad \tau_4^{NT} = \{0^{NT}, 1^{NT}, A_4^{NT}\},$$
$$\tau_5^{NT} = \{0^{NT}, 1^{NT}, A_5^{NT}\}, \qquad \tau_6^{NT} = \{0^{NT}, 1^{NT}, A_6^{NT}\},$$
$$\tau_7^{NT} = \{0^{NT}, 1^{NT}, A_7^{NT}\}.$$

3.4 Number of NTSs having 4-Neutrosophic Open Sets

An arbitrary NT with 4-NOSs is a NT consisting of 1^{NT}, 0^{NT} and other two NSubs. These NSubs are either chain of 2-elements or antichain of size 2 having 1^{NT} and 0^{NT} as union and intersection respectively.

Proposition 3.4.1

In $\mathscr{N}_{\mathscr{X}}$*, the NNTs with* 4*-NOSs is*

$$\mathscr{T}_{\mathscr{X}}^{NT}(n,m,4) = \left(\frac{m(m+1)}{2}\right)^n - 3m^n + 2^{n-1} + 2.$$

Proof:

There are $\binom{m+1}{2}^n - 3m^n + 3$ NCs of length four having both 0^{NT} and 1^{NT} (Following Proposition 2.1.4 and Corollary 2.1.1). These chains form NTs with 4-NOSs. So, the NNT with 4-NOSs which are in chain is $\binom{m+1}{2}^n - 3m^n + 3$.

Also, there are $2^{n-1}-1$ antichains of size 2 with 1^{NT} as union and 0^{NT} as intersection (Following Lemma 2.1.1). These antichains together with 0^{NT} and 1^{NT} form NT with 4-NOS. So, the number of antichain NTs with 4-NOSs is $2^{n-1}-1$.

Hence, in total, the NNTs with 4-NOSs is

$$\mathscr{T}_{\mathscr{X}}^{NT}(n,m,4)=\left(\frac{m(m+1)}{2}\right)^{n}-3m^{n}+2^{n-1}+2.$$

Example 3.4.1

Let $\mathscr{X}=\{u,v\}$ *and* $\mathscr{M}=\{(0,1,1),(0.1,0.3,0.8),(1,0,0)\}$. *Therefore,* $|\mathscr{N}_{\mathscr{X}}|=3^2=9$. *These NSubs are*

$0^{NT}=\langle\frac{u}{(0,1,1)},\frac{v}{(0,1,1)}\rangle$, $1^{NT}=\langle\frac{u}{(1,0,0)},\frac{v}{(1,0,0)}\rangle$,
$\mathscr{A}_1^{NT}=\langle\frac{u}{(0,1,1)},\frac{v}{(1,0.3,0.8)}\rangle$, $\mathscr{A}_2^{NT}=\langle\frac{u}{(0,1,1)},\frac{v}{(1,0,0)}\rangle$,
$\mathscr{A}_3^{NT}=\langle\frac{u}{(0.1,0.3,0.8)},\frac{v}{(0,1,1)}\rangle$, $\mathscr{A}_4^{NT}=\langle\frac{u}{(0.1,0.3,0.8)},\frac{v}{(0.1,0.3,0.8)}\rangle$,
$\mathscr{A}_5^{NT}=\langle\frac{u}{(0.1,0.3,0.8)},\frac{v}{(1,0,0)}\rangle$, $\mathscr{A}_6^{NT}=\langle\frac{u}{(1,0,0)},\frac{v}{(0,1,1)}\rangle$,
$\mathscr{A}_7^{NT}=\langle\frac{u}{(1,0,0)},\frac{v}{(0.1,0.3,0.8)}\rangle$.

In this case, $n=2,m=3$, *therefore,*

$\mathscr{T}_{\mathscr{X}}^{NT}(2,3,4)=\left(\frac{3(3+1)}{2}\right)^{2}-3.3^2+2^{2-1}+2=6^2-23=13.$

These NTs with 4-*NOSs are*

$\tau_1^{NT}=\{0^{NT},1^{NT},\mathscr{A}_1^{NT},\mathscr{A}_2^{NT}\}$, $\tau_2^{NT}=\{0^{NT},1^{NT},\mathscr{A}_1^{NT},\mathscr{A}_4^{NT}\}$,
$\tau_3^{NT}=\{0^{NT},1^{NT},\mathscr{A}_1^{NT},\mathscr{A}_5^{NT}\}$, $\tau_4^{NT}=\{0^{NT},1^{NT},\mathscr{A}_1^{NT},\mathscr{A}_7^{NT}\}$,
$\tau_5^{NT}=\{0^{NT},1^{NT},\mathscr{A}_2^{NT},\mathscr{A}_5^{NT}\}$, $\tau_6^{NT}=\{0^{NT},1^{NT},\mathscr{A}_2^{NT},\mathscr{A}_6^{NT}\}$,
$\tau_7^{NT}=\{0^{NT},1^{NT},\mathscr{A}_3^{NT},\mathscr{A}_4^{NT}\}$, $\tau_8^{NT}=\{0^{NT},1^{NT},\mathscr{A}_3^{NT},\mathscr{A}_5^{NT}\}$,
$\tau_9^{NT}=\{0^{NT},1^{NT},\mathscr{A}_3^{NT},\mathscr{A}_6^{NT}\}$, $\tau_{10}^{NT}=\{0^{NT},1^{NT},\mathscr{A}_3^{NT},\mathscr{A}_7^{NT}\}$,
$\tau_{11}^{NT}=\{0^{NT},1^{NT},\mathscr{A}_4^{NT},\mathscr{A}_5^{NT}\}$, $\tau_{12}^{NT}=\{0^{NT},1^{NT},\mathscr{A}_4^{NT},\mathscr{A}_7^{NT}\}$,
$\tau_{13}^{NT}=\{0^{NT},1^{NT},\mathscr{A}_6^{NT},\mathscr{A}_7^{NT}\}$.

