CHAPTER 5

Number of Neutrosophic Crisp Topological Spaces on a Finite Set

In Chapter 2, the formula to find the number of neutrosophic crisp subsets in a nonempty finite set $\mathscr{X}$ has been obtained, and some propositions are also explored. The present chapter aims to find formulae to compute the number of neutrosophic crisp topological spaces having 2-NCrOSs, 3-NCrOSs, and 4-NCrOSs.

Remark 5.0.1 *(Salama, 2013)*
Let $\mathscr{A} = \langle \mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3 \rangle$ and $\mathscr{B} = \langle \mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3 \rangle$ be any two neutrosophic crisp sets on $\mathscr{X}$ (Using Definition 1.9.1). To perform intersection and union on $\mathscr{A}$ and $\mathscr{B}$, the following operations has been taken

(i) $\mathscr{A} \cap \mathscr{B} = \langle \mathscr{A}_1 \cap \mathscr{B}_1, \mathscr{A}_2 \cap \mathscr{B}_2, \mathscr{A}_3 \cup \mathscr{B}_3 \rangle$,

(ii) $\mathscr{A} \cup \mathscr{B} = \langle \mathscr{A}_1 \cup \mathscr{B}_1, \mathscr{A}_2 \cap \mathscr{B}_2, \mathscr{A}_3 \cap \mathscr{B}_3 \rangle$.

Definition 5.0.1
A NCrT having k-NCrOSs on a non-empty set $\mathscr{X}$ is said to be a NCrT of cardinality k. The number of NCrTs of cardinality k on $\mathscr{X}$ with $|\mathscr{X}| = n$ will be denoted by $\mathscr{T}_{\mathscr{C}r}(n, k)$.

Example 5.0.1

Let $\mathscr{X} = \{u, v, w\}$ *and* $\mathscr{A}_1 = \langle \emptyset, \emptyset, \{u\} \rangle$*, then* $\tau^{NCr} = \{\phi_{\mathscr{N}}, \mathscr{X}_{\mathscr{N}}, \mathscr{A}_1\}$ *form a NCrT on* $\mathscr{X}$*. So,* τ^{NCr} *is a NCrT of cardinality* 3 *as it has* 3*-NCrOSs.*

5.1 Neutrosophic Crisp Topological Spaces with 2-NCrOSs

Proposition 5.1.1

For a non-empty finite set $\mathscr{X}$ *with* $|\mathscr{X}| = n$*,*

(a) $\mathscr{T}_{\mathscr{C}r}(n, 2) = 1$*,*

(b) $\mathscr{T}_{\mathscr{C}r}(n, k) = 1$*, where* $k = |\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})|$*.*

Proof:

(a) The NCrT having 2-NCrOSs is the indiscrete NCrT which is $\mathscr{T}_{\mathscr{N}} = \{\phi_{\mathscr{N}}, \mathscr{X}_{\mathscr{N}}\}$. Therefore, $(\mathscr{X}, \mathscr{T}_{\mathscr{N}})$ is the only NCrTS having 2-NCrOSs as $\mathscr{T}_{\mathscr{N}}$ contains only two members $\phi_{\mathscr{N}}$ and $\mathscr{X}_{\mathscr{N}}$. Hence, the number of neutrosophic crisp topological spaces (NCrTSs) having 2-NCrOSs is 1 i.e., $\mathscr{T}_{\mathscr{C}r}(n, 2) = 1$.

(b) The NCrT of cardinality $k = |\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})|$ is the discrete NCrT only. Hence, $\mathscr{T}_{\mathscr{C}r}(n, k) = 1$, for $k = |\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})|$.

Example 5.1.1

Let $\mathscr{X} = \{u, v\}$*, then,* $|\mathscr{X}| = n = 2$*. Here, the neutrosophic crisp subsets on* $\mathscr{X}$ *are*

$\phi_{\mathscr{N}}$, $\mathscr{X}_{\mathscr{N}}$, $\mathscr{A}_1 = \langle \emptyset, \emptyset, \{u\} \rangle$, $\mathscr{A}_2 = \langle \emptyset, \{u\}, \emptyset \rangle$, $\mathscr{A}_3 = \langle \{u\}, \emptyset, \emptyset \rangle$,
$\mathscr{A}_4 = \langle \emptyset, \emptyset, \{v\} \rangle$, $\mathscr{A}_5 = \langle \emptyset, \{v\}, \emptyset \rangle$, $\mathscr{A}_6 = \langle \{v\}, \emptyset, \emptyset \rangle$,
$\mathscr{A}_7 = \langle \emptyset, \{u\}, \{v\} \rangle$, $\mathscr{A}_8 = \langle \{u\}, \emptyset, \{v\} \rangle$, $\mathscr{A}_9 = \langle \{u\}, \{v\}, \emptyset \rangle$,
$\mathscr{A}_{10} = \langle \emptyset, \{v\}, \{u\} \rangle$, $\mathscr{A}_{11} = \langle \{v\}, \emptyset, \{u\} \rangle$, $\mathscr{A}_{12} = \langle \{v\}, \{u\}, \emptyset \rangle$.

In this case, the only NCrT having 2-NCrOSs is $\{\phi_{\mathscr{N}}, \mathscr{X}_{\mathscr{N}}\}$ *and hence* $\mathscr{T}_{\mathscr{C}r}(n, 2) = 1$.

Also, the NCrT having $k = |\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})| = 14$*-NCrOSs is* $\{\phi_{\mathscr{N}}, \mathscr{X}_{\mathscr{N}}, \mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3, \mathscr{A}_4, \mathscr{A}_5, \mathscr{A}_6, \mathscr{A}_7, \mathscr{A}_8, \mathscr{A}_9, \mathscr{A}_{10}, \mathscr{A}_{11}, \mathscr{A}_{12}\}$ *and hence,* $\mathscr{T}_{\mathscr{C}r}(n, k) = 1$*, for* $k = |\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})| = 14$.

