Chapter 4

Five Dimensional Exact Bianchi Type-I Cosmological Models within the Framework of Sáez-Ballester Theory

4.1 Introduction

The universe is spherically symmetric in its current state of evolution and its matter distribution is mostly isotropic and homogeneous. However, it could not have had such a smoothed out picture in its early phases of evolution because the types of matter fields in the early universe are uncertain. Furthermore, no observable evidence exist that guarantee in an epoch prior to recombination. As a result, anisotropy is a very natural phenomenon to explore in order to solve difficulties like the local anisotropies that we see today in galaxies, clusters and superclusters. These anisotropies could be caused by a variety of factors, including cosmological magnetic or electric fields, long-wavelength gravitational waves and Yang-Mills fields (Barrow, 1997).

Because they are isotropic and homogenous, Friedmann-Robertson-Walker (FRW) models best approximate the current universe's large scale structure. However, models with anisotropic backgrounds are adequate for describing the early stages of the universe's evolution. One of the most basic models of the universe with anisotropic background that represents a spatially homogenous and flat universe is the Bianchi type-I model, which is a straightforward generalisation of the flat FRW model. The

Bianchi type-I universe has the unusual trait of behaving like a Kasner universe near the singularity even in the presence of matter and so falls under Belinskii et al. (Belinskii et al., 1970)'s general analysis of the singularity. It has also been proved that in a world packed with matter with $p = \gamma\rho$, $\gamma < 1$, any initial singularity in a Bianchi type-I universe quickly dies away and a Bianchi type-I universe evolves into a FRW universe (Marciano, 1968). This property of the Bianchi type-I universe makes it an excellent option for investigating the effects of anisotropy in the early universe on modern data. Anisotropic Bianchi type-I models have been studied by a number of authors (Kumar and Singh, 2007; Pradhan and Chouhan, 2011; Singh and et al., 2020) in various circumstances.

4.2 The metric and field equations

We consider the metric for this problem as given below

$$ds^2 = -dt^2 + A^2(dx^2 + dy^2) + B^2dz^2 + C^2d\psi^2 \tag{4.1}$$

where *A*, *B* and *C* are arbitrary function of cosmic time.

The scalar-tensor field equations in the S$\acute{a}$ez-Ballester theory are given by

$$R_{ij} - \frac{1}{2}g_{ij}R - \omega\phi^m(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}) = -T_{ij} \tag{4.2}$$

where the scalar field ϕ satisfies the equation

$$2\phi^m\phi^{;i}_{;i} + m\phi^{m-1}\phi_{,k}\phi^{,k} = 0 \tag{4.3}$$

Here T_{ij} is stress energy tensor of matter, comma and Semicolon represents partial and covariant differentiation respectively.

Also we have

$$T^{ij}_{;i} = 0 \tag{4.4}$$

which is a consequence of the field equation (4.2) and (4.3).

We consider the energy momentum tensor T_{ij} for perfect fluid in the following form

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij} \tag{4.5}$$

where ρ, p denotes the energy density and pressure of the fluid. Also u^i is the five-velocity vector satisfying $u^i u_i = 1$.

For this chapter we define the spatial volume *V*, expansion scalar θ, Hubble's parameter *H*, the deceleration parameter *q*, the shear scalar σ^2 and the main isotropy parameter $\triangle$ for the metric (4.1) are as follows

$$V = a^4 = A^2 BC \tag{4.6}$$

$$\theta = u^i_{;i} = 2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \tag{4.7}$$

$$H = \frac{\dot{a}}{a} = \frac{1}{4}(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}) \tag{4.8}$$

$$q = -\frac{a\ddot{a}}{\dot{a^2}} \tag{4.9}$$

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{2}(2\frac{\dot{A^2}}{A^2} + \frac{\dot{B^2}}{B^2} + \frac{\dot{C^2}}{C^2}) - \frac{\theta^2}{8} \tag{4.10}$$

$$\triangle = \frac{1}{4}\Sigma_{i=1}^{4}(\frac{H_i - H}{H})^2 \tag{4.11}$$

where an overhead dot represents differentiation with respect to cosmic time *t*.