Here, the only antichain NTs in $\mathscr{N}_{\mathscr{X}}$ *is* τ_6^{NT} *with* 0^{NT} *and* 1^{NT} *as intersection and union respectively.*

3.5 Number of Neutrosophic Bitopological Spaces

In this section, the NBTS having 3-NOSs in NTs defines the NBTS of the form $(\mathscr{X}, \tau_i^{NT}, \tau_j^{NT})$, where τ_i^{NT}, τ_j^{NT} are identical or non-identical NTs having 3-NOSs, and the NBTS having 3-NOSs in NTs without repetition defines the NBTS of the form $(\mathscr{X}, \tau_i^{NT}, \tau_j^{NT})$, where τ_i^{NT}, τ_j^{NT} are non-identical NTs having 3-NOSs. Similar meaning is extended to the NBTS having 4-NOSs in NTs.

Proposition 3.5.1

In $\mathscr{N}_{\mathscr{X}}$, the NNBTS with 2-NOSs in NTs is

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n, m, 2) = 1.$$

Proof:

Following Proposition 3.2.1, $\mathscr{T}_{\mathscr{X}}^{NT}(n, m, 4)(n, m, 2) = 1$ which is the indiscrete topology $\tau_1^{NT} = \{0^{NT}, 1^{NT}\}$. Hence, NBTS with 2-NOSs is only one i.e., $(\mathscr{X}, \tau_1^{NT}, \tau_1^{NT})$.

Proposition 3.5.2

In $\mathscr{N}_{\mathscr{X}}$, the NNBTSs having 3-NOSs in NTs is

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n, m, 3) = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(n, m, 3) + 1}{2} = \frac{m^{2n} - 3m^n + 2}{2}.$$

Proof:

Let $\mathscr{T}_{\mathscr{X}}^{NT}(n, m, 3) = n_t$ and $\tau_1^{NT}, \tau_2^{NT}, \ldots, \tau_{n_t}^{NT}$ be the NTs with 3-NOSs on $\mathscr{X}$.

Let $\mathfrak{B}_i^{NT}$ be the collection of NBTSs on $\mathscr{X}$ having 3-NOSs in NTs, i.e.,

$\mathfrak{B}_i^{NT} = \{(\mathscr{X}, \tau_i^{NT}, \tau_j^{NT}) : 1 \leq j \leq i\}, i = 1, 2, \ldots, n_t\}.$

Then,

$\mathfrak{B}_1^{NT} = \{(\mathscr{X}, \tau_1^{NT}, \tau_1^{NT})\},$

$\mathfrak{B}_2^{NT} = \{(\mathscr{X}, \tau_2^{NT}, \tau_1^{NT}), (\mathscr{X}, \tau_2^{NT}, \tau_2^{NT})\},$

$\mathfrak{B}_3^{NT} = \{(\mathscr{X}, \tau_3^{NT}, \tau_1^{NT}), (\mathscr{X}, \tau_3^{NT}, \tau_2^{NT}), (\mathscr{X}, \tau_3^{NT}, \tau_3^{NT})\},$

$$\vdots$$

$$\mathfrak{B}_{n_t}^{NT} = \{(\mathscr{X}, \tau_{n_t}^{NT}, \tau_1^{NT}), (\mathscr{X}, \tau_{n_t}^{NT}, \tau_2^{NT}), \ldots, (\mathscr{X}, \tau_{n_t}^{NT}, \tau_{n_t}^{NT})\}.$$

This shows that, $|\mathfrak{B}_1^{NT}| = 1$,$|\mathfrak{B}_2^{NT}| = 2$,...,$|\mathfrak{B}_{n_t}^{NT}| = n_t$ and $\bigcup_{i=1}^{n_t} \mathfrak{B}_i^{NT}$ contains all the NBTSs having 3-NOSs in NTs on $\mathscr{X}$.

Therefore,

$$\begin{aligned}(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n, m, 3) &= \text{cardinality of } \textstyle\bigcup_{i=1}^{n_t} \mathfrak{B}_i^{NT} \\ &= \textstyle\sum_{i=1}^{n_t} |\mathfrak{B}_i^{NT}|, \text{ as } \mathfrak{B}_i^{NT} \text{ are distinct} \\ &= |\mathfrak{B}_1^{NT}| + |\mathfrak{B}_2^{NT}| + \ldots + |\mathfrak{B}_{n_t}^{NT}| \\ &= 1 + 2 + 3 + \ldots + n_t \\ &= \tfrac{n_t(n_t+1)}{2} \\ &= \tfrac{n_t(n_t+1)(n_t-1)!}{2(n_t-1)!} \\ &= \tbinom{n_t+1}{2}\end{aligned}$$

Hence, the NNBTSs having 3-NOSs in NTs on $\mathscr{X}$ is

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n, m, 3) = \tbinom{n_t+1}{2} = \tbinom{\mathscr{T}_{\mathscr{X}}^{NT}(n,m,3)+1}{2}.$$

Since, $\mathscr{T}_{\mathscr{X}}^{NT}(n, m, 3) = m^n - 2$.

We have,

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n, m, 3) = \tfrac{m^{2n}-3m^n+2}{2}.$$

Example 3.5.1

Example 3.3.1 gives $\mathscr{T}_{\mathscr{X}}^{NT}(2, 3, 3) = 7$. *Therefore,*

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(2, 3, 3) = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(2, 3, 3) + 1}{2} = 28.$$

Then, these NBTSs are

$(\mathscr{X}, \tau_1^{NT}, \tau_1^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_2^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_3^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_4^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_2^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_3^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_4^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_3^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_4^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_4^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_5^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_5^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_5^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_6^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_6^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_7^{NT}, \tau_7^{NT})$.