5.2 Neutrosophic Crisp Topological Spaces with 3-NCrOSs

Proposition 5.2.1

The number of NCrTs of cardinality 3 *on a non-empty finite set* $\mathscr{X}$ *with* $|\mathscr{X}| = n$ *is given by the formula*

$$\begin{aligned}\mathscr{T}_{\mathscr{C}r}(n, 3) &= |\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})| - 2\\ &= 3(2^n - 2) + 3!\left[\sum_{i=2}^{n} \mathcal{S}(i, 2)\binom{n}{i} + \sum_{j=3}^{n} \mathcal{S}(j, 3)\binom{n}{j}\right].\end{aligned}$$

Proof:

The NCrTs having 3-NCrOSs necessarily consists of a chain containing $\phi_{\mathscr{N}}, \mathscr{X}_{\mathscr{N}}$ and any other neutrosophic crisp subset $\mathscr{A}_{\mathscr{N}}$ of $\mathscr{X}$ other than $\phi_{\mathscr{N}}$ and $\mathscr{X}_{\mathscr{N}}$. Clearly, $\phi_{\mathscr{N}} \subset \mathscr{A}_{\mathscr{N}} \subset \mathscr{X}_{\mathscr{N}}$. It is observed that the number of such $\mathscr{A}_{\mathscr{N}}$ is equal to $|\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})| - 2$. Since the set $\{\phi_{\mathscr{N}}, \mathscr{A}_{\mathscr{N}}, \mathscr{X}_{\mathscr{N}}\}$ form a NCrT and the total number of such NCrTs is $|\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})| - 2$.
Now, $|\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})| = (3.2^n - 4) + 3!\left\{\sum_{i=2}^{n} \mathcal{S}(i, 2)\binom{n}{i} + \sum_{j=3}^{n} \mathcal{S}(j, 3)\binom{n}{j}\right\}$.
Therefore,

$$\begin{aligned}|\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})| - 2 &= \left[(3.2^n - 4) + 3!\left\{\sum_{i=2}^{n} \mathcal{S}(i, 2)\binom{n}{i} + \sum_{j=3}^{n} \mathcal{S}(j, 3)\binom{n}{j}\right\}\right] - 2\\ &= (3.2^n - 6) + 3!\left\{\sum_{i=2}^{n} \mathcal{S}(i, 2)\binom{n}{i} + \sum_{j=3}^{n} \mathcal{S}(j, 3)\binom{n}{j}\right\}\\ &= 3(2^n - 2) + 3!\left\{\sum_{i=2}^{n} \mathcal{S}(i, 2)\binom{n}{i} + \sum_{j=3}^{n} \mathcal{S}(j, 3)\binom{n}{j}\right\}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathscr{T}_{\mathscr{C}r}(n, 3) &= |\mathscr{P}_{\mathscr{N}\mathscr{C}r}(\mathscr{X})| - 2\\ &= 3(2^n - 2) + 3!\left[\sum_{i=2}^{n} \mathcal{S}(i, 2)\binom{n}{i} + \sum_{j=3}^{n} \mathcal{S}(j, 3)\binom{n}{j}\right].\end{aligned}$$

Example 5.2.1

Let $\mathscr{X} = \{u, v\}$*, then*

$$\mathscr{T}_{\mathscr{C}r}(2, 3) = 3(2^2 - 2) + 3!\left\{\sum_{i=2}^{2} \mathcal{S}(i, 2)\binom{2}{i} + \sum_{j=3}^{2} \mathcal{S}(j, 3)\binom{2}{j}\right\}.$$

Clearly, $\sum_{j=3}^{2} \mathscr{S}(j,3)\binom{2}{j} = 0$.
So, $\mathscr{T}_{\mathscr{C}r}(2,3) = 6 + 6\left\{\mathscr{S}(2,2)\binom{2}{2} + 0\right\} = 12$.
Consequently, $\mathscr{T}_{\mathscr{C}r}(2,3) = 12$ *and these NCrTs having* 3*-NCrOSs are listed below*

$\{\phi_{\mathscr{N}}, \mathscr{A}_1, \mathscr{X}_{\mathscr{N}}\}$, $\{\phi_{\mathscr{N}}, \mathscr{A}_2, \mathscr{X}_{\mathscr{N}}\}$,$\{\phi_{\mathscr{N}}, \mathscr{A}_3, \mathscr{X}_{\mathscr{N}}\}$, $\{\phi_{\mathscr{N}}, \mathscr{A}_4, \mathscr{X}_{\mathscr{N}}\}$, $\{\phi_{\mathscr{N}}, \mathscr{A}_5, \mathscr{X}_{\mathscr{N}}\}$,$\{\phi_{\mathscr{N}}, \mathscr{A}_6, \mathscr{X}_{\mathscr{N}}\}$, $\{\phi_{\mathscr{N}}, \mathscr{A}_7, \mathscr{X}_{\mathscr{N}}\}$, $\{\phi_{\mathscr{N}}, \mathscr{A}_8, \mathscr{X}_{\mathscr{N}}\}$, $\{\phi_{\mathscr{N}}, \mathscr{A}_9, \mathscr{X}_{\mathscr{N}}\}$, $\{\phi_{\mathscr{N}}, \mathscr{A}_{10}, \mathscr{X}_{\mathscr{N}}\}$, $\{\phi_{\mathscr{N}}, \mathscr{A}_{11}, \mathscr{X}_{\mathscr{N}}\}$,$\{\phi_{\mathscr{N}}, \mathscr{A}_{12}, \mathscr{X}_{\mathscr{N}}\}$.

5.3 Neutrosophic Crisp Topological Spaces with 4-NCrOSs

The NCrT having 4-NCrOSs must have the form $\mathscr{T} = \{\phi_{\mathscr{N}}, \mathscr{A}, \mathscr{B}, \mathscr{X}_{\mathscr{N}}\}$, where $\mathscr{A} \neq \mathscr{B}$ such that $\mathscr{A} \cap \mathscr{B}, \mathscr{A} \cup \mathscr{B} \in \mathscr{T}$. To compute the number of NCrTs with exactly 4-NCrOSs, we need to compute formulae for following cases:

Case 1: $\mathscr{A} \cap \mathscr{B} = \phi_{\mathscr{N}}, \mathscr{A} \cup \mathscr{B} = \mathscr{X}_{\mathscr{N}}$
Case 2: $\mathscr{A} \cap \mathscr{B} = \phi_{\mathscr{N}},\ \mathscr{A} \cup \mathscr{B} = \phi_{\mathscr{N}}$
Case 3: ($\mathscr{A} \cap \mathscr{B} = \mathscr{A}$ or $\mathscr{B}$, $\mathscr{A} \cup \mathscr{B} = \phi_{\mathscr{N}}$) or
($\mathscr{A} \cap \mathscr{B} = \phi_{\mathscr{N}}, \mathscr{A} \cup \mathscr{B} = \mathscr{A}$ or $\mathscr{B}$)
Case 4: ($\mathscr{A} \cap \mathscr{B} = \mathscr{A}, \mathscr{A} \cup \mathscr{B} = \mathscr{A}$) or ($\mathscr{A} \cap \mathscr{B} = \mathscr{B}, \mathscr{A} \cup \mathscr{B} = \mathscr{B}$)
Case 5: ($\mathscr{A} \cap \mathscr{B} = \mathscr{A}, \mathscr{A} \cup \mathscr{B} = \mathscr{B}$) or ($\mathscr{A} \cap \mathscr{B} = \mathscr{B}, \mathscr{A} \cup \mathscr{B} = \mathscr{A}$).