For a co-moving coordinate system, the field equation (4.2), (4.3) for the metric (4.1) with the help of equation (4.5) can be written as follows

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{C}}{C} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} = -p + \omega\phi^m\frac{\dot{\phi}^2}{2} \tag{4.12}$$

$$2\frac{\ddot{A}}{A} + \frac{\dot{A^2}}{A^2} + \frac{\ddot{C}}{C} + 2\frac{\dot{A}}{A}\frac{\dot{C}}{C} = -p + \omega\phi^m\frac{\dot{\phi}^2}{2} \tag{4.13}$$

$$2\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A^2}}{A^2} + 2\frac{\dot{A}}{A}\frac{\dot{B}}{B} = -p + \omega\phi^m\frac{\dot{\phi}^2}{2} \tag{4.14}$$

$$\frac{\dot{A}^2}{A^2} + 2\frac{\dot{A}}{A}\frac{\dot{B}}{B} + 2\frac{\dot{A}}{A}\frac{\dot{C}}{C} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} = \rho - \omega\phi^m\frac{\dot{\phi}^2}{2} \tag{4.15}$$

$$\ddot{\phi} + \dot{\phi}(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}) + \frac{m}{2}\frac{\dot{\phi}^2}{\phi} = 0 \tag{4.16}$$

$$\dot{\rho} + (\rho + p)(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}) = 0 \tag{4.17}$$

The field equations (4.12-4.16) are a set of five equations with six unknown variables: *A*, *B*, *C*, *p*, ρ and ϕ. So, in order to achieve complete determinacy of the system, we need one more relation among the variables, which we will acquire in the following section by solving the field equations using a particular rule of variation for Hubble's parameter provided by the authors (Kumar and Singh, 2007).

4.3 Solution of the field equations

The field equations (4.12-4.16) are a system of connected very non-linear equations. We often assume a form for the matter content or that the space-time supports killer vector symmetries in order to solve the field equations. The field equations can also be solved using a rule of variation for Hubble's parameter, which was first proposed in FRW models by Berman (Berman, 1983a) and yields a constant value of deceleration parameter. In anisotropic Bianchi type-I space-time, the authors (Kumar and Singh, 2007) developed a similar law of variation for Hubble's parameter, which also provides a constant value of deceleration parameter. The observed fluctuations do not contradict the assumed fluctuation for Hubble's parameter.

The FRW metric has been applied to the most well-known versions of Einstein's theory and Brans-Dicke theory. Several authors have studied cosmological models with constant deceleration parameter in the literature (Bishi and et al., 2017; Kumar and Singh, 2007; Singh and Kumar, 2006). In scalar-tensor and scale covariant theories of gravity, Reddy et al. (Reddy et al., 2007, 2006b) proposed LRS Bianchi type-I models with constant deceleration parameter. The authors of this work looked into

LRS Bianchi type-II models in general relativity with constant deceleration parameter, Guth's inflationary theory, and self-creation theory of gravitation (Singh and Kumar, 2006, 2007a,b).

Using a specific law of variation for Hubble's parameter that provides a constant value of deceleration parameter, the authors (Kumar and Singh, 2007) have found a class of exact solutions for a spatially homogenous and in general relativity, as well as anisotropic Bianchi type-I space-time with perfect fluid.

The variation law of Hubble's parameter is

$$H = La^{-n} = L(A^2BC)^{-\frac{4}{n}} \tag{4.18}$$

here $L \geq 0$ and $n > 0$ are constants

The deceleration parameter q is given by

$$q = -\frac{a\ddot{a}}{\dot{a}^2} \tag{4.19}$$

From equation (4.8) and (4.18), we have

$$\frac{\dot{a}}{a} = La^{-n} \tag{4.20}$$

Integrating euation (4.20), we get

$$a = (nLt + k_1)^{\frac{1}{n}} \quad for \quad n \neq 0 \tag{4.21}$$

and

$$a = k_2 e^{Lt} \quad for \quad n = 0 \tag{4.22}$$

k_1 and k_2 are constants of integration.

Using (4.21) into (4.19) we get,

$$q = n - 1 \tag{4.23}$$

This demonstrates that the law (4.18) leads to a constant value of deceleration parameter. We may get the following three relations by subtracting (4.12) from (4.13), (4.12) from

(4.14), and (4.13) from (4.14) and taking the second integral of each

$$\frac{A^2}{B} = l_1 e^{(c_1 \int a^{-4}dt)} \tag{4.24}$$

$$\frac{A^2}{C} = l_2 e^{(c_2 \int a^{-4}dt)} \tag{4.25}$$

$$\frac{B}{C} = l_3 e^{(c_3 \int a^{-4}dt)} \tag{4.26}$$

l_1, l_2, l_3, c_1, c_2 and c_3 are integration constants..