Proposition 3.5.3

In $\mathscr{N}_{\mathscr{X}}$, the NNBTSs having 3-NOSs in NTs without repetition is

$$(\mathscr{T}_i^{NT},\mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n,m,3)=\binom{\mathscr{T}_{\mathscr{X}}^{NT}(n,m,3)}{2}.$$

Proof:

Let $\mathscr{T}_{\mathscr{X}}^{NT}(n,m,3)=n_t$ and $\tau_1^{NT},\tau_2^{NT},\ldots,\tau_{n_t}^{NT}$ be the NTs with 3-NOSs on $\mathscr{X}$.

Let $\mathfrak{B}'^{NT}_i$ be the collection of NBTSs on $\mathscr{X}$ having 3-NOSs in NTs without repetition, i.e.,

$$\mathfrak{B}'^{NT}_{\mathfrak{i}}=\{(\mathscr{X},\tau_i^{NT},\tau_j^{NT}):i<j\le\eta_t\},i=1,2,\ldots,n_t-1.$$

Then,

$$\mathfrak{B}_1^{NT}=\left\{\begin{matrix}(\mathscr{X},\tau_1^{NT},\tau_2^{NT}),(\mathscr{X},\tau_1^{NT},\tau_3^{NT}),\\(\mathscr{X},\tau_1^{NT},\tau_4^{NT}),\ldots,(\mathscr{X},\tau_1^{NT},\tau_{n_t}^{NT})\end{matrix}\right\},$$
$$\mathfrak{B}_2^{NT}=\left\{\begin{matrix}(\mathscr{X},\tau_2^{NT},\tau_3^{NT}),(\mathscr{X},\tau_2^{NT},\tau_4^{NT}),\\(\mathscr{X},\tau_2^{NT},\tau_5^{NT}),\ldots,(\mathscr{X},\tau_2^{NT},\tau_{n_t}^{NT})\end{matrix}\right\},$$
$$\mathfrak{B}_3^{NT}=\left\{\begin{matrix}(\mathscr{X},\tau_3^{NT},\tau_4^{NT}),(\mathscr{X},\tau_3^{NT},\tau_5^{NT}),\\(\mathscr{X},\tau_3^{NT},\tau_6^{NT}),\ldots,(\mathscr{X},\tau_3^{NT},\tau_{n_t}^{NT})\end{matrix}\right\},$$
$$\vdots$$
$$\mathfrak{B}_{n_t-2}^{NT}=\{(\mathscr{X},\tau_{n_t-2}^{NT},\tau_{n_t-1}^{NT}),(\mathscr{X},\tau_{n_t-2}^{NT},\tau_{n_t}^{NT})\},$$
$$\mathfrak{B}_{n_t-1}^{NT}=\{(\mathscr{X},\tau_{n_t-1}^{NT},\tau_{n_t}^{NT})\}.$$

This shows that,

$|\mathfrak{B}_1^{NT}|=n_t-1,|\mathfrak{B}_2^{NT}|=n_t-2,\ldots,|\mathfrak{B}_{n_t-2}^{NT}|=2,|\mathfrak{B}_{n_t-1}^{NT}|=1$ and $\bigcup_{i=1}^{n_t-1}\mathfrak{B}_i^{NT}$ contains all the NBTSs having 3-NOSs in NTs without repetition on $\mathscr{X}$. Therefore,

$$\begin{aligned}(\mathscr{T}_i^{NT},\mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n,m,3)&=\text{cardinality of }\textstyle\bigcup_{i=1}^{n_t-1}\mathfrak{B}_i^{NT}\\&=\textstyle\sum_{i=1}^{n_t-1}|\mathfrak{B}_i^{NT}|,\text{ as }A_i\text{ are distinct}\\&=|\mathfrak{B}_1^{NT}|+|\mathfrak{B}_2^{NT}|+\ldots+|\mathfrak{B}_{n_t-1}^{NT}|\\&=(n_t-1)+(n_t-2)+\ldots+2+1\\&=1+2+\ldots+(n_t-2)+(n_t-1)\\&=\tfrac{n_t(n_t-1)}{2}\end{aligned}$$

$= \frac{n_t(n_t-1)(n_t-2)!}{2(n_t-2)!}$
$= \binom{n_t}{2}$
$= \binom{\mathscr{T}_{\mathscr{X}}^{NT}(n,m,3)}{2}$.

Hence, the NNBTSs having 3-NOSs in NTs without repetition is

$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n,m,3) = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(n,m,3)}{2}$.

Example 3.5.2

Following Example 3.3.1 and Proposition 3.5.3, the NNBTSs without repetition is $21 = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(2,3,3)}{2} = \binom{7}{2}$.

Proposition 3.5.4

In $\mathscr{N}_{\mathscr{X}}$, the NNBTSs having 4-NOSs in NTs is

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n,m,4) = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(n,m,4)+1}{2}.$$

Proof:

The proof of this proposition can be done in the same manner as Proposition 3.5.2.

Example 3.5.3

Let $\mathscr{X} = \{u,v\}$ and $\mathscr{M} = \{(0,1,1),(0.1,0.3,0.8),(1,0,0)\}$.
Then, $\mathscr{T}_{\mathscr{X}}^{NT}(2,3,4) = 13$ and the NNBTSs is

$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(2,3,4) = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(2,3,4)+1}{2} = 91$.