Proposition 5.3.1

For a non-empty finite set $\mathscr{X}$ *with* $|\mathscr{X}| = n$*, the number of NCrTs having* 4*-NCrOSs satisfying the condition in case 1 is obtained by the formula*

$$\mathscr{S}(n,2)(2^n + 1).$$

Proof:

In general, the number of partitions of a non-empty set $\mathscr{X}$ with $|\mathscr{X}| = n$ into two blocks is given by $\mathscr{S}(n,2)$. To obtain $\mathscr{A} \cap \mathscr{B} = \phi_{\mathscr{N}}$ and $\mathscr{A} \cup \mathscr{B} = \mathscr{X}_{\mathscr{N}}$, clearly $\mathscr{A}$ and $\mathscr{B}$ must have the following two forms:

(i) $\mathscr{A} = \langle \mathscr{A}_1, \mathscr{A}_2, \emptyset \rangle$ & $\mathscr{B} = \langle \mathscr{B}_1, \mathscr{B}_2, \emptyset \rangle$,

(ii) $\mathscr{A} = \langle \mathscr{A}_1, \emptyset, \mathscr{A}_3 \rangle$ & $\mathscr{B} = \langle \mathscr{B}_1, \emptyset, \mathscr{B}_3 \rangle$.

Let us count the ways that they can be chosen.

(i) We have, $\mathscr{A} \cap \mathscr{B} = \langle \mathscr{A}_1 \cap \mathscr{B}_1, \mathscr{A}_2 \cap \mathscr{B}_2, \emptyset \rangle$, and $\mathscr{A} \cup \mathscr{B} = \langle \mathscr{A}_1 \cup \mathscr{B}_1, \mathscr{A}_2 \cap \mathscr{B}_2, \emptyset \rangle$. Now, to get $\mathscr{A} \cap \mathscr{B} = \phi_{\mathscr{N}}$ and $\mathscr{A} \cup \mathscr{B} = \mathscr{X}_{\mathscr{N}}$, we must have, $\mathscr{A}_1 \cap \mathscr{B}_1 = \emptyset$, $\mathscr{A}_1 \cup \mathscr{B}_1 = \mathscr{X}$ and $\mathscr{A}_2 \cap \mathscr{B}_2 = \emptyset$. This implies that $\mathscr{A}_1$, $\mathscr{B}_1$ is a partition of $\mathscr{X}$ and so, $\mathscr{B}_1 = \mathscr{X} - \mathscr{A}_1$. Therefore, $\mathscr{A}_1$, $\mathscr{B}_1$ can be chosen in $\mathcal{S}(n, 2)$ ways. Now, if $|\mathscr{A}_1| = i$ then $|\mathscr{B}_1| = n-i$. Since $\mathscr{A}_2 \cap \mathscr{B}_2 = \emptyset$, then the neutrosophic crisp subset $\mathscr{A}_2$ can be chosen out of $n-i$ elements in $\binom{n-i}{k}, k = 0, 1, 2, \dots, n-i$ ways with $k = 0$ representing the empty set. Therefore, $\mathscr{A}_2$ can be chosen in $\sum_{k=0}^{n-i} \binom{n-i}{k} = 2^{n-i}$ ways. Similarly, $\mathscr{B}_2$ can be chosen out of $n - (n - i) = i$ elements in $\sum_{k=0}^{i} \binom{i}{k} = 2^i$ ways. Hence, the total number of ways is $\mathcal{S}(n, 2).2^{n-i}.2^i = \mathcal{S}(n, 2).2^n$.

(ii) We have, $\mathscr{A} \cap \mathscr{B} = \langle \mathscr{A}_1 \cap \mathscr{B}_1, \emptyset, \mathscr{A}_3 \cup \mathscr{B}_3 \rangle$, and $\mathscr{A} \cup \mathscr{B} = \langle \mathscr{A}_1 \cup \mathscr{B}_1, \emptyset, \mathscr{A}_3 \cap \mathscr{B}_3 \rangle$. Now, to get $\mathscr{A} \cap \mathscr{B} = \phi_{\mathscr{N}}$ and $\mathscr{A} \cup \mathscr{B} = \mathscr{X}_{\mathscr{N}}$, we must have, $\mathscr{A}_1 \cap \mathscr{B}_1 = \emptyset$, $\mathscr{A}_3 \cup \mathscr{B}_3 = \mathscr{X}$ and $\mathscr{A}_1 \cup \mathscr{B}_1 = \mathscr{X}$, $\mathscr{A}_3 \cap \mathscr{B}_3 = \emptyset$ simultaneously. This shows that $\mathscr{A}_1$ and $\mathscr{B}_1$ is a partition of $\mathscr{X}$ and $\mathscr{A}_3 = \mathscr{A}_1^C = \mathscr{B}_1$, $\mathscr{B}_3 = \mathscr{B}_1^C = \mathscr{A}_1$. Therefore, we can take $\mathscr{A}_1$ and $\mathscr{B}_1$ or $\mathscr{A}_3$ and $\mathscr{B}_3$ in $\mathcal{S}(n, 2)$ ways.

From (i) and (ii), the total number of ways is $\mathcal{S}(n, 2)(2^n + 1)$.

Hence, the number of NCrTs having 4-NCrOSs satisfying the condition in case 1 is obtained by the formula

$$\mathcal{S}(n, 2)(2^n + 1).$$

Proposition 5.3.2

The number of NCrTs having 4-NCrOSs on a non-empty set $\mathscr{X}$ satisfying the condition in case 2 is obtained by the formula

$$\frac{n(n-1)}{2} + \{\mathscr{S}(n,2) \times 2^n\} + \sum_{i=3}^{n} \left\{ \binom{n}{i} \mathscr{S}(i,2) \right\}$$

where $|\mathscr{X}| = n$.

Proof:

To obtain $\mathscr{A} \cap \mathscr{B} = \phi_{\mathscr{N}}$ and $\mathscr{A} \cup \mathscr{B} = \phi_{\mathscr{N}}$, clearly, $\mathscr{A}$ and $\mathscr{B}$ must have the following two forms

(i) $\mathscr{A} = \langle \emptyset, \mathscr{A}_2, \mathscr{A}_3 \rangle$ & $\mathscr{B} = \langle \emptyset, \mathscr{B}_2, \mathscr{B}_3 \rangle$ such that $\mathscr{A}_3 \cup \mathscr{B}_3 = \mathscr{X}$ and $\mathscr{A}_3 \cap \mathscr{B}_3 = \emptyset$ and $\mathscr{A}_2 \cap \mathscr{B}_2 = \emptyset$.