The metric functions from (4.24-4.26) can be defined as

$$A = a^{\frac{2}{3}} x_1 e^{d_1 \int a^{-4}dt} \tag{4.27}$$

$$B = a^{\frac{4}{3}} x_2 e^{d_2 \int a^{-4}dt} \tag{4.28}$$

$$C = a^{\frac{4}{3}} x_3 e^{d_3 \int a^{-4}dt} \tag{4.29}$$

Here,

$x_1 = \sqrt[6]{l_1 l_2}, \quad x_2 = \sqrt[3]{l_3 l_1^{-1}}$ and $x_3 = \sqrt[3]{(l_2 l_3)^{-1}}$

and

$d_1 = \frac{c_1+c_2}{6}, \quad d_2 = \frac{c_3-c_1}{3}$ and $d_3 = -\frac{(c_2+c_3)}{3}$

where these constants fulfil the following two relationships

$$x_1 x_2 x_3 = 0 \quad and \quad d_1 + d_2 + d_3 = 0 \tag{4.30}$$

(4.16)'s second integral gives us

$$\phi = [\frac{h(m+2)}{2} \int a^{-4}dt]^{\frac{2}{m+2}} \tag{4.31}$$

4.3.1 Case(I): for $n \neq 0$,

The following scale factor expressions are obtained by using (4.21) in (4.27-4.29)

$$A = x_1(nLt+k_1)^{\frac{2}{3n}} e^{\frac{d_1(nLt+k_1)^{\frac{n-4}{n}}}{L(n-4)}} \tag{4.32}$$

$$B = x_2(nLt+k_1)^{\frac{4}{3n}} e^{\frac{d_2(nLt+k_1)^{\frac{n-4}{n}}}{L(n-4)}} \tag{4.33}$$

$$C = x_3(nLt+k_1)^{\frac{4}{3n}} e^{\frac{d_3(nLt+k_1)^{\frac{n-4}{n}}}{L(n-4)}} \tag{4.34}$$

The scalar field is given by using (4.21) in (4.31)

$$\phi = [\frac{h(m+2)}{2L(n-4)}]^{\frac{2}{m+2}} [nLt+k_1]^{\frac{2(n-4)}{n(m+2)}} \tag{4.35}$$

The pressure and energy density of the model are as follows when (4.32)-(4.35) are substituted in (4.14) and (4.15)

$$p = \frac{L^2}{9}(24n-44)(nLt+k_1)^{-2} - (3d_1^2+d_2^2+2d_1d_2 - \frac{\omega h^2}{2})(nLt+k_1)^{-\frac{8}{n}} + \frac{4}{3}Ld_1(nLt+k_1)^{-1-\frac{4}{n}} \tag{4.36}$$

$$\rho = (d_1^2+2d_1d_2+2d_1d_3+d_2d_3+\frac{\omega h^2}{2})(nLt+k_1)^{-\frac{8}{n}} + \frac{52}{9}L^2(nLt+k_1)^{-2} + 4Ld_1(nLt+k_1)^{-1-\frac{4}{n}} \tag{4.37}$$

Using (4.30), the solutions (4.32-4.37) fulfil the energy conservation equation (4.17) in the same way and hence represents exact solutions of Einstein's field equations (4.12-4.16). Furthermore, if we assume $x_2 = x_3$ and $d_2 = d_3$, i.e. $B = C$, the above solutions reduce to the Reddy et al. (Reddy et al., 2006b) solutions. As a result, the model presented above generalises the model explored by Reddy et al. (Reddy et al., 2006b).

The Hubble's parameter is given by

$$H = \frac{d_1}{4}(nLt+k_1)^{-\frac{4}{n}} + L(nLt+k_1)^{-1} \tag{4.38}$$

The expansion scalar is define by

$$\theta = \frac{3d_1}{4}(nLt + k_1)^{-\frac{4}{n}} + 3L(nLt + k_1)^{-1} \tag{4.39}$$