These NBTSs are

$(\mathscr{X},\tau_1^{NT},\tau_1^{NT})$, $(\mathscr{X},\tau_1^{NT},\tau_2^{NT})$, $(\mathscr{X},\tau_1^{NT},\tau_3^{NT})$, $(\mathscr{X},\tau_1^{NT},\tau_4^{NT})$,
$(\mathscr{X},\tau_1^{NT},\tau_5^{NT})$, $(\mathscr{X},\tau_1^{NT},\tau_6^{NT})$, $(\mathscr{X},\tau_1^{NT},\tau_7^{NT})$, $(\mathscr{X},\tau_1^{NT},\tau_8^{NT})$,
$(\mathscr{X},\tau_1^{NT},\tau_9^{NT})$, $(\mathscr{X},\tau_1^{NT},\tau_{10}^{NT})$, $(\mathscr{X},\tau_1^{NT},\tau_{11}^{NT})$, $(\mathscr{X},\tau_1^{NT},\tau_{12}^{NT})$,
$(\mathscr{X},\tau_1^{NT},\tau_{13}^{NT})$, $(\mathscr{X},\tau_2^{NT},\tau_2^{NT})$, $(\mathscr{X},\tau_2^{NT},\tau_3^{NT})$, $(\mathscr{X},\tau_2^{NT},\tau_4^{NT})$,
$(\mathscr{X},\tau_2^{NT},\tau_5^{NT})$, $(\mathscr{X},\tau_2^{NT},\tau_6^{NT})$, $(\mathscr{X},\tau_2^{NT},\tau_7^{NT})$, $(\mathscr{X},\tau_2^{NT},\tau_8^{NT})$,
$(\mathscr{X},\tau_2^{NT},\tau_9^{NT})$, $(\mathscr{X},\tau_2^{NT},\tau_{10}^{NT})$, $(\mathscr{X},\tau_2^{NT},\tau_{11}^{NT})$, $(\mathscr{X},\tau_2^{NT},\tau_{12}^{NT})$,
$(\mathscr{X},\tau_2^{NT},\tau_{13}^{NT})$, $(\mathscr{X},\tau_3^{NT},\tau_3^{NT})$, $(\mathscr{X},\tau_3^{NT},\tau_4^{NT})$, $(\mathscr{X},\tau_3^{NT},\tau_5^{NT})$,
$(\mathscr{X},\tau_3^{NT},\tau_6^{NT})$, $(\mathscr{X},\tau_3^{NT},\tau_7^{NT})$, $(\mathscr{X},\tau_3^{NT},\tau_8^{NT})$, $(\mathscr{X},\tau_3^{NT},\tau_9^{NT})$,

$(\mathscr{X}, \tau_3^{NT}, \tau_{10}^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_{11}^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_{12}^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_{13}^{NT})$,
$(\mathscr{X}, \tau_4^{NT}, \tau_4^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_7^{NT})$,
$(\mathscr{X}, \tau_4^{NT}, \tau_8^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_9^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_{10}^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_{11}^{NT})$,
$(\mathscr{X}, \tau_4^{NT}, \tau_{12}^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_{13}^{NT})$, $(\mathscr{X}, \tau_5^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_5^{NT}, \tau_6^{NT})$,
$(\mathscr{X}, \tau_5^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_5^{NT}, \tau_8^{NT})$, $(\mathscr{X}, \tau_5^{NT}, \tau_9^{NT})$, $(\mathscr{X}, \tau_5^{NT}, \tau_{10}^{NT})$,
$(\mathscr{X}, \tau_5^{NT}, \tau_{11}^{NT})$, $(\mathscr{X}, \tau_5^{NT}, \tau_{12}^{NT})$, $(\mathscr{X}, \tau_5^{NT}, \tau_{13}^{NT})$, $(\mathscr{X}, \tau_6^{NT}, \tau_6^{NT})$,
$(\mathscr{X}, \tau_6^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_6^{NT}, \tau_8^{NT})$, $(\mathscr{X}, \tau_6^{NT}, \tau_9^{NT})$, $(\mathscr{X}, \tau_6^{NT}, \tau_{10}^{NT})$,
$(\mathscr{X}, \tau_6^{NT}, \tau_{11}^{NT})$, $(\mathscr{X}, \tau_6^{NT}, \tau_{12}^{NT})$, $(\mathscr{X}, \tau_6^{NT}, \tau_{13}^{NT})$, $(\mathscr{X}, \tau_7^{NT}, \tau_7^{NT})$,
$(\mathscr{X}, \tau_7^{NT}, \tau_8^{NT})$, $(\mathscr{X}, \tau_7^{NT}, \tau_9^{NT})$, $(\mathscr{X}, \tau_7^{NT}, \tau_{10}^{NT})$, $(\mathscr{X}, \tau_7^{NT}, \tau_{11}^{NT})$,
$(\mathscr{X}, \tau_7^{NT}, \tau_{12}^{NT})$, $(\mathscr{X}, \tau_7^{NT}, \tau_{13}^{NT})$, $(\mathscr{X}, \tau_8^{NT}, \tau_8^{NT})$, $(\mathscr{X}, \tau_8^{NT}, \tau_9^{NT})$,
$(\mathscr{X}, \tau_8^{NT}, \tau_{10}^{NT})$, $(\mathscr{X}, \tau_8^{NT}, \tau_{11}^{NT})$, $(\mathscr{X}, \tau_8^{NT}, \tau_{12}^{NT})$, $(\mathscr{X}, \tau_8^{NT}, \tau_{13}^{NT})$,
$(\mathscr{X}, \tau_9^{NT}, \tau_9^{NT})$, $(\mathscr{X}, \tau_9^{NT}, \tau_{10}^{NT})$, $(\mathscr{X}, \tau_9^{NT}, \tau_{11}^{NT})$, $(\mathscr{X}, \tau_9^{NT}, \tau_{12}^{NT})$,
$(\mathscr{X}, \tau_9^{NT}, \tau_{13}^{NT})$, $(\mathscr{X}, \tau_{10}^{NT}, \tau_{10}^{NT})$, $(\mathscr{X}, \tau_{10}^{NT}, \tau_{11}^{NT})$, $(\mathscr{X}, \tau_{10}^{NT}, \tau_{12}^{NT})$,
$(\mathscr{X}, \tau_{10}^{NT}, \tau_{13}^{NT})$, $(\mathscr{X}, \tau_{11}^{NT}, \tau_{11}^{NT})$, $(\mathscr{X}, \tau_{11}^{NT}, \tau_{12}^{NT})$, $(\mathscr{X}, \tau_{11}^{NT}, \tau_{13}^{NT})$,
$(\mathscr{X}, \tau_{12}^{NT}, \tau_{12}^{NT})$, $(\mathscr{X}, \tau_{12}^{NT}, \tau_{13}^{NT})$, $(\mathscr{X}, \tau_{13}^{NT}, \tau_{13}^{NT})$.