(ii) $\mathscr{A} = \langle \emptyset, \mathscr{A}_2, \emptyset \rangle$ & $\mathscr{B} = \langle \emptyset, \mathscr{B}_2, \emptyset \rangle$ such that $\mathscr{A}_2 \cap \mathscr{B}_2 = \emptyset$.

From (i), $\mathscr{A} \cap \mathscr{B} = \langle \emptyset, \mathscr{A}_2 \cap \mathscr{B}_2, \mathscr{A}_3 \cup \mathscr{B}_3 \rangle$, and $\mathscr{A} \cup \mathscr{B} = \langle \emptyset, \mathscr{A}_2 \cap \mathscr{B}_2, \mathscr{A}_3 \cap \mathscr{B}_3 \rangle$. Since, $\mathscr{A}_3 \cup \mathscr{B}_3 = \mathscr{X}$ and $\mathscr{A}_3 \cap \mathscr{B}_3 = \emptyset$, which implies that $\mathscr{A}_3$ and $\mathscr{B}_3$ is a partition of $\mathscr{X}$ and say $\mathscr{B}_3 = \mathscr{X} - \mathscr{A}_3$. Therefore, $\mathscr{A}_3$ and $\mathscr{B}_3$ can be chosen in $\mathscr{S}(n,2)$ ways. Now, if $|\mathscr{A}_3| = i$, $|\mathscr{B}_3| = n-i$, $1 \leq i \leq n-1$, and $\mathscr{A}_2 \cap \mathscr{B}_2 = \emptyset$, then $\mathscr{A}_2$ can be chosen in $\sum_{k=0}^{n-i} \binom{n-i}{k} = 2^{n-i}$ ways, and similarly, $\mathscr{B}_2$ can be chosen out of $n-(n-i) = i$ elements in $\sum_{k=0}^{i} \binom{i}{k} = 2^i$ ways.

Therefore, the total number of ways is $\mathscr{S}(n,2) \times 2^{n-i} \times 2^i$ i.e., $\mathscr{S}(n,2) \times 2^n$.

From (ii), $\mathscr{A} \cap \mathscr{B} = \langle \emptyset, \mathscr{A}_2 \cap \mathscr{B}_2, \emptyset \rangle$, $\mathscr{A} \cup \mathscr{B} = \langle \emptyset, \mathscr{A}_2 \cap \mathscr{B}_2, \emptyset \rangle$, and $\mathscr{A}_2 \cap \mathscr{B}_2 = \emptyset$. If $|\mathscr{A}_2 \cup \mathscr{B}_2| = i$, $2 \leq i \leq n$, then $\mathscr{A}_2 \cup \mathscr{B}_2$ is chosen in $\binom{n}{i}$ different ways and then it is partitioned into two disjoint blocks: this is done in $\mathscr{S}(i,2)$ different ways. Therefore, the number of ways for form (ii) is $\sum_{i=2}^{n} \binom{n}{i} \mathscr{S}(i,2)$.

Hence, the number of NCrTs having 4-NCrOSs satisfying condition in case 2 is obtained by the formula

$$\{\mathscr{S}(n,2) \times 2^n\} + \sum_{i=2}^{n} \left\{ \binom{n}{i} \mathscr{S}(i,2) \right\}$$

i.e.,

$$\frac{n(n-1)}{2} + \{\mathscr{S}(n,2) \times 2^n\} + \sum_{i=3}^{n} \left\{ \binom{n}{i} \mathscr{S}(i,2) \right\}.$$

Proposition 5.3.3

For a non-empty finite set $\mathscr{X}$ with $|\mathscr{X}| = n$, the number of NCrTs having 4-NCrOSs satsifying conditions in case 3 is obtained by the formula $2(2^n - 2)^2$.

Proof:

There are two forms

(i) $\mathscr{A} = \langle \emptyset, \emptyset, \mathscr{A}_3 \rangle$ & $\mathscr{B} = \langle \emptyset, \mathscr{B}_2, \emptyset \rangle$,

(ii) $\mathscr{A} = \langle \mathscr{A}_1, \emptyset, \emptyset \rangle$ & $\mathscr{B} = \langle \emptyset, \mathscr{B}_2, \emptyset \rangle$.

Let us count the ways that they can be chosen.

Clearly, these two forms agree with the conditions in case 3, i.e., for the first kind, we have, $\mathscr{A} \cap \mathscr{B} = \langle \emptyset, \emptyset, \mathscr{A}_3 \rangle = \mathscr{A}$ and $\mathscr{A} \cup \mathscr{B} = \langle \emptyset, \emptyset, \emptyset \rangle = \phi_{\mathscr{N}}$, and for the second kind $\mathscr{A} \cap \mathscr{B} = \langle \emptyset, \emptyset, \emptyset \rangle = \phi_{\mathscr{N}}$ and $\mathscr{A} \cup \mathscr{B} = \langle \mathscr{A}_1, \emptyset, \emptyset \rangle = \mathscr{A}$. Now, since $\emptyset \subset \mathscr{A}_3 \subset \mathscr{X}$, $\emptyset \subset \mathscr{B}_2 \subset \mathscr{X}$ such that $|\mathscr{A}_3| = |\mathscr{B}_2| = i, 1 \leq i \leq n-1$ so, $\mathscr{A}_3$ and $\mathscr{B}_2$ are chosen in $\binom{n}{i}$ different ways. This implies that $\mathscr{A}$ and $\mathscr{B}$ are chosen in $\binom{n}{i}$ different ways. Therefore, the number of ways in this kind is $\left\{\sum_{i=1}^{n-1} \binom{n}{i}\right\} \times \left\{\sum_{i=1}^{n-1} \binom{n}{i}\right\} = \left(\sum_{i=1}^{n-1} \binom{n}{i}\right)^2 = (2^n - 2)^2$.

Similarly, the second kind is computed and is equal to $(2^n - 2)^2$.

Finally, the desired number of ways is $2\left(2^n - 2\right)^2$.

Proposition 5.3.4

For a non-empty set $\mathscr{X}$ with $|\mathscr{X}| = n$, the number of NCrTs having 4-NCrOSs satisfying condition in case 4 is obtained by the formula

$$\sum_{i=1}^{n-2} \left\{ \binom{n}{i} \left(2^{n-i} - 2\right) \right\} + 2\mathscr{T}_1 + 6\left(\mathscr{T}_2 + \mathscr{T}_3\right),$$

where $\mathscr{T}_k = \sum_{i=k}^{n-1} \left\{\binom{n}{i} \mathscr{S}(i,k) \left(2^{n-i} - 1\right)\right\}$, $k = 1, 2, 3$.