The anisotropy parameter $\triangle$ is calculated as follows

$$\begin{aligned}\triangle = \frac{1}{4}[\frac{d_1}{4}(nLt + k_1)^{-\frac{4}{n}} + L(nLt + k_1)^{-1}]^{-2} \\ [(\frac{5}{4}d_1^2 + d_2^2 + d_3^2 - \frac{d_1 d_2}{2} - \frac{d_1 d_3}{2})(nLt + k_1)^{-\frac{8}{n}} \\ + \frac{4}{9}L^2(nLt + k_1)^{-2} + \frac{2L}{3}(d_2 + d_3 - 2d_1)(nLt + k_1)^{-1-\frac{4}{n}}]\end{aligned} \tag{4.40}$$

volume is given by

$$V = (nLt + k_1)^{\frac{4}{n}} \tag{4.41}$$

and the shear scalar is

$$\begin{aligned}\sigma^2 = \frac{(nLt + k_1)^{-\frac{8}{n}}}{2}(\frac{247}{128}d_1^2 + d_2^2 + d_3^2) + \frac{79}{9}L^2(nLt + k_1)^{-2} \\ - \frac{9}{16}d_1 L(nLt + k_1)^{-1-\frac{4}{n}}\end{aligned} \tag{4.42}$$

4.3.2 Case(II): for $n = 0$,

The model's scale factors are as follows when using (4.22) in (4.27-4.29)

$$A = x_1 k_2^{\frac{2}{3}} e^{(\frac{2Lt}{3} - \frac{d_1}{4Lk_2^4}e^{-4Lt})} \tag{4.43}$$

$$B = x_2 k_2^{\frac{4}{3}} e^{(\frac{4Lt}{3} - \frac{d_2}{4Lk_2^4}e^{-4Lt})} \tag{4.44}$$

$$C = x_3 k_2^{\frac{4}{3}} e^{(\frac{4Lt}{3} - \frac{d_3}{4Lk_2^4}e^{-4Lt})} \tag{4.45}$$

The scalar field ϕ is defined as follows

$$\phi = [\frac{h(m+2)}{8Lk_2^4}]^{\frac{2}{m+2}} e^{-\frac{8Lt}{m+2}} \tag{4.46}$$

The pressure and energy density are calculated as follows

$$p = \frac{4}{3}d_1 L e^{-4Lt} k_2^{-4} - \frac{44}{9}L^2 - k_2^{-8} e^{-8Lt}(3d_1^2 + d_2^2 + 2d_1 d_2 - \frac{\omega h^2}{2}) \quad (4.47)$$

$$\rho = \frac{52}{9}L^2 + 4d_1 L e^{-4Lt} k_2^{-4} + e^{-8Lt} k_2^{-8}(d_1^2 + 2d_1 d_2 + 2d_1 d_3 + d_2 d_3 + \frac{\omega h^2}{2}) \quad (4.48)$$

The results (4.43-4.48) satisfy (4.17) in the same way, indicating that they are exact solutions to the field equations (4.12-4.16).

The Hubble's parameter, expansion scalar, anisotropy parameter, volume and the shear scalar are given by in the following way

$$H = \frac{1}{4}[d_1 e^{-4Lt} k_2^{-4} + 4L] \quad (4.49)$$

$$\theta = [d_1 e^{-4Lt} k_2^{-4} + 4L] \quad (4.50)$$

$$\triangle = \frac{1}{4}[\frac{1}{4}(d_1 e^{-4Lt} k_2^{-4} + 4L)]^{-2}[(\frac{5}{4}d_1^2 + d_2^2 + d_3^2 - \frac{d_1 d_2}{2} - \frac{d_1 d_3}{2})e^{-8Lt} k_2^{-4} + \frac{4L^2}{9} + \frac{2L}{3}(d_3 + d_2 - 2d_1)e^{-4Lt} k_2^{-4}] \quad (4.51)$$

$$V = k_2^4 e^{(4Lt)} \quad (4.52)$$

$$\sigma^2 = \frac{e^{-8Lt}}{2} k_2^{-8}(\frac{7}{4}d_1^2 + d_2^2 + d_3^2) + \frac{2L^2}{9} - L d_1 e^{-4Lt} k_2^{-4} \quad (4.53)$$

Here, we plot in all the graphs by taking $L = d_1 = d_2 = d_3 = h = m = k_2 = 1$ and $\omega = 500$, $n = 5$, $k_1 = 0$.

