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63/2 (SEM-1) MAT 104

2021

(held in 2022)

MATHEMATICS

Paper Code : MAT-104

(Real Analysis)

Full Marks – 80

Time – Three hours

The figures in the margin indicate full marks for the questions.

1. (a) Show that there does not exists a set with countably infinite power set. 5

Or

Show that the cardinality of the set of all countable subsets of $\mathbb{R}$ is c. 5

(b) Show that the set of all continuous functions $f : \mathbb{R} \mapsto \mathbb{R}$ is equipotent with $\mathbb{R}$. 5

[Turn over

2\. (a) By using the properties of uniform convergence of sequence of functions, show that $\lim_{n\mapsto\infty}\int_0^x g_n(t)dt = \int_0^x \frac{dt}{1-t}$, where $g_n(t)$ stands for the geometric series and $|x|<1$. 5

Or

Let $\{f_n\}$ be a sequence of continuous functions converges uniformly on [a,b], show by means of an example that $\lim_{n\mapsto\infty}\left(\frac{d}{dx}f_n(x)\right) \neq \frac{d}{dx}\left(\lim_{n\mapsto\infty} f_n(x)\right)$. State a condition under which equality will hold good. 3+2=5

(b) Let $\{P_n\}$ be a sequence of polynomial such that $p_0(x) = 0$ and $\forall n \geq 0$, $p_{n+1}(x) = p_n(x) + (x - p_n(x)^2)/2$. Then show that $q_n(x) = p_n(x^2)$ converges uniformly to $f(x) = |x|$ on $[-1, 1]$. 5

(c) Let $Q(x) = |x|$ for $x \in [-1, 1]$, which is extended periodically with period 2, to $\mathbb{R}$ by setting $Q(x+2) = Q(x)$ $(x \in \mathbb{R})$. Then show that

(i) the function $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n Q(4^n x)$$ is continuous.

(ii) for every $x \in \mathbb{R}$, there exists a sequence $\{\delta_n\}$ converging to 0 such that $|(f(x+\delta_m) - f(x))/\delta_n| \to \infty$ as $m \to \infty$.

Also write the physical interpretation of this result. 2+2+1=5

Or

Let $f : [a, b] \to \mathbb{R}$ be a Riemann-integrable function, show that there exists a sequence $\{p_n\}$ of polynomials such that $\int_a^b |p_n(x) - f(x)|^2 dx \to 0$ as $n \mapsto \infty$. 5

(b) By using Fourier series, show that

$$\sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}.$$ 5

Or

Write a short note on the application of Fourier series in periodically forced oscillation. 5

3\. (a) Let u, v, w are respectively functions of x, y and z such that $u := xyz$, $v := x^2 + y^2 + z^2$, $w := x + y + z$, find $J(x, y, z)$. 5

(b) A container with an open top is to have $10m^3$ capacity and be made of thin sheet metal. Calculate the dimensions of the box if it is to use the minimum possible amount of metal. 5

Or

Equal angle bends are made at equal distances from the two ends of a 100 metre long fence so the resulting three segment fence can be placed along an existing wall to make an enclosure of trapezoidal shape. What is the largest possible area for such an enclosure ?

4. (a) Let $f : [a, b] \to \mathbb{R}$ be a function. Then show that $f \in V[a, b]$ if and only if there exists an increasing function $\varphi : [a,b] \to \mathbb{R}$ such that for any $a \le x' < x'' \le b$, $f(x'') - f(x') \le \varphi(x'') - \varphi(x')$. 5

Or

Show that the function f on [0, 1] defined by

$$f(x) := \begin{cases} x^p \sin\left(\frac{1}{x}\right), & \text{if } 0 < x \le 1,\ p \ge 2 \\ 0, & \text{if } x = 0 \end{cases}$$

is of bounded variation on [0, 1]. 5

(b) Let $f : [a, b] \to \mathbb{R}$ be a function and $P_a^b(f)$, $Q_a^b(f)$, $V_a^b(f)$ be respectively the positive, negative and total variation of f on [a, b]. Show that if one of the magnitudes $P_a^b(f)$, $Q_a^b(f)$, $V_a^b(f)$ is finite, then so are the two others. 5

Or

Let $f : [-2, 2] \to \mathbb{R}$ be function defined by $f(x) := 4x^3 - 3x^4$. Find $P_a^b(f)$, $Q_a^b(f)$, $V_a^b(f)$, where $P_a^b(f)$, $Q_a^b(f)$ and $V_a^b(f)$ are respectively the positive, negative and total variation of f on [–2, 2]. 5

(c) $f : [a, b] \to \mathbb{R}$ be a absolutely continuous function. Then show that f is of bounded variation on [a, b]. 5

(d) Let c be the Cantor function defined on [0, 1]. By using integration by parts, find the value of $\int_0^1 x\,dc(x)$. 5

5. (a) Let A and B are two nonempty subsets of $\mathbb{R}$ such that $A \subseteq B \subseteq \mathbb{R}$. Show that A is dense in B if and only if $B \subseteq \bar{A}$. 5

Or

Show that in a metric space a closed subset is nowhere dense if and only if it does not contain any open sphere. 5

(b) Let (X, d) be a complete metric space and F be a nonempty subset of X. Show that F is closed if and only if every Cauchy sequence in F converges to an element in F. 5

(c) Show that any convex set in a metric space is connected. 5

(d) By using Baire Category Theorem, show that there exists no function $f :]0, 1[\to \mathbb{R}$, which is continuous at rationals and discontinuous at irrational. 5

Or

By using Baire Category Theorem, show that a function is continuous except at the points of a first category set if and only if it is continuous at a dense set of points. 5

(Symbols are in their usual meaning.)