Proof:

Let $\mathscr{A} = \langle \mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3\rangle$ and $\mathscr{B} = \langle \mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3\rangle$. Then to satisfy the condition $(\mathscr{A} \cap \mathscr{B} = \mathscr{A}, \mathscr{A} \cup \mathscr{B} = \mathscr{A})$ or $(\mathscr{A} \cap \mathscr{B} = \mathscr{B}, \mathscr{A} \cup \mathscr{B} = \mathscr{B})$, we must have, $\mathscr{A}_1 = \mathscr{B}_1, \mathscr{A}_2 \subset \mathscr{B}_2, \mathscr{A}_3 = \mathscr{B}_3$ or $\mathscr{A}_1 = \mathscr{B}_1, \mathscr{B}_2 \subset \mathscr{A}_2, \mathscr{A}_3 = \mathscr{B}_3$ respectively . Then, we obtain four forms

(i) $\mathscr{A} = \langle \emptyset, \mathscr{A}_2, \emptyset\rangle$ & $\mathscr{B} = \langle \emptyset, \mathscr{B}_2, \emptyset, \rangle$ such that $\mathscr{A}_2 \subset \mathscr{B}_2$ or $\mathscr{B}_2 \subset \mathscr{A}_2$,

(ii) $\mathscr{A} = \langle \emptyset, \emptyset, \mathscr{A}_3\rangle$ & $\mathscr{B} = \langle \emptyset, \mathscr{B}_2, \mathscr{A}_3\rangle$ and $\mathscr{A} = \langle \mathscr{A}_1, \emptyset, \emptyset\rangle$ & $\mathscr{B} = \langle \mathscr{A}_1, \mathscr{B}_2, \emptyset\rangle$,

(iii) $\mathscr{A} = \langle \mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3\rangle$ & $\mathscr{B} = \langle \mathscr{A}_1, \mathscr{B}_2, \mathscr{A}_3\rangle$; exactly one of $\mathscr{A}_i, i = 1, 2, 3$ is $\emptyset$ and $\mathscr{A}_2 \subset \mathscr{B}_2$.

(iv) $\mathscr{A} = \langle \mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3\rangle$ & $\mathscr{B} = \langle \mathscr{A}_1, \mathscr{B}_2, \mathscr{A}_3\rangle$ such that all $\mathscr{A}_i, i = 1, 2, 3$ are non-empty and $\mathscr{A}_2 \subset \mathscr{B}_2$.

Let us count the ways that they can be chosen.

(i) Let $\mathscr{A}_2 \subset \mathscr{B}_2$ and if $|\mathscr{A}_2| = i, 1 \le i \le n-2$ then $i < |\mathscr{B}_2| = k \le n-1$. Therefore, $\mathscr{A}_2$ is chosen in $\binom{n}{i}$ ways and $\mathscr{B}_2$ is chosen in $\sum_{j=1}^{(n-i)-1} \binom{n-i}{j} = 2^{n-i} - 2$ different ways. Since, i varies from 1 to $n-2$, $\mathscr{A}_2$ and $\mathscr{B}_2$ are chosen in $\sum_{i=1}^{n-2} \left\{ \binom{n}{i} \left(2^{n-i} - 2\right)\right\}$ different ways. Hence, the neutrosophic crisp subsets $\mathscr{A}$ and $\mathscr{B}$ are chosen in $\sum_{i=1}^{n-2} \left\{ \binom{n}{i} \left(2^{n-i} - 2\right)\right\}$ different ways.

(ii) Let $|\mathscr{A}_3| = i, 1 \le i \le n-1$ then $\mathscr{A}_3$ is chosen in $\binom{n}{i}$ different ways then it is partitioned into one block: this is done in $\mathcal{S}(i, 1)$ different ways and hence $\mathscr{A}$. Next, in $\mathscr{B}$, $\mathscr{A}_3 \cap \mathscr{B}_2 = \emptyset$ and so, $\mathscr{B}_2$ is chosen from $n-i$ elements in $\sum_{j=1}^{n-i} \binom{n-i}{j} = 2^{n-i} - 1$ different ways and hence $\mathscr{B}$. Since i varies from 1 to $n-1$, we obtain $\sum_{i=1}^{n-1} \binom{n}{i} \mathcal{S}(i, 1)(2^{n-i} - 1)$ different ways for $\mathscr{A}$ and $\mathscr{B}$.

Similarly, for $\mathscr{A} = \langle \mathscr{A}_1, \emptyset, \emptyset\rangle$ & $\mathscr{B} = \langle \mathscr{A}_1, \mathscr{B}_2, \emptyset\rangle$, we have $\sum_{i=1}^{n-1} \binom{n}{i} \mathcal{S}(i, 1)(2^{n-i} - 1)$ different ways.

(iii) We have, $\mathscr{A}_1 \cap \mathscr{A}_3 = \emptyset$. If $|\mathscr{A}_1 \cup \mathscr{A}_3| = i, 2 \leq i \leq n-1$ then $\mathscr{A}_1, \mathscr{A}_3$ is chosen in $\binom{n}{i}\mathcal{S}(i,2)$ different ways. Since $\mathscr{A}_1 \cap \mathscr{B}_2 = \mathscr{A}_3 \cap \mathscr{B}_2 = \emptyset$, so, $\mathscr{B}_2$ is chosen in $\binom{n-i}{j}, 1 \leq j \leq n-i$ different ways. Therefore, $\mathscr{B}_2$ is chosen in $\sum_{j=1}^{n-i}\binom{n-i}{j} = 2^{n-i} - 1$ different ways. Together $\mathscr{A}$ and $\mathscr{B}$ is chosen in $\sum_{i=2}^{n-1}\binom{n}{i}\mathcal{S}(i,2)(2^{n-i}-1)$ different ways. It is known that we can arrange three element into three places in six different ways, so, $\mathscr{A}$ has six forms, as three components of $\mathscr{A}$ are the neutrosophic crisp subsets $\mathscr{A}_1, \mathscr{A}_3$ and $\emptyset$.