Proposition 3.5.5

In $\mathscr{N}_{\mathscr{X}}$, the NNBTSs having 4-NOSs in NTs without repetition is

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n, m, 4) = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(n, m, 4)}{2}.$$

Proof:

The proof of this proposition can be done in the same manner as Proposition 3.5.3.

Example 3.5.4

Following Example 3.4.1 and Proposition 3.5.5, the NNBTSs without repetition is $78 = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(2,3,4)}{2} = \binom{13}{2}$.

3.6 Number of Neutrosophic Tritopological Spaces

In this section, the NTRS having 3-NOSs in NTs defines the NTRS of the form $(\mathscr{X}, \tau_i^{NT}, \tau_j^{NT}, \tau_k^{NT})$, where τ_i^{NT}, τ_j^{NT}, and τ_j^{NT} are identical or non-identical NTs having 3-NOSs and the NTRS having 3-NOSs in NTs without repetition defines the NTRS of the form $(\mathscr{X}, \tau_i^{NT}, \tau_j^{NT}, \tau_k^{NT})$, where τ_i^{NT}, τ_j^{NT}, and τ_j^{NT} are non-identical NTs having 3-NOSs. Similar meaning is extended to the NTRS having 4-NOSs in NTs.

Proposition 3.6.1

In $\mathscr{N}_{\mathscr{X}}$, the NNTRS with 2-NOSs in NTs is

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(n, m, 2) = 1.$$

Proof:

In this case, NT with 2-NOSs is the indiscrete one i.e., $\tau_1^{NT} = \{0^{NT}, 1^{NT}\}$. Therefore, the NNTRS with 2-NOSs is exactly one, namely, $(\mathscr{X}, \tau_1^{NT}, \tau_1^{NT}, \tau_1^{NT})$. Hence, $(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(n, m, 2) = 1$.

Proposition 3.6.2

In $\mathscr{N}_{\mathscr{X}}$, the NNTRS with 3-NOSs in NTs is

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(n, m, 3) = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(n, m, 3) + 2}{3}.$$

Proof:

Let $\mathscr{T}_{\mathscr{X}}^{NT}(n, m, 3) = n_t$ and $\tau_1^{NT}, \tau_2^{NT}, \ldots, \tau_{n_t}^{NT}$ be the NTs with 3-NOSs on $\mathscr{X}$.

Let $\mathfrak{T}_i^{NT}$ be the collection of NTRSs on $\mathscr{X}$ having 3-NOSs in NTs, i.e.,

$$\mathfrak{T}_i^{NT} = \{(\mathscr{X}, \tau_p^{NT}, \tau_q^{NT}, \tau_r^{NT}) : p = i, 1 \leq q \leq p, q \leq r \leq p\}, \quad i = 1, 2, \ldots, n_t.$$

Then,

$\mathfrak{T}_1^{NT} = \{(\mathscr{X}, \tau_1^{NT}, \tau_1^{NT}, \tau_1^{NT})\}$,

$\mathfrak{T}_2^{NT} = \left\{\begin{matrix} (\mathscr{X}, \tau_2^{NT}, \tau_1^{NT}, \tau_1^{NT}), (\mathscr{X}, \tau_2^{NT}, \tau_1^{NT}, \tau_2^{NT}), \\ (\mathscr{X}, \tau_2^{NT}, \tau_2^{NT}, \tau_2^{NT}), \end{matrix}\right\}$,

$\mathfrak{T}_3^{NT} = \left\{\begin{matrix} (\mathscr{X}, \tau_3^{NT}, \tau_1^{NT}, \tau_1^{NT}), (\mathscr{X}, \tau_3^{NT}, \tau_1^{NT}, \tau_2^{NT}), (\mathscr{X}, \tau_3^{NT}, \tau_1^{NT}, \tau_3^{NT}), \\ (\mathscr{X}, \tau_3^{NT}, \tau_2^{NT}, \tau_2^{NT}), (\mathscr{X}, \tau_3^{NT}, \tau_2^{NT}, \tau_3^{NT}), \\ (\mathscr{X}, \tau_3^{NT}, \tau_3^{NT}, \tau_3^{NT}), \end{matrix}\right\}$,

$\vdots$

$\mathfrak{T}_{n_t}^{NT} = \left\{\begin{matrix} (\mathscr{X}, \tau_{n_t}^{NT}, \tau_1^{NT}, \tau_1^{NT}), (\mathscr{X}, \tau_{n_t}^{NT}, \tau_1^{NT}, \tau_2^{NT}), \ldots, (\mathscr{X}, \tau_{n_t}^{NT}, \tau_1^{NT}, \tau_{n_t}^{NT}), \\ (\mathscr{X}, \tau_{n_t}^{NT}, \tau_2^{NT}, \tau_2^{NT}), (\mathscr{X}, \tau_{n_t}^{NT}, \tau_2^{NT}, \tau_3^{NT}), \ldots, (\mathscr{X}, \tau_{n_t}^{NT}, \tau_2^{NT}, \tau_{n_t}^{NT}), \\ \vdots \\ (\mathscr{X}, \tau_{n_t}^{NT}, \tau_{n_t}^{NT}, \tau_{n_t}^{NT}) \end{matrix}\right\}$.

Here, $|\mathfrak{T}_1^{NT}| = 1$, $|\mathfrak{T}_2^{NT}| = 3 = 1+2$, $|\mathfrak{T}_3^{NT}| = 6 = 1+2+3, \ldots,$ $|\mathfrak{T}_{n_t}^{NT}| = 1+2+3+\ldots+n_t$. Also, $\bigcup_{i=1}^{n_t} \mathfrak{T}_i^{NT}$ contains all the NTRSs having 3-NOSs in NTs on $\mathscr{X}$.

