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63/2(SEM-2) MAT 203

# 2022

# MATHEMATICS

(Theory Paper)

Paper Code : MAT-203

**(Functional Analysis)**

Full Marks – 80

Time – Three hours

The figures in the margin indicate full marks for the questions.

1. Choose the correct option of the following : 1×6=6

(i) For infinite dimensional normed space X with the closed unit ball $M = \{x \in X : \|x\| \leq 1\}$, then by Riesz's lemma we can construct a sequence $(x_n)$, which

(a) have convergent subsequence such that

$$x_n \in M, (m \neq n) \|x_m - x_n\| \geq \frac{1}{2}$$

(b) have convergent subsequence such that $x_n \in M, (m \neq n) \| x_m - x_n \| = \frac{1}{2}$

(c) cannot have convergent subsequence such that $x_n \in M, (m \neq n) \| x_m - x_n \| \geq \frac{1}{2}$

(d) cannot have convergent subsequence such that $x_n \in M, (m \neq n) \| x_m - x_n \| \leq \frac{1}{2}$

(ii) Let $T: X \to Y$ be a bounded linear transformation on normed spaces then which of the following is true ?

(a) $\| T \| = \sup\{\| Tx \| : x \in X, \| x \| \leq 1\}$

(b) $\| T \| = \sup\left\{\frac{\| Tx \|}{x} : x \in X, x \neq 0\right\}$

(c) $\| T \| = \inf\{K : K \geq 0, \| Tx \| \leq K \| x \|, \forall \in X\}$

(d) All of the above are true

(iii) If $1 + 1 \neq 0$ in the field K, then the bilinear form f can be obtained from the quadratic form q by the following polar form

(a) $f(u, v) = \frac{1}{2}[q(u+v) - q(u) + q(v)]$



(b) $f(u, v) = \frac{1}{2}[q(u+v) - q(u) - q(v)]$

(c) $f(u, v) = \frac{1}{2}[q(u+v) + q(u) - q(v)]$

(d) $f(u, v) = \frac{1}{2}[q(u+v) + q(u) + q(v)]$

(iv) In a Fourier series $x(t) = a_0 + \Sigma_k[a_k \cos kt + b_k \sin kt]$, where $u_k(t) = \cos kt$, $v_k(t) = \sin kt$ then the series can be written $x(t) = a_0 + \Sigma_k[a_k u_k(t) + b_k v_k(t)]$. Let $(e_j)$ and $(\hat{e}_j)$ be two orthonormal bases, then the inner product form of the series will be

(a) $x = < x, e_0 > e_0 + \Sigma_{k=1}^{\infty}\left[< x, e_k > e_k + < x, \widehat{e_k} > \widehat{e_k}\right]$

(b) $x = < x, e_0 > e_0 + \Sigma_{k=1}^{\infty}\left[< x, e_k > e_k - < x, \widehat{e_k} > \widehat{e_k}\right]$

(c) $x = < x, e_0 > e_0 - \Sigma_{k=1}^{\infty}\left[< x, e_k > e_k - < x, \widehat{e_k} > \widehat{e_k}\right]$

(d) $x = < x, e_0 > e_0 - \Sigma_{k=1}^{\infty}\left[< x, e_k > e_k + < x, \widehat{e_k} > \widehat{e_k}\right]$

(v) Let M # $\phi$ be the subset of normed space X and $\bar{M}$ be its closure, then there exists an element $x \in \bar{M}$ if and only if there exists a sequence $(x_n) \in M$ such that

(a) $x_n$ not converges to x

(b) $(x_n)$ is closed

(c) $(x_n)$ is compact

(d) $x_n$ converges to x

(vi) Choose the correct statement :

(a) A normed linear space is metric space under $d(x, y) = \| x - y \|$.

(b) An inner product space is a normed space under $| x \|^2 = < x, x >$

(c) An inner product is a metric space under $d(x, y = \| x - y \| = \sqrt{< x - y, x - y >}$

(d) All of the above.

2. Answer the following questions : 2×5=10

(a) Define bounded linear functional and describe in $l^2$ space.