Hence, the total number of ways to choose $\mathscr{A}$ and $\mathscr{B}$ is

$$6\sum_{i=2}^{n-1}\binom{n}{i}\mathcal{S}(i,2)(2^{n-i}-1).$$

(iv) We have, $\mathscr{A}_1 \cap \mathscr{A}_2 = \mathscr{A}_1 \cap \mathscr{A}_3 = \mathscr{A}_2 \cap \mathscr{A}_3 = \emptyset$. If $|\mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3| = i, 3 \leq i \leq n-1$ then $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ are chosen in $\binom{n}{i}\mathcal{S}(i,3)$ different ways. Since $\mathscr{A}_1 \cap \mathscr{B}_2 = \mathscr{A}_3 \cap \mathscr{B}_2 = \emptyset$, so, $\mathscr{B}_2$ is chosen in $\binom{n-i}{j}, 1 \leq j \leq n-i$ different ways. Therefore, $\mathscr{B}_2$ is chosen in $\sum_{j=1}^{n-i}\binom{n-i}{j} = 2^{n-i} - 1$ different ways. Together $\mathscr{A}$ and $\mathscr{B}$ are chosen in $\sum_{i=3}^{n-1}\binom{n}{i}\mathcal{S}(i,3)2^{n-i} - 1$ different ways. It is known that we can arrange three elements into the three places in six different ways, so, $\mathscr{A}$ has 6 forms, as three components of $\mathscr{A}$ are different neutrosophic crisp subsets $\mathscr{A}_1, \mathscr{A}_2$ and $\mathscr{A}_3$.

Hence, the total number of ways to choose $\mathscr{A}$ and $\mathscr{B}$ is

$$6\sum_{i=3}^{n-1}\binom{n}{i}\mathcal{S}(i,3)(2^{n-i}-1).$$

Hence, we have the total

$$\sum_{i=1}^{n-2}\left\{\binom{n}{i}\left(2^{n-i}-2\right)\right\} + \sum_{i=1}^{n-1}\binom{n}{i}\mathcal{S}(i,1)(2^{n-i}-1) +$$

$$6\sum_{i=2}^{n-1}\binom{n}{i}\mathcal{S}(i,2)(2^{n-i}-1)+6\sum_{i=3}^{n-1}\binom{n}{i}\mathcal{S}(i,3)(2^{n-i}-1).$$

i.e.,

$$\sum_{i=1}^{n-2}\left\{\binom{n}{i}\left(2^{n-i}-2\right)\right\}+2\mathscr{T}_1+6\left(\mathscr{T}_2+\mathscr{T}_3\right),$$

where $\mathscr{T}_k=\sum_{i=k}^{n-1}\left\{\binom{n}{i}\mathcal{S}(i,k)\left(2^{n-i}-1\right)\right\}$, $k=1,2,3$. This formula gives the number of NCrTs having 4-NCrOSs satisfying condition in case 4.

Proposition 5.3.5

For a non-empty set $\mathscr{X}$ with $|\mathscr{X}|=n$, the number of NCrTs having 4-NCrOSs satisfying condition in case 5 is obtained by the formula

$$\sum_{i=1}^{n-1}\binom{n}{i}\left[(2^n-2)+2\left\{\left(\sum_{j=1}^{i-1}\binom{i}{j}2^{n-j}\right)+(2^{n-i}-1)\right\}\right]+$$

$$2\sum_{i=1}^{n-1}\binom{n}{i}(2^{n-i}-1)+\sum_{i=1}^{n-1}\binom{n}{i}(2^{n-i}-1)^2+2\mathscr{T}_n+\sum_{i=0}^{n-2}\binom{n}{i}\mathscr{T}_{n-i},$$

where $\mathscr{T}_n=\sum_{i=1}^{n-2}\binom{n}{i}\left\{\sum_{k=1}^{n-(i+1)}\binom{n-i}{k}(2^{n-(i+k)}-1)\right\}+$
$\sum_{i=1}^{n-2}\binom{n}{i}\left[\sum_{j=1}^{n-i}\binom{n-i}{j}\left\{\sum_{k=1}^{j}\binom{j}{k}(2^{n-(i+j)}-1)\right\}\right]$

or

$\mathscr{T}_n=\sum_{i=1}^{n-2}\binom{n}{i}\left\{\sum_{k=1}^{n-(i+1)}\binom{n-i}{k}(2^{n-(i+k)}-1)\right\}+$
$\sum_{i=1}^{n-2}\binom{n}{i}\left[\sum_{j=1}^{n-i}\left\{\sum_{k=j}^{n-i}\binom{n-i}{j}\binom{j}{k}\right\}(2^{n-(i+j)}-1)\right].$

Proof:

Here, the second component must always match to satisfy the conditions in case 5.

For $\mathscr{A}=\langle\emptyset,\emptyset,\mathscr{A}_3\rangle$ we can choose $\mathscr{B}$ in two forms which are $\mathscr{B}=\langle\mathscr{B}_1,\emptyset,\emptyset\rangle$ and $\mathscr{B}=\langle\mathscr{B}_1,\emptyset,\mathscr{B}_3\rangle$ such that $\mathscr{B}_3\subseteq\mathscr{A}_3$. For this kind of $\mathscr{A}$ we have $\binom{n}{i}$ different ways. For each $\mathscr{A}$, we can choose $\mathscr{B}=\langle\mathscr{B}_1,\emptyset,\emptyset\rangle$ in 2^n-2 different ways. Next if $\mathscr{B}_3\subset\mathscr{A}_3$, say $|\mathscr{B}_3|=j<i=|\mathscr{A}_3|$,

we can choose $\mathscr{B}$ in $\sum_{j=1}^{i-1}\binom{i}{j}2^{n-j}$ different ways and if $\mathscr{B}_3 = \mathscr{A}_3$, say $|\mathscr{B}_3| = |\mathscr{A}_3| = i$, then $\mathscr{B}$ can be chosen in $2^{n-i}-1$ different ways. Similarly, for $\mathscr{A} = \langle \mathscr{A}_1, \emptyset, \emptyset\rangle$, we have same number of choices for $\mathscr{B}$ satisfying conditions in case 5.
Therefore, in this part we have

$$\sum_{i=1}^{n-1}\binom{n}{i}\left[(2^n-2)+2\left\{\left(\sum_{j=1}^{i-1}\binom{i}{j}2^{n-j}\right)+(2^{n-i}-1)\right\}\right]$$

different ways.

For $\mathscr{A} = \langle \emptyset, \mathscr{A}_2, \emptyset\rangle$, we can choose $\mathscr{B} = \langle \emptyset, \mathscr{A}_2, \mathscr{B}_3\rangle$ and $\mathscr{B} = \langle \mathscr{B}_1, \mathscr{A}_2, \emptyset\rangle$. Since $\mathscr{A}_2$ can be chosen in $\binom{n}{i}, i = 1, 2, \ldots, n-1$ different ways then $\mathscr{B}_3$ can be chosen in $2^{n-i}-1$ different ways for each i and therefore, $\mathscr{B}$. As we have two forms of $\mathscr{B}$ and are symmetric, and i varies from 1 to $n-1$, we have the total $2\sum_{i=1}^{n-1}\binom{n}{i}(2^{n-i}-1)$.