Therefore,

$$\begin{aligned}(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(n,m,3) &= \text{cardinality of } \textstyle\bigcup_{i=1}^{n_t} \mathfrak{T}_i^{NT} \\ &= \textstyle\sum_{i=1}^{n_t} |\mathfrak{T}_i^{NT}|, \text{ as } \mathfrak{T}_i^{NT} \text{ are distinct} \\ &= |\mathfrak{T}_1^{NT}| + |\mathfrak{T}_2^{NT}| + \ldots + |\mathfrak{T}_{n_t}^{NT}| \\ &= 1 + (1+2) + (1+2+3) \\ &\quad + \ldots + (1+2+\ldots+n_t) \\ &= \tfrac{n_t(n_t+1)(n_t+2)}{6} \\ &= \tfrac{(n_t+2)(n_t+1)n_t(n_t-1)!}{3!(n_t-1)!} \\ &= \tbinom{n_t+2}{3} \\ &= \tbinom{\mathscr{T}_{\mathscr{X}}^{NT}(n,m,3)+2}{3}.\end{aligned}$$

Hence, the NNTRSs having 3-NOSs in NTs on $\mathscr{X}$ is

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(n,m,3) = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(n,m,3)+2}{3}.$$

Example 3.6.1

Example 3.3.1 implies that $\mathscr{T}_{\mathscr{X}}^{NT}(2,3,3) = 7$.

Therefore, $(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(2,3,3) = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(2,3,3)+2}{3} = \frac{9\times 8\times 7}{6} = 84$.

These are

$(\mathscr{X}, \tau_1^{NT}, \tau_1^{NT}, \tau_1^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_1^{NT}, \tau_2^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_1^{NT}, \tau_3^{NT})$,
$(\mathscr{X}, \tau_1^{NT}, \tau_1^{NT}, \tau_4^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_1^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_1^{NT}, \tau_6^{NT})$,
$(\mathscr{X}, \tau_1^{NT}, \tau_1^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_2^{NT}, \tau_2^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_2^{NT}, \tau_3^{NT})$,
$(\mathscr{X}, \tau_1^{NT}, \tau_2^{NT}, \tau_4^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_2^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_2^{NT}, \tau_6^{NT})$,
$(\mathscr{X}, \tau_1^{NT}, \tau_2^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_3^{NT}, \tau_3^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_3^{NT}, \tau_4^{NT})$,
$(\mathscr{X}, \tau_1^{NT}, \tau_3^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_3^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_3^{NT}, \tau_7^{NT})$,
$(\mathscr{X}, \tau_1^{NT}, \tau_4^{NT}, \tau_4^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_4^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_4^{NT}, \tau_6^{NT})$,
$(\mathscr{X}, \tau_1^{NT}, \tau_4^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_5^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_5^{NT}, \tau_6^{NT})$,
$(\mathscr{X}, \tau_1^{NT}, \tau_5^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_6^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_6^{NT}, \tau_7^{NT})$,
$(\mathscr{X}, \tau_1^{NT}, \tau_7^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_2^{NT}, \tau_2^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_2^{NT}, \tau_3^{NT})$,
$(\mathscr{X}, \tau_2^{NT}, \tau_2^{NT}, \tau_4^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_2^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_2^{NT}, \tau_6^{NT})$,
$(\mathscr{X}, \tau_2^{NT}, \tau_2^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_3^{NT}, \tau_3^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_3^{NT}, \tau_4^{NT})$,
$(\mathscr{X}, \tau_2^{NT}, \tau_3^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_3^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_3^{NT}, \tau_7^{NT})$,
$(\mathscr{X}, \tau_2^{NT}, \tau_4^{NT}, \tau_4^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_4^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_4^{NT}, \tau_6^{NT})$,
$(\mathscr{X}, \tau_2^{NT}, \tau_4^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_5^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_5^{NT}, \tau_6^{NT})$,
$(\mathscr{X}, \tau_2^{NT}, \tau_5^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_6^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_6^{NT}, \tau_7^{NT})$,
$(\mathscr{X}, \tau_2^{NT}, \tau_7^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_3^{NT}, \tau_3^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_3^{NT}, \tau_4^{NT})$,
$(\mathscr{X}, \tau_3^{NT}, \tau_3^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_3^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_3^{NT}, \tau_7^{NT})$,
$(\mathscr{X}, \tau_3^{NT}, \tau_4^{NT}, \tau_4^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_4^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_4^{NT}, \tau_6^{NT})$,
$(\mathscr{X}, \tau_3^{NT}, \tau_4^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_5^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_5^{NT}, \tau_6^{NT})$,
$(\mathscr{X}, \tau_3^{NT}, \tau_5^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_6^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_6^{NT}, \tau_7^{NT})$,
$(\mathscr{X}, \tau_3^{NT}, \tau_7^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_4^{NT}, \tau_4^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_4^{NT}, \tau_5^{NT})$,
$(\mathscr{X}, \tau_4^{NT}, \tau_4^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_4^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_5^{NT}, \tau_5^{NT})$,
$(\mathscr{X}, \tau_4^{NT}, \tau_5^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_5^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_6^{NT}, \tau_6^{NT})$,
$(\mathscr{X}, \tau_4^{NT}, \tau_6^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_7^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_5^{NT}, \tau_5^{NT}, \tau_5^{NT})$,

$(\mathscr{X}, \tau_5^{NT}, \tau_5^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_5^{NT}, \tau_5^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_5^{NT}, \tau_6^{NT}, \tau_6^{NT})$,
$(\mathscr{X}, \tau_5^{NT}, \tau_6^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_5^{NT}, \tau_7^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_6^{NT}, \tau_6^{NT}, \tau_6^{NT})$,
$(\mathscr{X}, \tau_6^{NT}, \tau_6^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_6^{NT}, \tau_7^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_7^{NT}, \tau_7^{NT}, \tau_7^{NT})$.

