Total No. of printed pages = 5

63/2 (SEM-1) MAT 101

2021

(held in 2022)

MATHEMATICS

(Theory Paper)

Paper Code : MAT-101

(Algebra)

Full Marks – 80

Time – Three hours

The figures in the margin indicate full marks for the questions.

1. Answer any *four* from the following questions : 5×4=20

(a) Find the all cyclic subgroups of U (30). Give an example of non-cyclic group whose all subgroups are cyclic.

[Turn over

(b) What is the maximum order of any element in A_{10}? Let $\gamma = (1\,3\,5\,7\,9\,8\,6)\,(2\,4\,10)$ then what is the smallest positive integer n for which $\gamma^n = \gamma^{-5}$?

(c) Define group homomorphism. How many homomorphisms are there from z_{20} onto z_{10}? How many are there to z_{10}?

(d) Suppose $\varphi : \mathbb{Z}_{50} \to \mathbb{Z}_{15}$ is group homomorphism with $\varphi(7) = 6$. Then determine the following :

(i) $\varphi(x)$, (ii) Image of φ

(iii) Ker φ and (iv) $\varphi^{-1}(3)$.

(e) Show that a group of order 72 must have a proper, nontrivial normal subgroup.

(f) State and prove Sylow's second theorem.

2. Answer any *four* from the following questions : 5×4=20

(a) Let in a ring R with unity $(xy)^2 = x^2$, $\forall x, y \in R$, then show that R is commutative.

(b) Let R be a commutative ring with unity. If every ideal of R is prime, show that R is a field.

(c) Let $f : R \to R^{|}$ be an onto homomorphism, then show that $R^{|}$ is isomorphic to a quotient ring of R.

(d) Prove that an ideal P of a commutative ring R is prime if and only if for two ideals A, B of R, $AB \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$.

(e) Let R be commutative ring with unity. Show that an ideal M of R is maximal ideal of R iff R/M is field.

3. Answer any *four* from the following questions : 5×4=20

(a) Let R be a commutative ring with unity and g is a non-zero polynomial in R[x] of degree n with leading coefficient a unit in R. Show that for any $f \in R[x]$ there exist unique polynomials h and r in R[x] such that $f = hg + r$ where either $r = 0$ or $\deg r < \deg g$.

(b) Let R be an integral domain with unity. If $l_1 = \text{l.c.m.}(a,b)$ in R then l_2 is also an l.c.m. (a,b) if and only if l_1 and l_2 are associates.

(c) Let R be an Euclidean domain and let A be an ideal of R then show that there exist $a_0 \in A$ such that $A = \{a_0 x \mid x \in R\}$.

(d) Let R be a Euclidean domain. For all $a, b \in R$, $a \neq 0$, $b \neq 0$, there exist t and r in R such that $a = tb + r$ where either $r = 0$ or $d(r) < d(b)$. Show that t and r are uniquely determined if and only if $d(a+b) \leq \text{Max}\{d(a), d(b)\}$.

(e) Prove that in a PID an element is prime if and only if it is irreducible.

4. Answer any *four* from the following questions : $5 \times 4 = 20$

(a) Let K be a finite extension of F and L, a finite extension of K. Then show that L is a finite extension of F and $[L : F] = [L : K][K : F]$.

(b) Let $\alpha \in K$ be algebraic over F. Then show that $F(a) = F[a]$.

(c) If L is an algebraic extension of K and K, an algebraic extension of F, then show that L is an algebraic extension of F.

(d) Show that any prime field is either isomorphism to the field of rational numbers or to the field of integers modulo some prime number.

(e) If the rational number r is also an algebraic integer, prove that r must be an ordinary integer.