For $\mathscr{A} = \langle \emptyset, \mathscr{A}_2, \mathscr{A}_3\rangle$, we can choose $\mathscr{B} = \langle \mathscr{B}_1, \mathscr{A}_2, \mathscr{B}_3\rangle, \mathscr{A}_3 \subseteq \mathscr{B}_3$ and $\mathscr{B}_1$ is any subset of $\mathscr{X}$ different from $\mathscr{A}_2$ and $\mathscr{B}_3$. Then $\mathscr{A}$ and $\mathscr{B}$ can be chosen in $\sum_{i=1}^{n-2}\binom{n}{i}\left\{\sum_{k=1}^{n-(i+1)}\binom{n-i}{k}(2^{n-(i+k)}-1)\right\}+$ $\sum_{i=1}^{n-2}\binom{n}{i}\left[\sum_{j=1}^{n-i}\binom{n-i}{j}\left\{\sum_{k=1}^{j}\binom{j}{k}(2^{n-(i+j)}-1)\right\}\right]$ different ways. Let us take $\sum_{i=1}^{n-2}\binom{n}{i}\left\{\sum_{k=1}^{n-(i+1)}\binom{n-i}{k}(2^{n-(i+k)}-1)\right\}+$ $\sum_{i=1}^{n-2}\binom{n}{i}\left[\sum_{j=1}^{n-i}\binom{n-i}{j}\left\{\sum_{k=1}^{j}\binom{j}{k}(2^{n-(i+j)}-1)\right\}\right] = \mathscr{T}_n$ for further use. Also, for $\mathscr{A} = \langle \mathscr{A}_1, \mathscr{A}_2, \emptyset\rangle$, we have equal number of choices as it is symmetric to $\mathscr{A} = \langle \emptyset, \mathscr{A}_2, \mathscr{A}_3\rangle$. Hence, a total of $2\mathscr{T}_n$ different ways.

For $\mathscr{A} = \langle \emptyset, \mathscr{A}_2, \mathscr{A}_3\rangle$, we can also choose $\mathscr{B} = \langle \mathscr{B}_1, \mathscr{A}_2, \emptyset\rangle$ which can be done in $\sum_{i=1}^{n-1}\binom{n}{i}(2^{n-i}-1)^2$ different ways.

For $\mathscr{A} = \langle \mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3\rangle$, we can choose $\mathscr{B} = \langle \mathscr{B}_1, \mathscr{A}_2, \mathscr{B}_3\rangle$ such that $\mathscr{A}_1 \subseteq \mathscr{B}_1, \mathscr{B}_3 \subseteq \mathscr{A}_3$ and $|\mathscr{A}_2| = i, 0 \leq i \leq n-2$. If $|\mathscr{A}_2| = 0$ i.e., $\mathscr{A}_2 = \emptyset$ then $\mathscr{B}$ can be chosen in $\binom{n}{0}\mathscr{T}_n$ different ways. Further, if $|\mathscr{A}_2| = 1$ then $\mathscr{B}$ can be chosen in $\binom{n}{1}\mathscr{T}_{n-1}$ different ways. Continuing in the similar way for $|\mathscr{A}_2| = n-2$, we have $\binom{n}{n-2}\mathscr{T}_{n-(n-2)}$ i.e., $\binom{n}{n-2}\mathscr{T}_2$ different ways. Thus, for $\mathscr{A} = \langle \mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3 \rangle$, we can choose $\mathscr{B}$ in $\sum_{i=0}^{n-2} \binom{n}{i} \mathscr{T}_{n-i}$ different ways.

Hence, the number of NCrTs having 4-NCrOSs satisfying conditions in case 5 is obtained by the formula

$$\sum_{i=1}^{n-1} \binom{n}{i} \left[(2^n - 2) + 2 \left\{ \left(\sum_{j=1}^{i-1} \binom{i}{j} 2^{n-j} \right) + (2^{n-i} - 1) \right\} \right] +$$

$$2 \sum_{i=1}^{n-1} \binom{n}{i} (2^{n-i} - 1) + \sum_{i=1}^{n-1} \binom{n}{i} (2^{n-i} - 1)^2 + 2\mathscr{T}_n + \sum_{i=0}^{n-2} \binom{n}{i} \mathscr{T}_{n-i}.$$

Example 5.3.1

The follwing table gives the number of NCrTs having 4*-NCrOSs for* $\mathscr{X} \leq 5$.

Table 5.1: Number of NCrTSs having 4-NCrOSs on $\mathscr{X}$

$\mathscr{A} \cap \mathscr{B}, \mathscr{A} \cup \mathscr{B}$	Number of NCrTSs having 4-NCrOSs on $\mathscr{X}$				
	$\lvert\mathscr{X}\rvert = 1$	$\lvert\mathscr{X}\rvert = 2$	$\lvert\mathscr{X}\rvert = 3$	$\lvert\mathscr{X}\rvert = 4$	$\lvert\mathscr{X}\rvert = 5$
Case 1: $\mathscr{A} \cap \mathscr{B} = \phi_{\mathscr{N}}$, $\mathscr{A} \cup \mathscr{B} = \mathscr{X}_{\mathscr{N}}$	0	5	27	119	495
Case 2: $\mathscr{A} \cap \mathscr{B} = \phi_{\mathscr{N}}$, $\mathscr{A} \cup \mathscr{B} = \phi_{\mathscr{N}}$	0	5	30	137	570
Case 3: $\mathscr{A} \cap \mathscr{B} = \mathscr{A}$, $\mathscr{A} \cup \mathscr{B} = \phi_N$	0	8	72	392	1800
Case 4: $\mathscr{A} \cap \mathscr{B} = \mathscr{A}$, $\mathscr{A} \cup \mathscr{B} = \mathscr{A}$	0	4	48	340	2040
Case 5: $\mathscr{A} \cap \mathscr{B} = \mathscr{A}$, $\mathscr{A} \cup \mathscr{B} = \mathscr{B}$	0	14	216	1958	15240
The total number of NCrTSs having 4-NCrOSs on $\mathscr{X}$	0	36	393	2946	20145

Suppose, $\mathscr{X} = \{a, b\}$ *i.e.,* $|\mathscr{X}| = 2$*, then from Table 5.1, we have,* $\mathscr{T}_{\mathscr{C}r}(2, 4) = 36$*. These are*