Proposition 3.6.3

In $\mathscr{N}_{\mathscr{X}}$, the NNTRSs consisting 3-NOSs in NTs without repetition is

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(n,m,3) = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(n,m,3)}{3}.$$

Proof:

Let $\mathscr{T}_{\mathscr{X}}^{NT}(n,m,3) = n_t$ and $\tau_1^{NT}, \tau_2^{NT}, \dots, \tau_{n_t}^{NT}$ be the NTs with 3-NOSs on $\mathscr{X}$.

Let $\mathfrak{T}'^{NT}_i$ be the collection of NTRSs on $\mathscr{X}$ having 3-NOSs in NTs without repetition, i.e.,

$$\mathfrak{T}'_i = \{(\mathscr{X}, \tau_p^{NT}, \tau_q^{NT}, \tau_r^{NT}) : p = i, p < q < r \le n_t\}, \quad i = 1, 2, \dots, n_t - 2.$$

Then,

$$\mathfrak{T}'^{NT}_1 = \left\{\begin{matrix} (\mathscr{X}, \tau_1^{NT}, \tau_2^{NT}, \tau_3^{NT}), (\mathscr{X}, \tau_1^{NT}, \tau_2^{NT}, \tau_4^{NT}), \dots, (\mathscr{X}, \tau_1^{NT}, \tau_2^{NT}, \tau_{n_t}^{NT}), \\ (\mathscr{X}, \tau_1^{NT}, \tau_3^{NT}, \tau_4^{NT}), \dots, (\mathscr{X}, \tau_1^{NT}, \tau_3^{NT}, \tau_{n_t}^{NT}), \\ \vdots \\ (\mathscr{X}, \tau_1^{NT}, \tau_{n_t-1}^{NT}, \tau_{n_t}^{NT}) \end{matrix}\right\},$$

$$\mathfrak{T}'^{NT}_2 = \left\{\begin{matrix} (\mathscr{X}, \tau_2^{NT}, \tau_3^{NT}, \tau_4^{NT}), (\mathscr{X}, \tau_2^{NT}, \tau_3^{NT}, \tau_5^{NT}), \dots, (\mathscr{X}, \tau_2^{NT}, \tau_3^{NT}, \tau_{n_t}^{NT}), \\ (\mathscr{X}, \tau_2^{NT}, \tau_4^{NT}, \tau_5^{NT}), \dots, (\mathscr{X}, \tau_2^{NT}, \tau_4^{NT}, \tau_{n_t}^{NT}), \\ \vdots \\ (\mathscr{X}, \tau_2^{NT}, \tau_{n_t-1}^{NT}, \tau_{n_t}^{NT}) \end{matrix}\right\},$$

$$\vdots$$

$$\mathfrak{T}'^{NT}_{n_t-3} = \left\{\begin{matrix} (\mathscr{X}, \tau_{n_t-3}^{NT}, \tau_{n_t-2}^{NT}, \tau_{n_t-1}^{NT}), (\mathscr{X}, \tau_{n_t-3}^{NT}, \tau_{n_t-2}^{NT}, \tau_{n_t}^{NT}), \\ (\mathscr{X}, \tau_{n_t-3}^{NT}, \tau_{n_t-1}^{NT}, \tau_{n_t}^{NT}) \end{matrix}\right\},$$

$$\mathfrak{T}'^{NT}_{n_t-2} = \left\{(\mathscr{X}', \tau_{n_t-2}^{NT}, \tau_{n_t-1}^{NT}, \tau_{n_t}^{NT})\right\}.$$

Here,

$$|\mathfrak{T}'^{NT}_1| = 1 + 2 + \dots + (n_t - 2), |\mathfrak{T}'_2| = 1 + 2 + \dots + (n_t - 3), \dots,$$

$|\mathfrak{T}'_{n_t-3}| = 1+2$, $|\mathfrak{T}'_{n_t-2}| = 1$.

Also, $\bigcup_{i=1}^{n_t-2} \mathfrak{T}'^{NT}_i$ contains all the NTRSs having 3-NOSs without repetition on $\mathscr{X}$.

Therefore, in this case,

$$\begin{aligned}
(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(n,m,3) &= \text{cardinality of } \textstyle\bigcup_{i=1}^{n_t-2} \mathfrak{T}'^{NT}_i \\
&= \textstyle\sum_{i=1}^{n_t-2} |\mathfrak{T}'^{NT}_i|, \text{ as } \mathfrak{T}'^{NT}_i \text{ are distinct} \\
&= |\mathfrak{T}'^{NT}_1| + |\mathfrak{T}'^{NT}_2| + \ldots + |\mathfrak{T}'^{NT}_{n_t-2}| \\
&= 1 + (1+2) + (1+2+3) + \ldots \\
&\quad + \{1 + 2 + \ldots + (n_t-2)\} \\
&= \tfrac{n_t(n_t-1)(n_t-2)}{6} \\
&= \tfrac{n_t(n_t-1)(n_t-2)(n_t-3)!}{3!(n_t-3)!} \\
&= \tbinom{\mathscr{T}_{\mathscr{X}}^{NT}(n,m,3)}{3}.
\end{aligned}$$

Hence, the NNTRSs consisting 3-NOSs in NTs without repetition is

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(n,m,3) = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(n,m,3)}{3}.$$

Example 3.6.2

From Example 3.3.1, $\mathscr{T}_{\mathscr{X}}^{NT}(2,3,3) = 7$. In this case, the NTRSs having 3-NOSs in NTs without repetition are

$(\mathscr{X}, \tau_1^{NT}, \tau_2^{NT}, \tau_3^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_2^{NT}, \tau_4^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_2^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_2^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_2^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_3^{NT}, \tau_4^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_3^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_3^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_3^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_4^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_4^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_4^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_5^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_5^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_1^{NT}, \tau_6^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_3^{NT}, \tau_4^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_3^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_3^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_3^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_4^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_4^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_4^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_5^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_5^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_2^{NT}, \tau_6^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_4^{NT}, \tau_5^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_4^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_4^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_5^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_5^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_3^{NT}, \tau_6^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_5^{NT}, \tau_6^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_5^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_4^{NT}, \tau_6^{NT}, \tau_7^{NT})$, $(\mathscr{X}, \tau_5^{NT}, \tau_6^{NT}, \tau_7^{NT})$.