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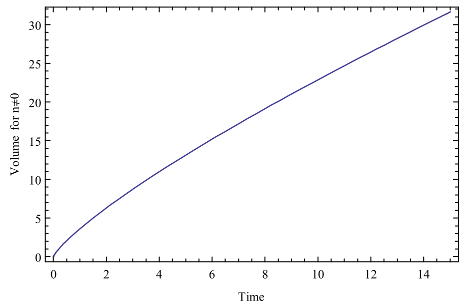

Figure 4.1: V for $n \neq 0$ vs. t (billion years)

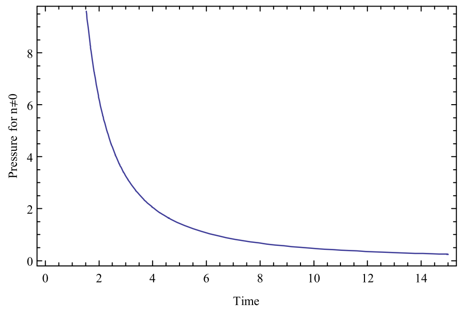

Figure 4.2: p for $n \neq 0$ vs. t (billion years)

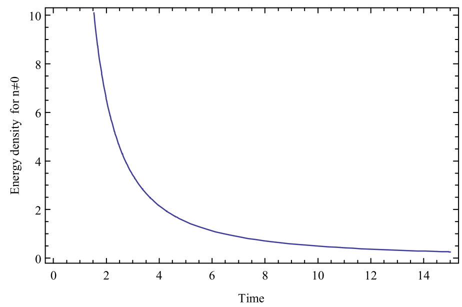

Figure 4.3: ρ for $n \neq 0$ vs. t (billion years)

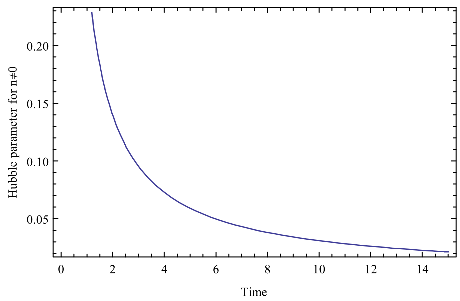

Figure 4.4: H for $n \neq 0$ vs. t (billion years)

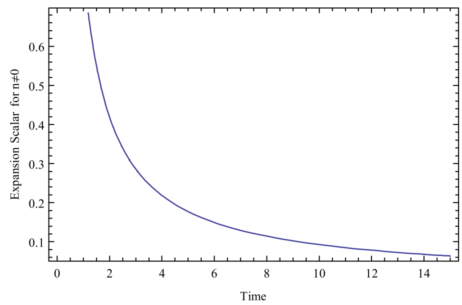

Figure 4.5: θ for $n \neq 0$ vs. t (billion years)

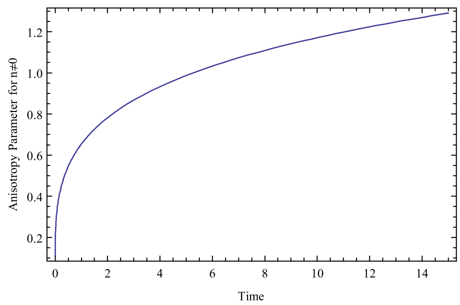

Figure 4.6: $\triangle$ for $n \neq 0$ vs. t (billion years)

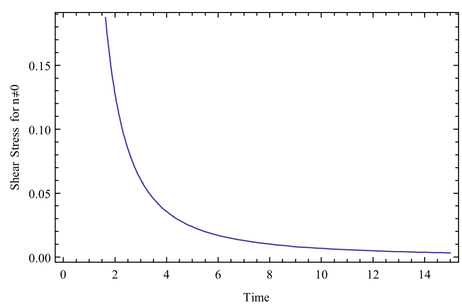

Figure 4.7: σ^2 for $n \neq 0$ vs. t (billion years)

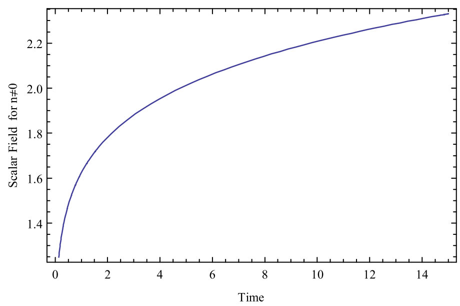

Figure 4.8: ϕ for $n \neq 0$ vs. t (billion years)