For Case 1:

$$\{\phi_{\mathscr{N}}, \mathscr{X}_{\mathscr{N}}, A_3 = \langle \{a\}, \emptyset, \emptyset \rangle, A_6 = \langle \{b\}, \emptyset, \emptyset \},$$
$$\{\phi_{\mathscr{N}}, \mathscr{X}_{\mathscr{N}}, A_3 = \langle \{a\}, \emptyset, \emptyset, A_{12} = \langle \{b\}, \{a\}, \emptyset \rangle \},$$
$$\{\phi_{\mathscr{N}}, \mathscr{X}_{\mathscr{N}}, A_6 = \langle \{b\}, \emptyset, \emptyset \rangle, A_9 = \langle \{a\}, \{b\}, \emptyset \rangle \},$$

$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_8 = \langle \{a\}, \emptyset, \{b\} \rangle, A_{11} = \langle \{b\}, \emptyset, \{a\} \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_9 = \langle \{a\}, \{b\}, \emptyset \rangle, A_{12} = \langle \{b\}, \{a\}, \emptyset \rangle\}.$$

For Case 2:

$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_2 = \langle \emptyset, \{a\}, \emptyset \rangle, A_5 = \langle \emptyset, \{b\}, \emptyset \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_1 = \langle \emptyset, \emptyset, \{a\} \rangle, A_4 = \langle \emptyset, \emptyset, \{b\} \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_1 = \langle \emptyset, \emptyset, \{a\} \rangle, A_7 = \langle \emptyset, \{a\}, \{b\} \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_4 = \langle \emptyset, \emptyset, \{b\} \rangle, A_{10} = \langle \emptyset, \{b\}, \{a\} \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_7 = \langle \emptyset, \{a\}, \{b\} \rangle, A_{10} = \langle \emptyset, \{b\}, \{a\} \rangle\}.$$

For Case 3:

$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_1 = \langle \emptyset, \emptyset, \{a\} \rangle, A_2 = \langle \emptyset, \{a\}, \emptyset \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_1 = \langle \emptyset, \emptyset, \{a\} \rangle, A_5 = \langle \emptyset, \{b\}, \emptyset \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_4 = \langle \emptyset, \emptyset, \{b\} \rangle, A_2 = \langle \emptyset, \{a\}, \emptyset \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_4 = \langle \emptyset, \emptyset, \{b\} \rangle, A_5 = \langle \emptyset, \{b\}, \emptyset \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_3 = \langle \{a\}, \emptyset, \emptyset \rangle, A_2 = \langle \emptyset, \{a\}, \emptyset \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_3 = \langle \{a\}, \emptyset, \emptyset \rangle, A_5 = \langle \emptyset, \{b\}, \emptyset \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_6 = \langle \{b\}, \emptyset, \emptyset \rangle, A_2 = \langle \emptyset, \{a\}, \emptyset \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_6 = \langle \{b\}, \emptyset, \emptyset \rangle, A_5 = \langle \emptyset, \{b\}, \emptyset \rangle\}.$$

For Case 4:

$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_1 = \langle \emptyset, \emptyset, \{a\} \rangle, A_{10} = \langle \emptyset, \{b\}, \{a\} \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_3 = \langle \{a\}, \emptyset, \emptyset \rangle, A_9 = \langle \{a\}, \{b\}, \emptyset \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_4 = \langle \emptyset, \emptyset, \{b\} \rangle, A_7 = \langle \emptyset, \{a\}, \{b\} \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_6 = \langle \{b\}, \emptyset, \emptyset \rangle, A_{12} = \langle \{b\}, \{a\}, \emptyset \rangle\}.$$

For Case 5:

$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_1 = \langle \emptyset, \emptyset, \{a\} \rangle, A_3 = \langle \{a\}, \emptyset, \emptyset \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_1 = \langle \emptyset, \emptyset, \{a\} \rangle, A_6 = \langle \{b\}, \emptyset, \emptyset \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_4 = \langle \emptyset, \emptyset, \{b\} \rangle, A_3 = \{a\}, \emptyset, \emptyset \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_4 = \langle \emptyset, \emptyset, \{b\} \rangle, A_6 = \langle \{b\}, \emptyset, \emptyset \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_7 = \langle \emptyset, \{a\}, \{b\} \rangle, A_2 = \langle \emptyset, \{a\}, \emptyset \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_7 = \langle \emptyset, \{a\}, \{b\} \rangle, A_{12} = \langle \{b\}, \{a\}, \emptyset \rangle\},$$
$$\{\phi_{\mathcal{N}}, \mathscr{X}_{\mathcal{N}}, A_{12} = \langle \{b\}, \{a\}, \emptyset \rangle, A_2 = \langle \emptyset, \{a\}, \emptyset \rangle\},$$

$\{\phi_{\mathscr{N}}, \mathscr{X}_{\mathscr{N}}, A_9 = \langle\{a\}, \{b\}, \emptyset\rangle, A_5 = \langle\emptyset, \{b\}, \emptyset\rangle\},$
$\{\phi_{\mathscr{N}}, \mathscr{X}_{\mathscr{N}}, A_9 = \langle\{a\}, \{b\}, \emptyset\rangle, A_{10} = \langle\emptyset, \{b\}, \{a\}\rangle\},$
$\{\phi_{\mathscr{N}}, \mathscr{X}_{\mathscr{N}}, A_{10} = \langle\emptyset, \{b\}, \{a\}\rangle, A_5 = \langle\emptyset, \{b\}, \emptyset\rangle\},$
$\{\phi_{\mathscr{N}}, \mathscr{X}_{\mathscr{N}}, A_8 = \langle\{a\}, \emptyset, \{b\}\rangle, A_3 = \langle\{a\}, \emptyset, \emptyset\rangle\},$
$\{\phi_{\mathscr{N}}, \mathscr{X}_{\mathscr{N}}, A_8 = \langle\{a\}, \emptyset, \{b\}\rangle, A_4 = \langle\emptyset, \emptyset, \{b\}\rangle\},$
$\{\phi_{\mathscr{N}}, \mathscr{X}_{\mathscr{N}}, A_{11} = \langle\{b\}, \emptyset, \{a\}\rangle, A_1 = \langle\emptyset, \emptyset, \{a\}\rangle\},$
$\{\phi_{\mathscr{N}}, \mathscr{X}_{\mathscr{N}}, A_{11} = \langle\{b\}, \emptyset, \{a\}\rangle, A_6 = \langle\{b\}, \emptyset, \emptyset\rangle\}.$

As a result, we have the total $\mathscr{T}_{\mathscr{C}r}(2, 4) = 36$*.*