Therefore, the NNTRSs consisting 3*-NOSs in NTs without repetition is* $(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(2,3,3) = 35 = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(2,3,3)}{3} = \binom{7}{3}$.

Proposition 3.6.4

For the NNBTSs and NNTRSs having 3*-NOSs in NTs,*
$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(n,m,3) = \frac{m^n}{3}(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n,m,3)$.

Proof:

Let $\mathscr{T}_{\mathscr{X}}^{NT}(n,m,3) = m^n - 2$.

Then,

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n,m,3) = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(n,m,3)+1}{2}$$

and

$$\begin{aligned}(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(n,m,3) &= \binom{\mathscr{T}_{\mathscr{X}}^{NT}(n,m,3)+2}{3}\\ &= \binom{m^n-2+2}{3}\\ &= \binom{m^n}{3}\\ &= \frac{m^n(m^n-1)(m^n-2)(m^n-3)!}{3!(m^n-3)!}\\ &= \frac{m^n((m^n-2)+1)(m^n-2)(((m^n-2)+1)-2)!}{3\times 2!(((m^n-2)+1)-2)!}\\ &= \frac{m^n}{3}\times\binom{(m^n-2)+1}{2}.\end{aligned}$$

Hence, $(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(n,m,3) = \frac{m^n}{3}\times(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n,m,3)$.

Example 3.6.3

Example 3.5.1 and 3.6.1 provides

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n,m,3) = 28$$

and

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(2,3,3) = 84.$$

Therefore,

$$\begin{aligned}\frac{3^2}{3}(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n,m,3) &= \frac{3^2}{3}\times 28\\ &= 84\\ &= (\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(2,3,3).\end{aligned}$$

Proposition 3.6.5

In $\mathscr{N}_{\mathscr{X}}$, the NNTRSs consisting 4-NOSs in NTs is

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(n, m, 4) = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(n, m, 4) + 2}{3}.$$

Proof:

The proof of this proposition can be done in the same manner as Proposition 3.6.2.

Example 3.6.4

Example 3.4.1 implies, $\mathscr{T}_{\mathscr{X}}^{NT}(2,3,4) = 13$. Then, the NNTRS having 4-NOSs is

$$\begin{aligned}(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(2,3,4) &= \tbinom{\mathscr{T}_{\mathscr{X}}^{NT}(2,3,4)+2}{3} \\ &= \tfrac{(13+2)(13+1)13}{6} \\ &= 455.\end{aligned}$$

Proposition 3.6.6

In $\mathscr{N}_{\mathscr{X}}$, the NNTRSs consisting 4-NOSs in NTs without repetition is

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(n, m, 4) = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(n, m, 4)}{3}.$$

Proof:

The proof of this proposition can be done in the same manner as Proposition 3.6.3.

Example 3.6.5

From Example 3.4.1, $\mathscr{T}_{\mathscr{X}}^{NT}(2,3,4) = 13$. Following Proposition 3.6.6, the NNTRSs consisting 4-NOSs in NTs without repetition is

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(2,3,4) = 286.$$

Proposition 3.6.7

For the NNBTSs and NNTRSs having 4-NOSs in NTs,

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(n, m, 4) = \tfrac{\mathscr{T}_{\mathscr{X}}^{NT}(n,m,4)+2}{3}(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n, m, 4).$$

Proof:

Let $\mathscr{T}_{\mathscr{X}}^{NT}(n,m,4) = n_t$.

Then,

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n,m,4) = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(n,m,4)+1}{2} = \binom{n_t+1}{2}$$

and

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(n,m,4) = \binom{\mathscr{T}_{\mathscr{X}}^{NT}(n,m,4)+2}{3} = \binom{n_t+2}{3}.$$

Now,

$$\begin{aligned}(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(n,m,4) &= \binom{n_t+2}{3} \\ &= \frac{(n_t+2)(n_t+1)(n_t)(n_t-1)!}{3!(n_t-1)!} \\ &= \frac{(n_t+2)}{3} \times \frac{(n_t+1)(n_t)(n_t-1)!}{2!(n_t-1)!} \\ &= \frac{n_t+2}{3} \times \binom{n_t+1}{2} \\ &= \frac{n_t+2}{3} \times (\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n,m,4).\end{aligned}$$

Therefore,

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(n,m,4) = \frac{\mathscr{T}_{\mathscr{X}}^{NT}(n,m,4)+2}{3}(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(n,m,4).$$

Example 3.6.6

Examples 3.4.1, 3.5.3 and 3.6.4 provides

$$\mathscr{T}_{\mathscr{X}}^{NT}(2,3,4) = 13,\ (\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(2,3,4) = 91,$$

and

$$(\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(2,3,4) = 455.$$

Therefore,

$$\begin{aligned}\frac{\mathscr{T}_{\mathscr{X}}^{NT}(2,3,4)+2}{3} \times (\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT})_{\mathscr{X}}^{NT}(2,3,4) &= \frac{13+2}{3} \times 91 \\ &= 455 \\ &= (\mathscr{T}_i^{NT}, \mathscr{T}_j^{NT}, \mathscr{T}_k^{NT})_{\mathscr{X}}^{NT}(2,3,4).\end{aligned}$$