(b) Let X be a normed space and let $x_0 (\neq 0) \in X$. Then show that there exists a bounded linear functional f^ on X such that $| f^\wedge \| = 1, f^\wedge(x_0) = \| x_0 \|$.

(c) If x is orthogonal to y then show that $\| x + y \|^2 = \| x \|^2 + \| y \|^2$.

(d) Let H be a Hilbert space and $\{e_i\}$ be an orthonormal set in H. If $x = \Sigma \langle x, e_i \rangle e_i$ then show that $\| x \|^2 = | \langle x, e_i \rangle |^2$.

(e) Find the symmetric matrix that corresponds to the following quadratic forms :

$$q(x, y, z) = 3x^2 + 4xy - y^2 + 8xz - 6yz + z^2.$$

3. Answer any *six* of the following questions : 5×6=30

(a) State and prove Riesz's Lemma.

(b) Prove that- in a finite dimensional Normed space X, any subset $M \subset X$ is compact if and only if M is closed and bounded.

(c) Let $(T_n)$ be a sequence of bounded linear operators $T_n : X \to Y$ from a Banach space X to a normed space Y such that $(\| T_n x \|)$ is bounded for every $x \in X$. Then show that the sequence of the norms $(\| T_n \|)$ is bounded.

(d) Let f be a bounded linear functional on a subspace Z of a normed space X. Then show that there exists a bounded linear functional f* on X which is a extension of f to X and has the same norm, $\| f^* \|_x = \| f \|_z$.

(e) Let λ be an eigen value of a linear operator T on V. Then prove the following :

(i) If $T^* = T^{-1}$ then $|\lambda| = 1$.

(ii) If $T^* = T$ then λ is real.

(iii) If $T^* = -T$ then λ is imaginary.

(iv) If $T = S^*S$ with $|S| \# 0$ then λ is real and positive.

(f) Prove: "A subspace Y of a Hilbert space is closed if and only if $Y = Y^{\perp\perp}$."

(g) Let $(e_k)$ be an orthonormal sequence in an inner product space X. Then show that $\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \le \| x \|^2$. (Bessel's Inequality)



(h) If P is a projection on a Hilbert space H with range M and null space N then show that $M \perp N$ if and only if P is self-adjoint.

(i) Let $T : X \to Y$ be a linear mapping where X and Y be normed linear spaces. Then show that the following statements are equivalent to one another

(i) T is continuous at any point $x_0$,

(ii) T is bounded,

(iii) If $S = \{x : \| x \| \le 1\}$ is closed unit sphere in X then its image is a bounded set in Y.

4. Answer any *two* of the following questions : 10×2=20

(A) State Gram-Schmidt Orthogonalization Process, and apply to find an orthogonal basis and then an orthonormal basis for the subspace U of $\mathbb{R}^4$ spanned by 3+7=10

$$v_1 = (1,1,1,1); v_2 = (1,2,4,5); v_3 = (1,-3,-4,-2)$$

(B) State and prove Riesz's Theroem in Hilbert space. 10

(C) Prove that : 5+5=10

(a) The space $l_p (p \neq 2)$ is not a Hilbert space.

(b) The followings are equivalent

(i) $P = T^2$ for some self-adjoint operator T.

(ii) $P = S^*S$ for some operator S, i.e., $P > 0$

(iii) P is self-adjoint and $< Pu, u > \geq 0$ for every $u \in V$.

5. Answer any *one* of the following questions : 14×1=14

(a) Prove that :

$L_p$, space is a normed space as well as complete under the norm

$\|f\|_p = \left[\int_x |f(x)|^p \, d\mu(x)\right]^{1/p}$, where $p > 0$ and f be a continuous complex valued measurable function on X with measure $\mu$ such that $\int_x |f(x)|^p \, d\mu(x) < \infty$.



(b) Let X be a real vector space and p be a sublinear functional on X. Furthermore, let f be a linear functional which is defined on a subspace z of X and satisfies $f(x) \leq p(x) \forall x \in z$. Then show that f has a linear extension $f^{\wedge}$ from Z to X, satisfying f $f^{\wedge}(x) \leq p(x), \forall x \in X$.

(All the symbols have their usual meaning.)