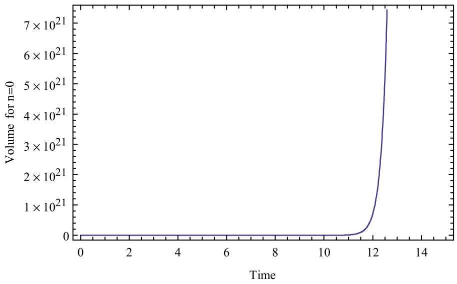

Figure 4.9: V for $n = 0$ vs. t (billion years)

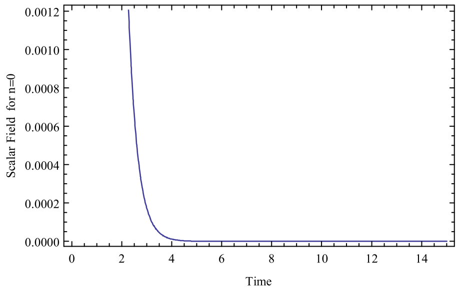

Figure 4.10: ϕ for $n = 0$ vs. t (billion years)

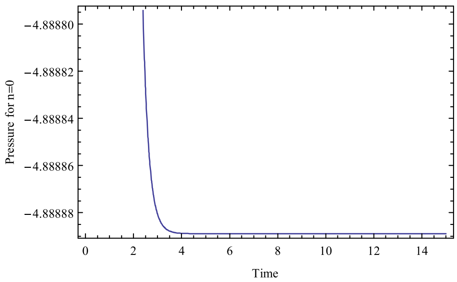

Figure 4.11: p for $n = 0$ vs. t (billion years)

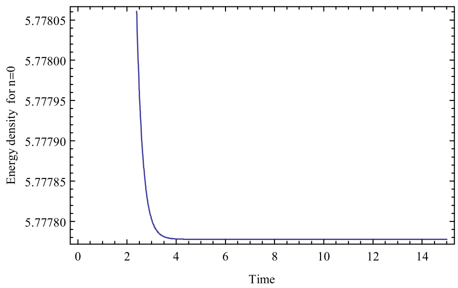

Figure 4.12: ρ for $n = 0$ vs. t (billion years)

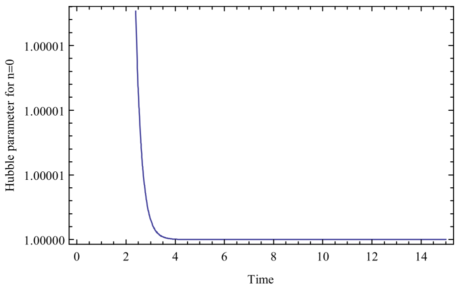

Figure 4.13: H for $n = 0$ vs. t (billion years)

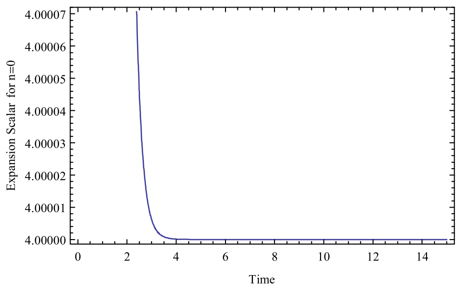

Figure 4.14: θ for $n = 0$ vs. t (billion years)

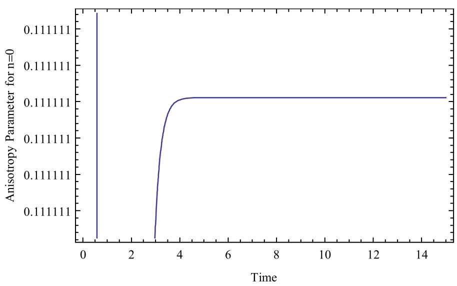

Figure 4.15: $\triangle$ for $n = 0$ vs. t (billion years)

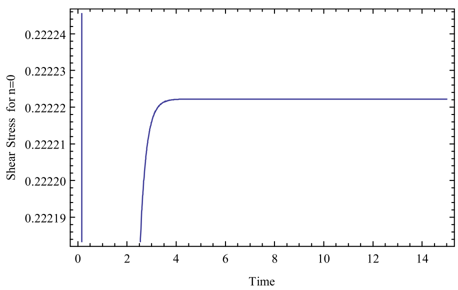

Figure 4.16: σ^2 for $n = 0$ vs. t (billion years)

4.4 Physical and geometrical interpretation

For case(I):

At $t = t_0$, where $t_0 = -k_1/nL$, the expansion scalar is infinite and the spatial volume is zero, indicating that the universe begins to evolve with zero volume at $t = t_0$ and an unlimited rate of expansion. At $t = t_0$, the scale factors vanishes as well, therefore the model has a point singularity at the beginning. At the initial singularity, the pressure, energy density, Hubble's parameters and shear scalar diverges. At the initial epoch, the scalar field and anisotropy parameter also tends to infinity, assuming $n < 4$. After the big bang, the universe expands in a power-law pattern. The scale factors and the spatial volume rises as *t* increases, but the expansion scalar decreases. As a result, when time passes, the pace of expansion slows. As *t* increases ϕ, ρ, *p*, *H*, *A* and σ^2 all decreases. Scale factors and volume grows indefinitely as $t \to \infty$, whereas ϕ, ρ, *p*, *H*, *A*, and σ^2 tends to zero [shown in fig-4.1, fig-4.2, fig-4.3, fig-4.4, fig-4.5, fig-4.6, fig-4.7, fig-4.8]. As a result, for large time *t*, the model would basically produce an empty universe. As $t \to \infty$, the ratio $\frac{\sigma}{\theta}$ approaches to zero, assuming $n < 4$. For large values of *t*, the model approaches isotropy. Hence, the model denotes a shearing, non-rotating and expanding universe with a big bang start that approaches isotropy at late periods.

The integral

$$\int_{t_0}^{t}[V(t')]dt' = \frac{1}{L(n-1)}[(nLt' + k_1)]_{t_0}^{t}$$

is finite if $n \neq 1$. Therefore, this model has a horizon. Also it is defined that the above solutions aren't applicable when $n = 4$. The spatial volume grows linearly with cosmic time for $n = 4$. For $n > 1$, $q > 0$; thus, the model describes a universe that is decelerating. We get, $-1 < q \leq 0$ when $n \leq 1$, which means that the universe is accelerating. The observations of type Ia supernovae (John, 2004; Knop and et al., 2003; Perlmutter and et al., 1997, 1998, 1999; Reiss and et al., 1998, 2004; Tonry and et al., 2003) further show that the universe is accelerating and that the value of deceleration parameter is somewhere between $-1 < q \leq 0$. As a result, the solutions produced in this model are compatible with the observations.

For case(II):

In this case the model has no initial singularity. At time $t = 0$, the spatial volume, scale factors, scalar field, pressure, energy density and other cosmic characteristics are all constant. As a result, the universe begins to evolve at a constant volume and expands at an exponential rate. The scale factors and spatial volume increases exponentially as time *t* increases, whereas the scalar field, pressure, energy density, anisotropy parameter and shear scalar decreases. Here, the expansion scalar remains constant throughout the evolution of the universe, implying that the universe expands uniformly exponentially in this scenario. The scale factors and volume of the universe increases indefinitely large as $t \to \infty$, whereas the scalar field, anisotropy parameter and shear scalar tend to zero. The pressure, energy density and Hubble's factors become constants such that $p = -\rho$ [shown in fig-4.9, fig-4.10, fig-4.11, fig-4.12, fig-4.13, fig-4.14, fig-4.15, fig-4.16]. This shows that the universe is dominated by vacuum energy at late times, which causes the universe's expansion. For large time *t*, the model approaches isotropy. As a result, the model represents a shearing, non-rotating, expanding universe with a finite start that eventually approaches isotropy when time passes.

Here we obtained that $\lim_{t\to 0} \frac{\rho}{\theta^2}$ appears to be a constant. Hence, the model approaches homogeneity and matter at the origin is dynamically negligible; this agrees with Collins (Collins, 1977). The observations of type Ia supernovae (John, 2004; Knop and et al., 2003; Perlmutter and et al., 1997, 1998, 1999; Reiss and et al., 1998, 2004; Tonry and et al., 2003) show that the universe's current state of evolution is accelerating. It is believed that because the universe is currently accelerating, it will continue to expand at the fastest possible rate in the future and forever. For $n = 0$, we get $q = -1$; by coincidence, this value of deceleration parameter leads to $\frac{dH}{dt} = 0$, implying the largest value of Hubble's parameter and the universe's fastest rate of expansion. As a result, the solutions presented in this model are consistent with observations and may have applications in the analysis of late-time evolution of the actual universe in S$\acute{a}$ez-Ballester's theory of gravitation.